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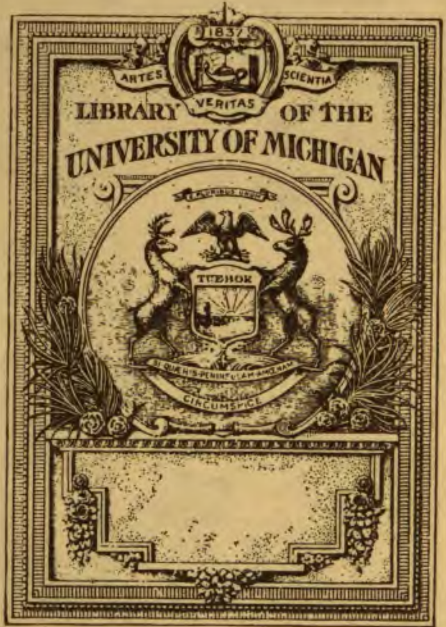
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ELEMENTARY ALGEBRA

FIRST COURSE

BY

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PREFACE

The work in this book is that comprised in the usual first-year course in algebra in secondary schools. The material is so arranged as to adapt the book to the use of those schools that give only a brief course in the subject, as well as schools giving more extended or more difficult courses. A more advanced book, a companion to this, called the SECOND COURSE, reviews the important topics covered in the FIRST COURSE, amplifies the treatment of these topics where necessary, and covers all additional topics that are required for college entrance.

In the preparation of this book, the authors have combined what teachers have found to be the permanently valuable features of their *Essentials of Algebra* with those features which characterize the most modern ideas in the teaching of secondary school mathematics, and which, through widespread test by application in the classroom, have become generally accepted. It is believed that this text will be found teachable and well balanced. It is free from "fads" and untried experiments, yet sufficiently in harmony with present tendencies to meet the demands of the most progressive teacher.

The book contains a minimum of theory and a maximum of practice. Abundance of drill is provided throughout the book. New processes and principles are psychologically and adequately developed, but they are finally mastered by the pupil through plentiful application in drill exercises and problems.

Throughout the text the work has been planned so as to introduce the pupil *to only one new difficulty at a time*, and this is mastered through copious drill before proceeding to the next. This controlling principle in the development and sequence of subject matter is one of the fundamentally important features

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of the book. Observe, for example, the introductory chapter, in which one new point of notation is developed at a time. Observe, also, the entire treatment of the equation.

The pupil is led to see the subject, not merely as a system of exercises to be pursued for the purpose of mental discipline, although the authors are thorough believers in the mental discipline feature of the study, but rather as a scientific instrument for solving certain types of problems such as are actually encountered in the world's work. See §§ 1, 2, 15, 16, 17, 22, etc.

To this end adequate use has been made of real applied problems of the various types encountered in practical life. Thus, in Chapter I computation by use of practical formulæ is introduced. In Chapter II the equation is introduced as a scientific instrument used in the solution of practical problems. In Chapter III many practical applications of negative number are made, etc. These real applied problems lend intense interest to the work. Through them the work is legitimately motivated. Furthermore, it is through the application of the pupil's knowledge of algebra in the solution of these real problems that the knowledge is led to function. The old-time useless puzzles have been eliminated from this book.

Algebra has been correlated with arithmetic and geometry. Review and use of the various process with whole numbers, common and decimal fractions, have been made in the practical computation with formulæ, and in the process of checking work by evaluation of the literal expressions involved. Short methods of multiplication of arithmetical numbers are taught in connection with algebraic multiplication and factoring, as in §§ 75, 77, etc. Principles and processes in arithmetic have been made the starting points in the development of many principles and processes of algebra, as in §§ 50, 51, 54, etc. Considerable use has been made of percentage in the solution of certain types of business problems. The facts of mensuration have been much used, and some empirical knowledge of simple facts in geometry with which the pupil is familiar has been drawn upon in problems.

The graph has been used as a natural means of solving certain types of problems and of interpretation of algebraic principles, rather than as an excrescence in the form of topics of analytic geometry.

Supplementary exercises have been given at the end of each chapter. They adapt the text to the use of schools with different kinds of courses. These exercises, as a rule, are more difficult than those in the body of the chapters. Schools wanting a brief course may omit these entirely. They may be used for reviews or in classes wanting a more difficult course.

The authors wish to acknowledge their indebtedness to all those whose timely suggestions and criticisms have helped in the preparation of this textbook, and especially to Mr. R. G. Kinkead, Assistant Superintendent of Schools, Columbus, Ohio; who has read critically all of the manuscript.

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JAMES F. MILLIS.

MAY, 1911.

CONTENTS

CHAPTER		PAGE
I.	THE FORMULA: GENERAL NUMBER	1
II.	THE EQUATION	19
III.	POSITIVE AND NEGATIVE NUMBERS	34
IV.	ADDITION AND SUBTRACTION OF LITERAL EXPRESSIONS	60
V.	MULTIPLICATION AND DIVISION OF LITERAL EXPRESSIONS	74
VI.	LINEAR EQUATIONS: PROBLEMS	92
VII.	SPECIAL PRODUCTS AND QUOTIENTS	117
VIII.	FACTORS. MULTIPLES. EQUATIONS SOLVED BY FACTORING	136
IX.	FRACTIONS	164
X.	FRACTIONAL EQUATIONS. PROBLEMS. FORMULÆ	189
XI.	PROPORTION. VARIABLES	206
XII.	SYSTEMS OF LINEAR EQUATIONS	232
XIII.	SQUARE ROOT. QUADRATIC SURDS	255
XIV.	QUADRATIC EQUATIONS	269
XV.	SYSTEMS INVOLVING QUADRATIC EQUATIONS	286
XVI.	EXPONENTS	294
	MISCELLANEOUS EXERCISES	300

ELEMENTARY ALGEBRA

FIRST COURSE

CHAPTER I

THE FORMULA: GENERAL NUMBER

1. **Algebra.** — Of the many kinds of problems encountered in the world's work, some are solved by arithmetic alone, some by *algebra*, and some by other branches of mathematics. **Algebra**, like arithmetic, deals with numbers. There is no clear line of distinction between these two subjects. But in addition to the Hindu symbols of arithmetic, algebra employs letters to represent numbers. In addition to the integers and fractions of arithmetic, it deals with new kinds of numbers. And, as we shall see, it employs some new and interesting principles in the use of numbers.

2. **The Formula.** — Many of the *rules* which the student has already encountered in arithmetic, as well as practical rules which are used in the various fields of manufacturing, the trades, the sciences, etc., are expressed often by **formulæ**, in which the numbers are represented by *letters*. A **formula** expresses a rule in a sort of shorthand. This will be seen clearly in the following problems.

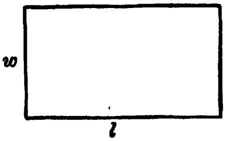
EXERCISES

1. How many square feet in a floor 12 feet wide and 15 feet long?

2. A concrete walk is 4 feet 6 inches wide and 24 feet long. Find the area of its surface.

3. What is the area of a rectangle 3.4 inches wide and 7.8 inches long?

4. In general, the rule for finding the area of any rectangle that is l feet long and w feet wide may be expressed by the formula: Area $= l \times w$ square feet.



Find the area when $l = 3\frac{1}{2}$ feet and $w = 2$ feet. When $l = 16$ feet and $w = 2\frac{1}{2}$ feet. When $l = 50$ feet and $w = 4\frac{1}{2}$ feet. When $l = 136$ feet and $w = 80$ feet.

3. Particular and General Numbers. — In Problem 4, § 2, the letter l is used to represent the length of any rectangle, and hence if we consider all possible rectangles, l may have many particular values. Similarly, if we consider all possible rectangles, w may have many particular values. It is seen, then, that in expressing a rule by a formula, letters are used to represent numbers, and to such letters may be assigned many particular values.

Compared with the **particular numbers** of arithmetic, 1, 2, 3, 4, etc., numbers represented by letters are called **general numbers.**

Numbers represented by letters are also called **literal numbers.**

4. Factors. Signs of Multiplication. — The numbers which are multiplied to form a given number are called its **factors,** as l and w in $l \times w$. In addition to use of the sign × to indicate multiplication, as in Problem 4, § 2, multiplication is sometimes expressed by placing a *dot* between the two factors, halfway up from the lower edge of the number-symbols.

Thus, $l \times w$ may be written also $l \cdot w$.

But unless both of the numbers multiplied are represented by Hindu symbols, the multiplication usually is expressed by *omitting* the multiplication sign.

Thus, the product $l \times w$ is written lw, and $2 \times w$ is written $2w$.

Express in three ways the product of 5 and l.

EXERCISES

1. Express the product of a and b; of 2, x, and y.

Give the value of $2ab$ when a and b have the following values:

2. $a = 2$, $b = 3$.

3. $a = 5$, $b = 6$.

4. $a = 3$, $b = 7$.

5. $a = 10$, $b = 30$.

6. $a = 40$, $b = 20$.

7. $a = 50$, $b = 30$.

8. $a = 25$, $b = 4$.

9. $a = 75$, $b = 6$.

10. $a = 60$, $b = 3\frac{1}{2}$.

11. Read Exercise 4, page 2, and give a meaning for $A = lw$.

Find the value of A when l and w have the following values:

12. $l = 18$, $w = 10$.

13. $l = 12.5$, $w = 8$.

14. $l = 13\frac{1}{3}$, $w = 12$.

15. $l = 6.2$, $w = 4$.

16. $l = 2.4$, $w = 1.5$.

17. $l = 8.5$, $w = 3.2$.

18. When b represents the base of a triangle, h the altitude, and A the area, the area is found by the formula $A = \frac{1}{2}bh$. Give the rule which this formula expresses.

In the formula $A = \frac{1}{2}bh$, give the value of A when b and h have the following values:

19. $b = 12$, $h = 10$.

20. $b = 14$, $h = 12$.

21. $b = 20$, $h = 16.5$.

22. $b = 8.5$, $h = 12$.

23. $b = 16$, $h = 9.5$.

24. $b = 12.5$, $h = 8$.

25. If c is the circumference and d the diameter of a circle, then $c = \pi d$. Compute the circumference of a circle from the formula $c = \pi d$ when $d = 10$. ($\pi = 3.1416$, approximately.)

26. When $r =$ the radius of a circle, $c = 2\pi r$. Find c when $r = 40$; $r = 60$; $r = 7.5$.

27. The volume of a prism is found by multiplying the area of the base by the height or altitude. If $V =$ volume, $b =$ area of base, and $h =$ altitude, then $V = bh$. Find V when

$b = 40$ and $h = 30$; $b = 48$ and $h = 12$; $b = 96$ and $h = 10\frac{1}{2}$.

28. If the base of a prism is a rectangle whose length is l and whose width is w, then $b = lw$. Hence bh becomes lwh. Then we have $V = lwh$. Express the meaning of this formula.

In the formula $V = lwh$, find V when l, w, and h have the values:

29. $l = 42$, $w = 16$, $h = 14$.

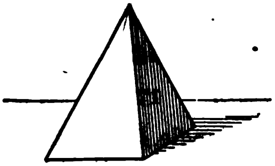
30. $l = 38$, $w = 20$, $h = 12\frac{1}{2}$.

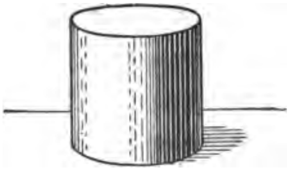
31. $l = 36$, $w = 30$, $h = 18$.

32. $l = 42$, $w = 38$, $h = 12$.

33. The volume of a pyramid is computed by the formula $V = \frac{1}{3}bh$, where $V =$ volume, $b =$ area of base, and $h =$ altitude. State in words the rule which this formula expresses.

Find V when $b = 12$ and $h = 4\frac{1}{2}$.

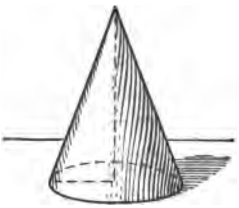




34. The volume of a cylinder is equal to the product of the altitude by the area of the base. Express this by a formula.

35. The volume of a cone is one third of the volume of a cylinder of the same base and the same altitude. If $V =$ volume, $b =$ area of base, and $h =$ altitude, write the formula for computing the volume of a cone.

If the base of a cone is 36 square inches, and the altitude 1 foot 6 inches, find its volume.



36. The rule for finding the area of a circle whose radius is r is expressed by the formula, $A = \pi r^2$. Give the rule which this formula expresses.

Find A when $r = 4$. When $r = 10$.

37. Find the area of a circle whose radius is 6 inches.

38. When there is a steam pressure of 90 pounds per square inch in the cylinder of an engine, what is the total pressure on a 12-inch piston? (By a 12-inch piston is meant a piston whose diameter is 12 inches.)

5. Powers. — In Problem 36, § 4, r^2 is a short way of writing rr. Similarly, rrr is written r^3, $aaaa$ is written a^4, $10 \times 10 \times 10 \times 10 \times 10$ is written 10^5, etc. In general, when all of the factors of a product are equal, the product is called **a power** of one of the factors, and is written in this abbreviated form

x^2 is read "x square," or "x second power";
x^3 is read "x cube," or "x third power";
x^4 is read "x fourth power";
x^5 is read "x fifth power"; etc.

In a power such as m^7, the factor m is called **the base**, and the number 7 is called the **exponent**. The exponent tells how many times the base is used as a factor to make the power.

EXERCISES

1. Name the bases and exponents in the following: a^3, y^{10}, C^6, A^{25}, 9^5, x^n.

2. Express with exponents: $mmmmmm$, $bbbbbbbb$, $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$.

3. Find the value of $2 \times 2 \times 2$; of 2^3; of 3^3; of 2^4; of 3^4; of 2^5.

4. In the formula, $A = \pi r^2$, find A when $r = 6.5$.

5. The area of a circle is found also by use of the formula, $A = \frac{1}{4}\pi d^2$, where d is the diameter. Express this in words as a rule. Find A when $d = 4\frac{1}{2}$.

6. The area of the surface of a sphere is computed by the formula $A = 4\pi r^2$, where $A =$ area, $r =$ radius. Find the area if the radius is 12 inches.

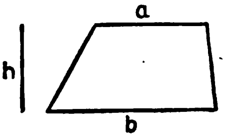
7. The earth is approximately a sphere whose radius is 4000 miles. Find its area.

8. The rule for computing the volume of a cone of which the base is a circle is expressed by $V = \frac{1}{3}\pi r^2 h$, where $r =$ radius of base and $h =$ altitude. Find V when $r = 6$ and $h = 4$.

9. The volume of a sphere is computed by the formula, $V=\frac{4}{3}\pi r^3$, where $V=$ volume and $r=$ radius. Find V when $r=3$.

10. The diameter of the moon is 2160 miles. Find its volume.

11. The bases or parallel sides of a trapezoid are 6 inches and 10 inches, respectively, and the altitude or distance between them is 8 inches. Find the area.



12. The area of a trapezoid is equal to the product of one half of the altitude by the sum of the bases. This is expressed by the formula, $A=\frac{1}{2}h(a+b)$, where h is the altitude and a and b, respectively, are the bases. Find A when $h=4$ feet, $a=3$ feet, and $b=6$ feet.

6. Use of Parentheses. — In the formula in Problem 12, § 5, the symbols ( ), which inclose $a+b$, or the sum of the bases, are called **parentheses.** These symbols are used to inclose two or more numbers which are to be added or subtracted, etc., and indicate that those numbers are not to be used singly, but are first to be combined into one number and then the result used.

Thus, $\frac{1}{2}h(a+b)$ indicates that a and b are first to be added, then the result multiplied by $\frac{1}{2}h$. Similarly, $x(a^2-b^2)$ indicates that a^2-b^2, as one number, is to be multiplied by x; that is, we must first square the values of a and b and subtract, then multiply the difference by x. And $(m+n)(m-n)$ indicates that we are first to find the sum of m and n, then their difference, then multiply these results.

In using a formula containing parentheses, *the operations with the numbers within the parentheses must be performed first.*

7. Other Signs of Grouping. — There are other symbols having the same meaning as parentheses that are used sometimes to inclose numbers. They are **brackets** [ ] and **braces** { }. A fourth symbol, called the **vinculum**, is sometimes used. It consists of a straight line drawn above the numbers, as $\overline{a+b}$, and is used especially in expressing roots and fractions.

Thus, $B(M-N)$, $B[M-N]$, $B\{M-N\}$, and $B\cdot\overline{M-N}$ all indicate the same thing, viz. that N is to be subtracted from M, and the remainder multiplied by B.

All of these symbols are called **signs of grouping** or **signs of aggregation.**

EXERCISES

1. In the formula $A=\frac{1}{2}h(a+b)$, find A when $h=4$, $a=6$, and $b=7$.

2. Find the value of $d(x-y)$ when $d=12$, $x=5$, $y=1$.

3. Find the value of $(m-n)(m+n)$ when $m=10$ and $n=6$.

4. Find the value of $D^2\{D-5\}$ when $D=8$.

5. Find the value of $[A+B][A-B][A^2+B^2]$ when $A=5$ and $B=3$.

6. Find the value of $(x+y)^2$ when $x=4$ and $y=2$.

7. Find the value of $5(R+S)^2(R-S)^2$ when $R=10$ and $S=2$.

8. Find the value of $(g+10)\div(5-g)$ when $g=3$.

9. In the formula, $S=a(t-\frac{1}{2})$, find the value of S when $a=32$ and $t=3$.

10. In the formula $A=\pi(R^2-r^2)$, find the value of A when $R=14$ and $r=6$.

11. In the formula $S=2\pi R(H+R)$, find the value of S when $R=4$ and $H=5$.

Letting the hypotenuse of a right triangle $=z$, the base $=x$, and the altitude $=y$, it has been shown in arithmetic that $z=\sqrt{x^2+y^2}$. Find the value of z when:

12. $x=3$, $y=4$.
13. $x=6$, $y=8$.
14. $x=40$, $y=30$.
15. $x=15$, $y=20$.
16. $x=12$, $y=16$.
17. $x=18$, $y=24$.

In the formula $x=\sqrt{z^2-y^2}$, find the value of x when:

18. $z=5$, $y=4$.
19. $z=10$, $y=6$.
20. $z=25$, $y=20$.
21. $z=15$, $y=12$.
22. $z=50$, $y=30$.
23. $z=35$, $y=28$.

8. Number Expressions. — Any number symbol, or combination of number symbols, indicating one or more of the operations of addition, subtraction, multiplication, etc., such as those used in the formulæ in the preceding sections, is called **a number expression**, or simply an **expression.**

Thus, πr^2, $2A - 3B + C$, and $\frac{1}{2}h(b + b')$ are expressions.

An expression involving one or more numbers represented by letters is often called a **literal expression.**

9. Terms. — In the expression $5x^2 - 2xy + 7y^2$, the parts $5x^2$, $2xy$, and $7y^2$ are called **terms.** In general, the parts of an expression connected by the signs + or − are called the *terms* of the expression. An expression such as $6a^2$, or $10x(m - n)$, which is not formed by two or more parts connected by the signs + or −, consists of *one term.*

10. Names of Expressions. — How many terms in $5a^2\sqrt{b}$? In $2P^2 - Q^2$? In $4A - B - C$? In $n^3 - 2n^2 + 3n - 1$?

An expression which consists of *only one* term is a **monomial.**

An expression which consists of *two* terms is a **binomial.**

An expression which consists of *three* terms is a **trinomial.**

The name **polynomial** is applied to an expression which consists of *more than one* term.

It must be remembered that an expression within a *sign of grouping* is to be considered as one term.

Thus, $4a - (b - 3c)$ is a *binomial.* Name the two terms.

EXERCISES

Tell how many terms in each of the following expressions, and apply the name "monomial," "binomial," etc., to each:

1. $25A^2BC$.
2. $x^3 - y^3$.
3. $a^2 + 2ab + b^2$.
4. $\pi h(r + r')$.
5. $(m + n)^2$.
6. $a^4 + b^4$.
7. $\frac{1}{6Rt^3}$.
8. $110 - 5\frac{H}{D}$.
9. $\frac{62.5QH}{550}$.

10. $.0622\frac{W}{V}(v-v')^2$.

11. $.7854\,D^2p\left(1-\frac{F}{100}\right)-W$.

12. $\sqrt{A}+C$.

13. y^2-5y+6.

14. $a^2-(b-c)^2$.

15. $m^2+n^2+p^2+2mn+2mp+2np$.

16. $R^2-D^2+2DC-C^2$.

17. $2xy-(x^2+y^2)+z^2$.

18. $r_1r_2+r_2r_3+r_1r_3$.

19. $(a+b)(a-b)$.

20. $x^3+3x^2y+3xy^2+y^3$.

11. Evaluation of Polynomials. — The computations by the formulæ in the preceding sections have given practice in finding the values of expressions of *one term.*

In finding the value of an expression containing more than one term, the value of each term must be found before the additions or subtractions indicated between the terms.

Thus, if $x=2$, $3x^2-4x+7=12-8+7=11$.

And, if $a=12$, $b=3$, $c=2$, and $d=4$,

$$\begin{aligned}2c(a-b)-(b+c)d &= 2\times 2\times 9-5\times 4\\ &= 36-20\\ &= 16.\end{aligned}$$

Here the values of $a-b$ and $b+c$ are computed first. See § 6.

EXERCISES

1. Find the value of $2P-3Q+7R-2S$ when $P=10$, $Q=6$, $R=1$, $S=4$.

2. Find the value of $M^2+2MN+N^2$ when $M=4$, $N=5$.

3. Find the value of $ab+bc-cd-5$ when $a=10$, $b=6$, $c=1$, $d=3$.

4. Find the value of x^3+5x^2-2x-3 when $x=4$.

5. Find the value of y^3-x^3 when $y=4$, $x=3$.

6. Find the value of $(u-v)^2-(w-x)^2$ when $u=12$, $v=8$, $w=5$, $x=3$.

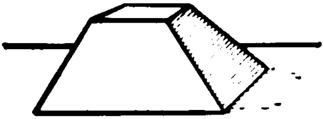
7. Find the value of $[2p \div$ [illegible] $q^2 -$ [illegible] -2 [illegible] $4p \div q]$ when $p=1$, $q=4$.

8. Find the value of $\sqrt{A^2+B^2+2}-\sqrt{C^2-75}$ when $A=5$, $B=3$, $C=10$.

9. $V=\frac{1}{3}\pi h(r^2+r'^2+rr')$. Find V when $r=12$, $r'=15$, $h=18$.

10. In § 10, Exercise 8, find value of expression when $H=96$, $D=15$.

11. In § 10, Exercise 11, find value of expression when $D=18$, $p=60$, $F=10$, $W=1140$.



12. The volume of the frustum of a pyramid is computed by the formula $V=\frac{1}{3}h\ (b+B+\sqrt{bB})$, where $V=$ volume, $h=$ altitude, $b=$ area of one base, and $B=$ area of other base. Find V when $h=6$ inches, $b=16$ square inches, and $B=49$ square inches.

In the formula $V=\frac{1}{3}h(b+B+\sqrt{bB})$, find the value of V when:

13. $h=6$, $b=16$, $B=25$.

14. $h=8$, $b=9$, $B=36$.

15. $h=10$, $b=25$, $B=49$.

16. $h=12$, $b=49$, $B=64$.

17. $h=16$, $b=25$, $B=36$.

18. $h=18$, $b=9$, $B=25$.

In the formula $V=\frac{1}{3}\pi h(r^2+r'^2+rr')$, find the value of V when:

19. $r=3$, $r'=6$, $h=9$.

20. $r=8$, $r'=9$, $h=6$.

21. $r=12$, $r'=15$, $h=18$.

22. $r=10$, $r'=16$, $h=12$.

23. **Coefficients.** — If an expression is separated into two factors, either factor is called the **coefficient** (co-factor) of the other. If a coefficient is an arithmetical number, it is called a **numerical coefficient.** Otherwise, it is a **literal coefficient.**

Thus, 6 is the numerical coefficient of xy; x is the coefficient of $6y$; and y is the coefficient of $6x$. In $4x(a-b)$, what is the numerical coefficient? What is the coefficient of $4(a-b)$? Of $a-b$?

If no numerical coefficient is written, the numerical coefficient 1 is understood.

Thus, since x is the same as $1x$, the numerical coefficient of x is 1. Likewise, the numerical coefficient of PQ is 1, and of $(m-n)$ it is 1.

13. Similar Terms. — In what respect do $2A$, $7A$, and $10A$ differ? Terms which do not differ at all, or which differ only in their coefficients, are called **like** or **similar terms.**

Thus, x^2y, $4x^2y$, and $20x^2y$ are similar. The terms, ax, bx, and cx are similar terms in x.

14. Addition and Subtraction of Similar Terms. — Just as the sum of 2 apples, 5 apples, and 8 apples is 15 apples, so the sum of $2a$, $5a$, and $8a$ is $15a$. And just as 3 men + 4 men = 7 men, so $3m + 4m = 7m$. Similarly, $6x + 9x = (6+9)x$, or $15x$; $C + 5C + 3C = (1+5+3)C$, or $9C$.

To add similar terms, add their coefficients and to the result attach the common letters with their exponents.

Just as 12 lb. − 7 lb. = 5 lb., so $12l - 7l = 5l$. And $32xy - 18xy = (32-18)xy$, or $14xy$; $15a^2b^2 - 9a^2b^2 = (15-9)a^2b^2$, or $6a^2b^2$.

To subtract similar terms, subtract their coefficients and to the result attach the common letters with their exponents.

EXERCISES

Find the sums in 1–12:

1. $2a, 3a, 4a$.
2. $5x, x, 2x$.
3. $10W, 3W, 8W$.
4. $P, 6P, 4P, 2P$.
5. $3t, t, 7t, 6t$.
6. $5S, 12S, S, 2S$.
7. $4k, 2k, 9k, 5k$.
8. $6A, 12A, A, A$.
9. $3x, 5x, 9x$.
10. $mn, 4mn, 3mn, 10mn$.
11. $abc, 9abc, 3abc, abc$.
12. $4E, 3E, 15E, 10E$.
13. $2a + 6a + 4a = ?$
14. $4b + b + 12b = ?$
15. $60z + 25z + 5z + z = ?$
16. $12d - 4d = ?$
17. $9r_1 - 3r_1 = ?$
18. $16m' - 12m' = ?$
19. $13\pi R^2 - 5\pi R^2 = ?$
20. $10y - 10y = ?$
21. $25D - 17D = ?$
22. $69w - 37w = ?$
23. $20 - 8 + 10 = ?$
24. $12 + 3 - 9 = ?$
25. $16 - 2 - 12 = ?$
26. $6n - 2n + 3n - 4n = ?$
27. $12t + 2t - 7t - t = ?$
28. $3\pi D + 5\pi D - \pi D - 4\pi D = ?$

29. Compare the values of $3+5$ and $5+3$. Of $6a+3a$ and $3a+6a$. Of $4P+7P+2P$ and $7P+2P+4P$, or $2P+4P+7P$.

30. Compare the values of $5+2+6$ and $5+(2+6)$. Of $6H+4H+2H$ and $6H+(4H+2H)$.

Combine the similar terms in the following:

31. $3a+2a+4+7$.

Solution. $3a+2a=5a$; $4+7=11$.
Hence, $3a+2a+4+7=5a+11$.

32. $5N-2N+9-3$.

33. $12z+4+6z+2$.

34. $9R-3+2R+7$.

35. $20+v+5+4v$.

36. $10m-8+10-8m$.

37. $3F-12-F+20$.

38. $K+10+7K-3-4K-6$

39. $12s+18-s-12-5s$.

SUPPLEMENTARY EXERCISES

Evaluate the following

1. $A=lw$, when $l=16$ and $w=14$.
2. $A=\frac{1}{2}bh$, when $b=18$ and $h=14\frac{1}{2}$.
3. $C=\pi d$, when $d=16$ and $\pi=3\frac{1}{7}$.*
4. $C=2\pi r$, when $r=12$, and $\pi=3\frac{1}{7}$.
5. $A=\pi r^2$, when $r=9$ and $\pi=3.1416$.
6. $A=\frac{1}{4}\pi d^2$, when $d=8$ and $\pi=3.1416$.
7. $A=4\pi r^2$, when $r=16$, and $\pi=3\frac{1}{7}$.
8. $A=\pi d^2$, when $d=25$ and $\pi=3.1416$.
9. $V=\frac{1}{3}\pi r^2h$, when $r=6$, $h=12$, and $\pi=3.1416$.
10. $V=\frac{4}{3}\pi r^3$, when $r=8$ and $\pi=3.1416$.
11. $A=\frac{1}{2}h(a+b)$, when $h=8$, $a=10$, and $b=15$.
12. $V=\frac{1}{3}h(b+B+\sqrt{bB})$, when $h=16$, $b=25$, and $B=49$.
13. $V=\frac{1}{3}\pi h(r^2+r'^2+rr')$, when $\pi=3\frac{1}{7}$, $h=10$, $r=6$, and $r'=9$.
14. Recall the rules you studied in arithmetic, and give the meanings of as many of the above formulæ as possible.

---

* **In practical work $3\frac{1}{7}$, instead of 3.1416, is often used for the value of π.**

In the formula for simple interest, $i = prt$, find i when:

15. $p = \$500$, $r = 5\%$, $t = 2$.
16. $p = \$1650$, $r = 6\%$, $t = 3\frac{1}{2}$.
17. $p = \$780$, $r = 5\frac{1}{2}\%$, $t = 2\frac{1}{2}$.
18. $p = \$900$, $r = 4\frac{1}{2}\%$, $t = 3$.
19. $p = \$1200$, $r = 5\%$, $t = 1\frac{3}{8}$.
20. $p = \$1500$, $r = 5\%$, $t = \frac{3}{4}$.

21. The sum to which p dollars placed at compound interest will amount in t years at the rate per cent r is $p(1 + r)^t$.

To what sum will \$500 amount if placed at compound interest for 2 years at 6%? For 3 years?

22. To what sum will \$2000 amount if placed at compound interest for 3 years at 5%?

23. I invest \$100 in the stock of a building and loan association which pays 7% interest compounded annually. To what will it amount in 3 years?

24. The number of ways that a committee of 3 persons may be selected from a group of n persons is $\frac{n(n-1)(n-2)}{6}$. In how many ways may a committee of 3 be selected from 4 persons? From 5 persons? From 6 persons?

25. If n gymnastic exercises may be taken in any order, the total number of different ways in which they may be selected to follow each other is $n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$. In how many different ways may a series of 5 exercises be arranged? 6 exercises? 7 exercises?

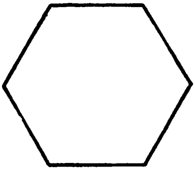
NOTE. — The expression $n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$ means for any number n, the product of this number and each consecutive number to 1. Thus, if $n = 4$, the expression becomes $4 \times 3 \times 2 \times 1$.

26. A party of 8 people secure a row of seats at a theater. In how many different ways may they be seated? (See Problem 25.)

27. The area of an equilateral triangle, each of whose sides is s, is found by the formula $A = \frac{1}{4}s^2\sqrt{3}$. Find A when $s = 8$. ($\sqrt{3} = 1.732$).

28. By drawing the figure of a race track with two straight parallel sides S, and with semicircular ends each with radius r,

you will see from your knowledge of arithmetic that the distance, D, around the track is expressed by: $D = 2(S + \pi r)$. Find D when $S = \frac{1}{6}$ mi. and $r = \frac{1}{8}$ mi.



29. In a figure (polygon) with n equal angles, the number of degrees in each angle is $\frac{180(n-2)}{n}$. How many degrees in each of the equal angles of a triangle? Of a figure with 5 equal angles (pentagon)? Of a figure with 6 equal angles (hexagon)? Of a figure with 12 equal angles?

SUPPLEMENTARY VOCATIONAL FORMULÆ

NOTE. — The following exercises are added for those who wish to give a more extensive course in the *evaluation of formulæ.* It is not intended that the teacher should make any attempt to explain the meaning of any of them. The uses of the various formulæ are stated in the belief that students will be more interested in the process of evaluation if they know that such a thing must be done by people doing the world's work. *They may be omitted without interfering with the subsequent work.*

1. The formula $C = \frac{E}{R}$ is much used in work with electricity. Compute C when $E = 20.5$ and $R = 16.75$.

2. The velocity of the recoil of guns is computed by the formula $V = \frac{wv}{W}$, where V = velocity of recoil, W = weight of gun and carriage, in pounds, w = weight of projectile, and v = muzzle velocity of projectile. A 10-inch gun on a battleship fires a 400-pound projectile with a muzzle velocity of 1600 feet per second. Weight of gun and carriage, 22 tons. Find velocity of recoil.

3. The force of pressure P of the wind, in pounds per square foot, is computed from $P = .005\,V^2$, where V = velocity of wind in miles per hour. Find the force of the wind when blowing at 40 miles an hour. What would be the total pressure of this wind against the side of a house 20 feet high and 60 feet long?

4. If an object, such as a brick dislodged from the wall, starts from rest and falls towards the earth, the distance that it will fall in a given length of time is computed by the formula $s = \frac{1}{2}at^2$, where $s =$ distance in feet, $a = 32$, and $t =$ number of seconds elapsed. Find the distance an object will fall in 1 second; 2 seconds; 3 seconds; 4 seconds; 10 seconds; 60 seconds; 5 minutes.

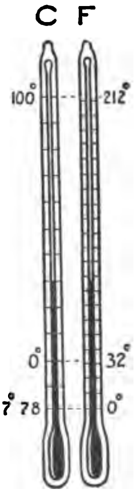
NOTE. — This formula holds accurately only for bodies falling in a perfect vacuum. For bodies falling through the air, the velocity is somewhat diminished by the resistance of the air.

5. How far would a body fall during the sixth second?

SUGGESTION. — Find the distance it would fall in 5 seconds and in 6 seconds.

6. How far would a falling body move during the thirtieth second?

7. To measure temperature, two different kinds of thermometers are in use: the Fahrenheit and the Centigrade. On the former the freezing point is marked 32° and the boiling point 212°. On the latter these are marked 0° and 100° respectively. If the temperature is read on a Fahrenheit thermometer, the corresponding temperature on the Centigrade thermometer is computed by the formula $C = \frac{5}{9}(F - 32)$, where $C =$ temperature in degrees on Centigrade scale and $F =$ temperature in degrees on Fahrenheit scale.

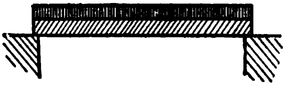


When it is 70° by the Fahrenheit thermometer, what is the temperature on the Centigrade thermometer? When 64°? When 48°? When 80°?

8. The strength or capacity for work of engines is expressed by horse power. The horse power of steam engines is found by the formula $H.P. = \frac{plan}{33000}$, where $p =$ pressure of steam in pounds per square inch, $l =$ length of stroke in feet, $a =$ area of piston in square inches, and $n =$ twice the number of revolutions per minute. Compute the horse power of an engine in which a test shows $p = 95$ pounds, $l = 30$ inches, $a = 706.8$ square inches, and $n = 100$.

9. Find the horse power of a steam engine in which $p = 110$ pounds, $l = 24$ inches, $n = 120$, and the diameter of the piston is 16 inches.

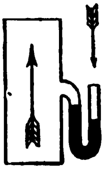
10. The horse power of automobile engines is computed by the formula H. P. $= KND(D - 1)(R + 2)$, where $K = .197$ for commercial touring cars, $N =$ number of cylinders, $D =$ diameter of cylinders, and $R =$ ratio of the stroke to the diameter. What is the horse power of a 4-cylinder engine of a touring car in which the diameter is 4 inches and the stroke 5 inches?



11. If a beam L feet long is supported at both ends and loaded uniformly throughout its length with W pounds per foot, the greatest bend or deflection D at the middle, in inches, is obtained from the formula $D = \frac{5}{384}\frac{WL}{EI}$, where E and I have particular values depending upon the material used. This formula is used by architects in designing buildings. If a beam in which $E = 30{,}000{,}000$ and $I = \frac{5}{8}$ is 12 feet long, and the load 200 pounds per foot, find the deflection.

12. The elevation of a point above sea level is obtained by use of the thermometer from the formula $H = 513\,t + t^2$, where $H =$ height in feet above sea level, and $t =$ difference (in degrees Fahrenheit) between 212° and the temperature at which water boils at the place of observation.

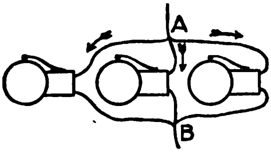
The temperature of boiling water at a certain place is 210°. Find the elevation of the place.



13. The relation between the height of a chimney and the pressure of draft which it produces is given by the formula $P = H\left(\frac{7.6}{460 + T} - \frac{7.9}{460 + t}\right)$, where P = pressure of draft as measured by the height in inches of a column of water that it will support in a tube, $H =$ height of chimney in feet, $T =$ temperature outside, and $t =$ temperature of air in chimney.

If $H = 150$ feet, $T = 50°$, and $t = 600°$, find P.

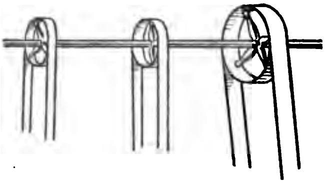
14. In electrical work problems of the following kind are encountered: An electric current flowing from a point A to a point B is divided at A into three branches, each passing through an electric bell, and the branches are united again into one current at B. The total resistance R of the circuit from A to B is computed by the formula $R = \frac{r_1 r_2 r_3}{r_1 r_2 + r_1 r_3 + r_2 r_3}$, where r_1, r_2, and r_3 are the respective resistances of the three branches. If $r_1 = 1.2$ ohms, $r_2 = 1.4$ ohms, and $r_3 = 1.6$ ohms, find the total resistance R.



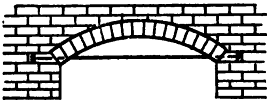
NOTE. — Problem 14 shows how the system of representing numbers by letters may be extended by attaching *subscripts* to the letters of the alphabet. Thus, by attaching subscripts to the letter a we can create any number of new symbols for representing numbers, as a_1 (read a sub 1), a_2, a_3, a_4, etc. The symbols a_1 and a_2 represent distinct and unrelated values, just as a and b do.

Similarly, by attaching *superscripts* to letters, new number symbols may be formed, as x' (read x prime), x'' (read x second), x''', x^{iv}, x^{v}, etc. Use is sometimes made of letters of the Greek alphabet, α (alpha), β (beta), γ (gamma), δ (delta), etc.

15. The horse power that may be transmitted safely by a certain kind of shafting without breaking or twisting is computed by the formula H. P. $= \frac{nd^3}{64}$, where $n =$ number of revolutions per minute, and $d =$ diameter of shafting in inches. How many horse power can be transmitted by such a shafting of 4 inch diameter, making 76 revolutions per minute? By one of 5 inch diameter making 100 revolutions per minute?



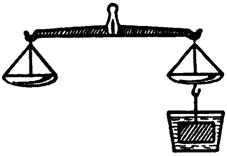
16. When a brick arch is supported by a tie rod to keep the walls from spreading, the "horizontal thrust," or stretching force exerted on the rod, in pounds per linear foot of arch, is obtained by the formula $F = \frac{1.5\ WS^2}{R}$, where $W =$ weight on arch in pounds per square foot, $S =$ span of arch in feet, $R =$ rise of arch in inches. Find the strain on the tie rod in an arch on which the weight is 360 pounds per square foot, the span 4 feet, and rise of arch 18 inches.



17. The discharge of a pump in gallons per minute is obtained from the formula $G = .03264\ Td^2$, where $G =$ number of gallons, $T =$ travel (total distance traveled) of piston in feet per minute, $d =$ diameter of cylinder in inches. Suppose that the diameter of the cylinder of a pump is 18 inches, that the stroke of the piston is 24 inches, and that it makes 40 revolutions per minute. Find the discharge.

18. The horse power of the pump required in Problem 17 is H.P. $= .00001238\ Td^2h$, where $h =$ vertical distance in feet between levels of water at source and point of discharge. Find the horse power of the pump in Problem 17 required to raise the water to a standpipe through a height of 216 feet.

19. By the *specific gravity* of a solid substance, such as iron, is meant the ratio of the weight of that substance to the weight of an equal volume of water. The specific gravity of an object may be found by first weighing it in air, then weighing it again when suspended under water. An object seems to lose weight when weighed in water due to the buoyancy of the water.



If the specific gravity is s, then $s = \frac{W}{W - w}$, where $W =$ weight in air and $w =$ weight in water.

A piece of glass weighing 40 grams in air weighs 24 grams in water. Find its specific gravity.

CHAPTER II

THE EQUATION

15. The Equation. — In solving some kinds of problems use is made of *equations*. An **equation** is the statement that two expressions are equal or that they have the same value.

Thus, $2 \times 5 = 10$, $a + b = b + a$, and $4n - 3 = 2n + 5$ are equations.

The two expressions which are connected by the sign $=$ are called the **members** of the equation.

For example, in $3P - 2 = 4$ the expression $3P - 2$ is called the **first** or **left member**, and the 4 is called the **second** or **right member**.

In some equations the members are equal for all particular values of the general number involved. These are called **identical equations** or **identities**.

Thus, in $a^2 + 2ab + b^2 = (a + b)^2$ the members are equal *for all values whatever* that may be given to a and b. When $a = 1$ and $b = 2$, the equation becomes $9 = 9$; when $a = 2$ and $b = 2$, it becomes $16 = 16$; when $a = 5$ and $b = 1$, it becomes $36 = 36$; etc.

In other equations the members are not equal for all particular values of the general numbers involved. These are called **conditional equations** or simply **equations**.

Thus, $5y - 2 = 3y$ is true only under the condition that $y = 1$. And $3R + 4 = 10$ is true only when R is 2.

16. Problems Expressed by Equations. — In every problem solved by use of equations, the values of one or more numbers are *unknown*.

For example, in the problem, "If eggs cost 34 cents a dozen, find the cost of 5 dozen," the cost of 5 dozen is an *unknown number*.

To solve a problem is to find the values of the unknown numbers. In solving a problem algebraically it is first expressed in a sort of shorthand by an equation. The equation expresses the relation between the known and unknown numbers. An unknown number is expressed by a letter, and later the value represented by the letter in the equation is found.

EXAMPLE 1. A man's salary was increased by 10%, or $\frac{1}{10}$ of itself. After the increase he received, annually, \$1980. What was his salary before the increase?

Represent by d his salary before the increase.

Then the problem may be expressed by $d + \frac{1}{10}d = \$1980$.

EXAMPLE 2. Two men form a partnership in business in which the first invests twice as much as the second. Their profits are \$3600. What part of the profits should each receive in settlement?

Let a represent the amount the second should receive.

Then $2a$ represents the amount the first should receive.

Hence, the problem may be expressed by the equation

$$2a + a = \$3600.$$

To express a problem by an equation we take the following steps:

(1) *First read the problem to discover what numbers are unknown.*

(2) *Let some letter represent one of the unknown numbers.*

(3) *Then, from statements in the problem, express all the other unknown numbers in terms of this letter.*

(4) *Finally, from another statement of the problem, form the equation between these and the known numbers of the problem.*

ORAL EXERCISES

1. If n denotes a certain number, what will denote a number 10 larger? What will denote one 10 less? One 10 times as large? One $\frac{1}{10}$ as large?

2. The rainfall last year at a certain place was 6 inches more than on the year before. If x represents last year's rainfall, what will denote the rainfall of the year before? If x represents the rainfall of the year before, what will denote that of last year?

3. The melting temperature of glass is 136 degrees lower than 3 times that of zinc. If t represents the melting temperature of zinc, what will represent that of glass?

4. One boy sold 20 more than $\frac{1}{2}$ as many papers as another. By use of some letter represent the number sold by each.

5. If A has $\$y$, B $\frac{3}{8}$ as much, and C $\frac{1}{2}$ as much, what will represent the amount that all three have?

6. Of two candidates at an election one was defeated by 362 votes. By some letter express the number of votes received by each.

7. A rectangle is 2 inches more than 3 times as long as it is wide. By use of a letter represent both the length and the width.

8. A merchant sold coffee at a profit of 20%. If it cost C cents a pound, what will express the selling price?

9. If I lend $\$D$ at 5% for 4 years, what will express the amount at the end of that time?

10. The distance from Chicago to New York by rail is about 900 miles. If a train ran an average of 40 miles an hour, how long would it require to make the run between the two cities? If it ran V miles an hour, what will express the time required for the run?

In each of the above problems one number has been expressed in terms of one or more other numbers.

WRITTEN EXERCISES

Express the following problems by means of equations:

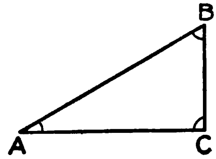
1. One of two partners in a business invests twice as much as the other. How should a profit of $1200 be divided?

2. A real estate dealer sold a lot for $1500, and thereby made a profit of 25%. What did it cost him?

3. It requires 2240 feet of wire fencing to inclose a rectangular piece of ground that is three times as long as it is wide. Find its width and length.

4. If I think of a number, double it, and add 8 to the result, I get 40. What is the number?

5. In the latitude of Chicago, on the longest day of the year, the day is 6 hr. 8 min. longer than the night. What is the length of each?



6. In the triangle ABC, the sum of the three angles is 180°. Angle B is twice as large as angle A, and angle C is three times as large as angle A. How many degrees in each angle?

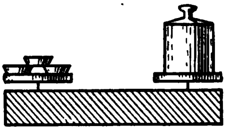
17. Solving an Equation. — It has been seen that a problem may be expressed by an equation, in which the unknown numbers are represented by use of a letter. It is evident that the problem may be solved if the value of the letter in the equation may be found. This is called *solving the equation.*

To solve an equation is to find the particular value or values of the unknown number that make the two members equal. A particular value of the unknown number thus found is called a **root** of the equation.

Thus, in $4y + 6 = 2y + 10$, if y is 2, each member equals 14. Hence, 2 is a *root.* The equation is said to "be satisfied" when $y = 2$.

Before proceeding to the solution of problems, we must discover how to solve an equation.

18. Axioms. — If the weights on the two pans of a balance are equal, they balance. If equal weights are added to, or taken from, the two pans, will the resulting weights balance? If the weights in the two pans are made twice as great, three times as great, etc., or one half as great, one third as great, etc., will the resulting weights balance?



These facts about the balance illustrate certain general principles in dealing with numbers which may be assumed. They are called **axioms**, and may be stated as follows:

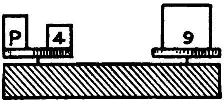
1. *If equal numbers are added to equal numbers, the sums are equal.*

2. *If equal numbers are subtracted from equal numbers, the remainders are equal.*

3. *If equal numbers are multiplied by equal numbers, the products are equal.*

4. *If equal numbers are divided by equal numbers (not zero), the quotients are equal.*

An equation may be compared to a balance. The members of the equation correspond to the weights on the two pans of the balance. In the figure, the fact that the weights balance is expressed by the equation $P + 4 = 9$. Just as the weights on the two pans may be equally increased, decreased, multiplied, or divided, and continue to balance, so the members of an equation may be equally increased, decreased, multiplied, or divided, according to the above axioms, and remain equal.



The use of these axioms in solving equations is shown in the following sections.

19. Equations Solved by a Single Addition, Subtraction, Multiplication, or Division.

EXAMPLE 1. — Solve $W + 6 = 10$.

Since $W + 6$ is 6 greater than W, what must be done to $W + 6$ to get W?
Subtracting 6 from each member, $W = 4$. Axiom 2.

EXAMPLE 2. — Solve $8a = 7a + 5$.

What must be done to $7a + 5$ to get 5?
Subtracting $7a$ from each member, $a = 5$. Axiom 2

EXAMPLE 3. — Solve $S - 4 = 12$.

Since $S - 4$ is 4 less than S, what must be done to $S - 4$ to get S?
Adding 4 to each member, $S = 16$. Axiom 1

EXAMPLE 4. — Solve $3x = 24$.

What must be done to $3x$ to get x?
Dividing each member by 3, $x = 8$. Axiom 4

Example 5. — Solve $\frac{n}{6} = 4$.

By what must $\frac{n}{6}$ be multiplied to get n?

Multiplying each member by 6, $n = 24$. **Axiom 3**

ORAL EXERCISES

Solve:

1. $a + 3 = 7$.
2. $P + 6 = 20$.
3. $v + 1 = 8$.
4. $R + 9 = 16$.
5. $3x = 2x + 4$.
6. $6A = 5A + 12$.
7. $8k = 23 + 7k$.
8. $9t = 3 + 8t$.
9. $16D = 5 + 15D$.
10. $B - 4 = 20$.
11. $x - 10 = 15$.
12. $M - 50 = 12$.
13. $p - 9 = 1$.
14. $R - 2 = 6$.
15. $w - 14 = 16$.
16. $x - \frac{1}{2} = \frac{1}{2}$.
17. $m - 1\frac{1}{3} = \frac{2}{3}$.
18. $5y = 15$.
19. $4H = 24$.
20. $2a = 18$.
21. $6Q = 48$.
22. $7S = 28$.
23. $24D = 72$.
24. $100z = 125$.
25. $12\frac{1}{2}c = 25$.
26. $\frac{y}{4} = 25$.
27. $\frac{n}{6} = 3$.
28. $\frac{A}{12} = 4$.
29. $\frac{r}{10} = 7$.
30. $\frac{Q}{5} = 8$.
31. $\frac{V}{20} = 18$.

WRITTEN EXERCISES

Solve:

1. $x + 3.25 = 5$.
2. $D - 0.426 = 4.32$.
3. $4.6A = 3.6A + 2.5$.
4. $P - 4.08 = 8$.
5. $2.16m = .24$.
6. $3.1416d = 9.5$.
7. $4\frac{1}{2}C = 7 + 3\frac{1}{2}C$.
8. $H + 3\frac{3}{8} = 7\frac{1}{16}$
9. $t - 7\frac{3}{7} = 8\frac{1}{3}$.
10. $y - \frac{3}{16} = \frac{5}{12}$.
11. $1\frac{1}{2}r' = 6$.
12. $2\frac{3}{8}v = 5\frac{8}{8}$.
13. $\frac{W}{3\frac{1}{4}} = 4\frac{1}{3}$.
14. $\frac{D}{3\frac{5}{6}} = 4\frac{2}{3}$.

20. Any Simple Equation Solved. — Any simple equation may be solved by using one or more of the processes employed in § 19, as shown by the following examples.

EXAMPLE 1. — Solve $4n + 15 + 13n = 5n + 99$.

Adding similar terms, $4n$ and $13n$, $17n + 15 = 5n + 99$.

Subtracting 15 and $5n$ from each member, $12n = 84$.

Dividing each member by 12, $n = 7$.

Notice that by subtracting 15 from each member we get the term free of n out of the first member, and by subtracting $5n$ from each member we get the term containing n out of the second member.

Check. — When $n = 7$, each member becomes 134, which shows that the answer is correct.

A number symbol put in place of another in an expression, as in checking the answer in the preceding example, is said to be **substituted** for it. The student is familiar with substitution in the formulæ in Chapter I.

The root of every equation should be checked, or tested for accuracy, by substituting the value found in place of the unknown number in each member of the equation.

EXAMPLE 2. — Solve $7V - 8 + 6V = 28 + 4V$.

By combining similar terms, what does the first member become?

What must be added to each member to free the first member of the term not containing V?

What must be subtracted from each member to free the second member of the term containing V?

Show that the resulting equation is $9V = 36$.

By what must each member now be divided?

Check the answer by substituting its value in each member of the given equation and seeing if they are equal.

EXAMPLE 3. — Solve $\frac{2x}{3} + \frac{3x}{4} + 9 = \frac{5x}{6} + 16$.

What is the least number divisible by each denominator?

By what, then, must each member be multiplied to free each term of fractions.

Show that the resulting equation is $8x + 9x + 108 = 10x + 192$.

Now show that $x = 12$.

Check by substituting 12 for x in the given equation.

It is evident from these examples that the steps in solving such equations are as follows:

(1) *If the equation contains one or more fractions, multiply both members by the L. C. M. of all the denominators to free the equation of fractions.*

(2) *Unite similar terms by adding, or subtracting, as the case may require.*

(3) *Free the first member of all terms that do not contain the unknown number, and the second member of all terms that do contain the unknown number, by adding the same number to both members or subtracting the same number from both members. To remove a term with a plus sign before it, subtract it, and to remove a term with a minus sign before it, add it to both members.*

(4) *Divide both members of the resulting equation by the coefficient of the unknown number.*

(5) *Check the work by substituting the value of the root found in each member of the given equation.*

EXERCISES

Solve:

1. $7a + 3 = a + 21$.
2. $4b + 4 = 25 + b$.
3. $5P - 4 = 12 - 3P$.
4. $2v - 5 = 7 - v$.
5. $3D - 7 = 14 - 4D$.
6. $5x + 3 = 3x + 9$.
7. $200A - 50 = 50A + 250$.
8. $5k - 5 = k + 3$.
9. $6M - 5 = 4M + 1$.
10. $5y + 5 = 8 - 3y$.
11. $20R - 25 = 5R + 5$.
12. $13h + 15 = 11h + 35$.
13. $4z - 3 = z$.
14. $12c - 13 + c = 35 + 7c$.
15. $13V - 7 = 5V + 2 - V$.
16. $3x = 10 + x$.

17. What is the least number divisible by both 2 and 3? By what one number may both of the fractions $\frac{x}{2}$ and $\frac{x}{3}$ be multiplied to change them to whole numbers?

18. By what one number may each of the fractions $\frac{2a}{3}$, $\frac{a}{4}$, and $\frac{5a}{6}$ be multiplied to change all of them to whole numbers?

19. What is 6 times 4 apples? 6 times $4a$? 3 times $5x$? 8 times $7n$? 10 times $3A$?

20. What is 6 times $\frac{2}{3}$? 8 times $\frac{3}{4}$? 10 times $\frac{3x}{5}$? 12 times $\frac{3t}{4}$?

Solve:

21. $\frac{t}{4}+\frac{t}{5}=\frac{t}{10}+7.$

22. $\frac{3n}{4}=\frac{2n}{3}+15.$

23. $\frac{2x}{3}+\frac{3x}{4}=\frac{5x}{6}+7.$

24. $\frac{a}{2}+6=17-\frac{3a}{5}.$

25. $2B+\frac{5B}{6}-17=0.$

26. $s+\frac{s}{2}+\frac{s}{3}=\frac{s}{6}+30.$

27. $x-\frac{x}{2}=30+\frac{x}{6}.$

28. $3W-\frac{W}{3}=\frac{W}{2}+13.$

29. $10-\frac{k}{8}=\frac{k}{4}+\frac{k}{2}+3.$

30. $R-10+\frac{R}{7}=\frac{R}{2}-\frac{R}{14}.$

21. Problems Solved by Equations. — The following example shows the complete process of solving a problem by use of an equation.

Example. — At an election two candidates, A and B, together received 2245 votes. A was elected by a majority of 285 votes. How many votes were cast for each?

Let n = number cast for A.

Then $n - 285$ = number cast for B.

Hence, $n + n - 285 = 2245.$

Solving, $2n - 285 = 2245.$

$2n = 2530.$

$n = 1265$, number cast for A.

Hence, $n - 285 = 980$, number cast for B.

EXERCISES

1. The initiation fee of a certain organization is \$40, and the annual dues \$5. In how many years will a member have paid into the treasury \$100?

2. It costs \$4 to have made the plates for printing a circular, and the cost of paper and printing is $\frac{1}{4}$ cent a copy. How many copies can I have made for \$8?

3. One partner has 4 times as much money invested in an enterprise as the other. How should a profit of \$2500 be divided?

4. Two boys agree to mow a man's lawn during a season for \$6. One mows it 3 times and the other 5 times. How should the money be divided between them?

5. A grocer bought some eggs at 16 cents a dozen. Thirty were broken, and he sold the remainder at 18 cents a dozen. He found that he got back just the cost of the whole lot. How many did he buy?

6. Divide \$75 between A and B so that A shall receive \$3 more than twice as much as B.

7. Two boys sold 45 papers. One sold $\frac{4}{5}$ as many as the other. How many did each boy sell?

8. A boy has \$1.05 in dimes and nickels. He has just as many dimes as nickels. How many has he of each?

9. Two trains start at the same time from stations 450 miles apart, one running at the rate of 35 miles per hour, and the other at the rate of 40 miles per hour. In how many hours do they meet?

10. The resistance of an electrical battery and the wire attached to it is found to be 3 ohms. By connecting to the battery a wire of the same size and material and three times as long, the resistance is found to be 6 ohms. Find the resistance of the battery.

11. The resistance of a battery and the wire attached to it is $5\frac{1}{2}$ ohms. When a wire of the same kind and 4 times as long is

connected to the battery, the resistance is found to be 16 ohms. Find the resistance of the battery.

12. The total number of admission tickets to a circus was 836. The number of tickets sold for adults was 136 less than twice the number of children's tickets. How many were sold of each?

13. The distance from Chicago to San Francisco by rail is 2563 miles, which is 397 miles more than twice as far as from Chicago to Denver. How far is it from Chicago to Denver?

14. In a class containing 24 boys and girls there are 6 more girls than boys. How many of each are there?

15. It takes 360 feet of wire fencing to inclose a rectangular lot 60 feet wide. How long is the lot?

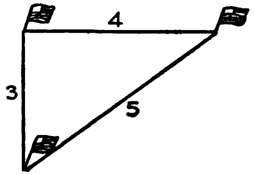
16. A boat steams against the current of a stream at a speed of 3 miles per hour, and with the current at a speed of 15 miles per hour. Find the speed of the current.

17. A train left a station and traveled at the rate of 30 miles per hour, and three hours later a second train left the station, following on the same track at the rate of 40 miles per hour. In how many hours from the time that the first train started was it overtaken by the second train?

18. A bar 40 inches long is to be cut into two pieces. The shorter piece is to be $\frac{2}{3}$ as long as the longer piece. Find the lengths of the pieces.

19. A board 62 inches long is to be sawed into two pieces so that one piece is 8 inches longer than the other. How long will each piece be?

20. Surveyors, in order to mark off a right angle on the ground, sometimes set three stakes so that when a tapeline is stretched around them so as to form a triangle, it is divided at the stakes into parts whose lengths are as 3, 4, and 5. The ancient Egyptians used this method in laying out their pyramids, etc. If a 100-foot tapeline is



used, find the lengths of the parts into which it must be divided at the stakes.

Suggestion. — Let $3n$ represent the number of feet in the shortest part.

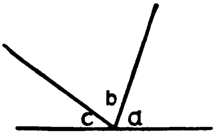
21. If, in Problem 20, the surveyors use a 50-foot tapeline, find the lengths of the parts into which it must be divided at the stakes.

22. A surveyor's chain which contains 60 links is divided into three parts whose lengths are as 3, 4, and 5, and used instead of the tapeline in Problem 20. How many links in each part?

Note. — Students who have time will find it interesting to use the method of Problems 20–22 in making out-of-door measurements.

23. A merchant sells tea for 70 cents a pound, and thereby makes a profit of 40% on the cost. What did it cost him?

24. How much money must I invest in a business yielding 20% profit in order that the investment and profit together may amount to $1000?



25. The sum of the angles a, b, and c is 180°. If angle b equals angle a, and angle c equals one-half of angle a, find the value of each.

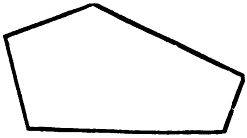
26. The sum of the angles of a triangle is 180°. If two of the angles are equal, and the other one twice as large as either of them, how many degrees in each?

27. An angle of 75° is made up of the sum of an angle of 15° and 5 equal angles. How many degrees in each of the equal angles?

28. Two angles are called complementary when their sum is 90°. If one of two complementary angles is 15° more than twice the other, how many degrees in each?

29. Three angles just cover all of the plane around a point. The difference between the second and the first is 60° and between the third and the second 60°. How many degrees in each?

30. If a polygon has n sides, the sum of all of its angles is $180° n - 360°$. How many sides has a polygon the sum of whose angles is 720°?



31. Whole numbers such as 5 and 6 are consecutive whole numbers. The sum of two consecutive whole numbers is 71. What are the numbers?

32. The sum of two consecutive odd numbers is 32. What are the numbers?

33. One number is twice as large as another. If I take 4 from the smaller and 16 from the larger, the remainders are equal. What are the numbers?

34. A number, its half, its third, and its fourth make 100. Find the number.

35. In the papyrus written by Ahmes, the Egyptian, about 1700 B.C., the unknown number was called "hau." This ancient book contained the following problem: "Hau, its $\frac{1}{7}$, its whole, it makes 19;" *i.e.* $\frac{1}{7}$ of the unknown number and the unknown number make 19. What was the unknown number?

SUPPLEMENTARY EXERCISES

1. A certain number exceeds n times the number x by m. What will express the number?

2. D dollars are invested at p per cent interest. What will represent the sum to which this will amount in 1 year? In 3 years? In t years?

3. A merchant sells sugar costing c cents a pound at a profit of 12%. What will represent the selling price? If it sells at a profit of p %, what will represent the selling price?

4. Solve $\frac{s}{12}+\frac{s}{8}=2\frac{1}{2}$.

5. Solve $\frac{3F}{4}+\frac{2F}{5}=\frac{5F}{6}+6\frac{1}{8}$.

6. Solve $7=\frac{y}{6}-\frac{2y}{9}+\frac{y}{4}$.

7. I paid $6 for an advertisement of 6 lines, as follows: 20 cents a line for the first insertion, 10 cents a line for each of the next 6 insertions, and 2 cents a line after that. Find the number of insertions at 2 cents a line.

8. How much per bushel must a merchant pay farmers for wheat in order to market it at $1.14 and make a profit of 4 %?

9. How much per bushel must a wholesale house pay for potatoes in order to sell them to retailers at 53 cents and make a profit of 6 %?

10. How much must a country merchant pay farmers for turkeys in order to make 6 %, if he can get 17 cents per pound for them in the city market?

11. In the city market, on a certain date, unwashed wool sold for 25 cents per pound. How much per pound must a wool buyer pay farmers for their unwashed wool in order to make a profit of 10 % when selling it at this price, if it costs him 3 cents per pound to market it?

12. On a certain date hogs sold for $9.75 per hundred pounds in the Chicago market. What must a buyer pay for hogs in the country in order to market them at this price and make a profit of 20 %, making no allowance for cost of delivery?

13. At what market price must 1-year 4 % bonds be offered for sale, in order that the buyer, by holding them until maturity, may make 8 % on his investment?

SUGGESTION. — The profit made must come from two sources, the interest on the par value, which is $4, and the excess of the maturity value over the price paid for the bonds. If x is the price paid, $100-x+4=.08x$.

14. At what market price must 1-year 7 % bonds be purchased, if by holding them until maturity the purchaser makes 10 % on his investment?

15. At what price must 3-year 5 % bonds (bonds that mature in 3 years) be purchased, if by holding them until maturity the purchaser makes 6 % on his investment?

SUGGESTION. — If x is the cost required, the annual profit due to excess of maturity value over cost is $\frac{100 - x}{3}$.*

16. At what price were Lake Shore 4's bought in 1910, which, if held until maturity in 1931, would earn the buyer $4\frac{3}{8}$ % on his investment?

17. Lackawanna 5's bought in 1910 and maturing in 1923, if held until maturity, would pay the purchaser $5\frac{9}{20}$ % on his investment. What was the price of Lackawanna 5's when bought?

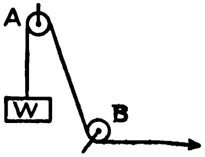
18. Chicago & N. W. stock bought May 13, 1913, and sold May 13, 1913, at 173, after paying three 12 % dividends, would yield the purchaser $12\frac{1}{2}$ % on his investment. At what was the stock bought May 13, 1913?

SUGGESTION. — If n = amount paid for a $100 share, show that

$$\frac{173 - n}{3} + 12 = .125\,n.$$

19. A merchant sold an article at a gain of 40 %. When the cost increases $90, the same selling price will be a loss of 50 %. Find the cost.

20. A weight W is lifted by means of a rope running over two pulleys, A and B. The friction in each pulley increases the tension in the rope 2 %. Find W if the force required to lift it is 400 pounds.



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[*] This method of distributing the loss or gain at maturity does not consider interest upon the annual loss or gain. For a full discussion, see the *Stone-Millis Secondary Arithmetic*, page 90.

CHAPTER III

POSITIVE AND NEGATIVE NUMBERS

22. Extension of the Number System. — We have seen in Chapter I a class of practical problems which are solved by use of the *formula,* and which require a knowledge of *literal notation.* In Chapter II we have seen another class of problems to solve which requires a knowledge of the *equation.* There are still other problems, now to be considered, which require us to extend our idea of number to include a *new kind of number.*

If a man deposits \$500 in a bank, and then withdraws \$300, the amount of his balance is \$500 − \$300, or \$200. It sometimes happens that a depositor *withdraws more money than he has deposited,* or makes an overdraft. For example, suppose that he has deposited only \$250, and withdraws \$300. There is now a balance of \$50 against him, showing that he owes the bank and not that the bank owes him. The balance is obtained by subtracting \$300, the amount withdrawn, from \$250; *i.e.* the balance is \$250 − \$300.

This problem, then, requires us *to subtract a number from a smaller number.* In order to solve such problems, our idea of number must be extended and this subtraction made possible.

23. Negative Number. — The remainder resulting from subtracting a number from a smaller number is called a **negative number.** We must now see how to represent a negative number.

If a person wishes to withdraw \$100 from a bank, he may withdraw part at a time. Thus, he may withdraw \$75 with one check and then \$25 with another. What are some ways that he may withdraw \$250 from a bank, by withdrawing only part at a time?

Similarly, in general, one number may be subtracted from another by separating the subtrahend into two or more parts, and subtracting the parts one at a time.

Hence, in attempting to subtract 7 from 4, by separating 7 into 4 and 3, we get

$$4-7=4-4-3=0-3.$$

In practice the zero is dropped, and the remainder $0-3$ written simply -3.

Similarly, $5-9=5-5-4=0-4$ or -4.

$$8-14=8-8-6=0-6 \text{ or } -6.$$

Hence, *a negative number is expressed by a number preceded by the minus sign.*

It is clear that a negative number, such as $-\$25$, indicates a **reserved subtraction**, there being nothing, when it stands alone, from which to subtract it. Thus, in $\$100-\$150=\$100-\$100-\$50=-\$50$, the remainder, $-\$50$, is a part of the subtrahend that is not subtracted because there is nothing from which to subtract it.

Hence, a negative number is in nature a subtrahend.

24. Positive Numbers.—For the sake of distinguishing from negative numbers, the arithmetical numbers with which we have always dealt up to this time are called **positive numbers.** All numbers with which we are familiar, therefore, are to be classified as either *positive* or *negative* numbers.

25. Use of Signs.—For distinction in writing positive and negative numbers, the positive or arithmetical numbers are often preceded by the sign + when standing alone.

Thus, 6 is written $+6$; a is written $+a$.

When clearness would not be sacrificed, however, the sign + may be omitted from a positive number. If no sign is written before a number, the number is *understood* to be positive.

It is clear, then, that the signs + and − have two uses: to indicate addition or subtraction, and to indicate the positive or negative quality or character of numbers. *They may always be considered as signs of addition or subtraction;* and when convenient, as signs of quality.

When standing alone, the positive numbers $+1$, $+2$, etc., are read either "plus 1," "plus 2," etc., or "positive 1," "positive 2," etc.; and the negative numbers -1, -2, etc., are read either "minus 1," "minus 2," etc., or "negative 1," "negative 2," etc.

EXERCISES

Read the following numbers:

1. 5; 90; -1; -12; x; $-n$; $\frac{1}{2}$; $-\frac{1}{n}$; $-\frac{a}{b}$.

What negative numbers result from the following subtractions?

2. $3-4$. 3. $8-13$. 4. $10-12$. 5. $7-16$. 6. $32-40$.

7. $100 − $125. 8. $500 − $650. 9. $400 deposited and $475 withdrawn. 10. $250 deposited and $325 withdrawn.

26. Opposite Numbers. — Since a negative number, such as -5, always implies a *reserved subtraction*, if it is combined with a positive number, into one number, *it tends to subtract from, or destroy of, the positive number a part equal to itself.*

Thus, when 6 and -4 are combined into one number, -4 destroys 4 of the 6, leaving 2 as the result. And when $10 and −$15 are combined, −$10 of the −$15 destroys the $10, leaving −$5 as the result.

Because of this neutralizing tendency of positive and negative numbers, they are sometimes called **opposite numbers.**

If two numbers such as $+8$ and -8, which differ only in their signs, are combined, they completely neutralize each other. Such numbers are said to have the same **absolute value.** Thus, 8 is the absolute value of both $+8$ and -8.

27. Positive and Negative Magnitudes. — As seen in the above illustrations, deposits and withdrawals of money in a bank tend to neutralize each other, and hence are positive and negative quantities, respectively. There are many other concrete quantities or magnitudes which are capable of existing in two *opposite states* or *senses* such that one tends to neutralize or destroy an equal amount of the other. Consequently, their numerical values are represented by positive and negative numbers. Several of these are illustrated in the following exercises.

EXERCISES

1. If in one of two sales of goods I lose \$10, and in the other I make a profit of \$15, what is the net result?

Loss and profit are thus neutralizing quantities, each tending to destroy an equal part of the other.

If profit is called positive, what will loss be called?

If \$15 profit is written +15, what will express \$10 loss?

2. Express with + and − signs the following: \$25 loss; \$40 profit; \$15 loss; \$6 loss; \$20 profit.

3. If I had \$20 in my pocket, then spent \$25 for a suit of clothes, I would have left \$20 − \$25, or − \$5. What does the − \$5 in this case mean?

4. Express with + and − signs the following: \$50 credit; \$18 debt; \$25 debt; \$60 cash in hand; \$100 borrowed; \$20 loaned; \$15 spent.

5. The number of people entering a store during a certain hour was 216, and the number coming out was 250. Express the relation of these numbers by + and − signs. Also express the excess of the number coming out over the number entering.

6. The amount of water pumped into a tank during an hour was 1900 gallons; and during the same hour three locomotives withdrew, respectively, 675 gallons, 750 gallons, and 650 gallons. Express with + and − signs these quantities, and also the excess withdrawn during the hour.

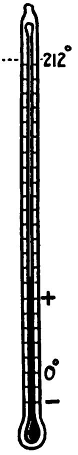
7. A locomotive pulls a train with a force of 35 tons. The resistance of the train is 20 tons. If the pull of the locomotive is positive, represent by + and − signs the two forces.

8. A balloon lifts up with a force of 200 pounds. How much weight must be attached to keep the balloon from rising? If the lifting force is +, what represents the weight?

9. A balloon lifts with a force of 250 pounds, and a weight of 175 pounds is attached to it. If the weight is called positive, what will represent the lifting force? What will represent the result when the weight is attached?

10. A battleship has a displacement of 20,000 tons (sinks into the water until it displaces 20,000 tons of water). Express with + and − signs the weight of the ship and the upward pressure of the water.

11. A diver who weighs 165 pounds displaces 156 pounds of water (is buoyed up with a force of 156 pounds). Represent by + and − signs these forces and the resultant force with which he tends to sink to the bottom.



12. If the thermometer is 5° above zero at noon, and falls 15° by night, what is the temperature by night? $5^\circ - 15^\circ =$ what? How else may 10° below zero be written? How would temperature above zero be written?

13. On Jan. 6, 1910, the Weather Bureau reported the temperature at several different places as follows: Chicago, -2°; Davenport, -18°; Des Moines, -12°; Kansas City, -2°; Madison, -14°; Milwaukee, -8°; Peoria, -6°; Pueblo, -6°; St. Paul, -16°. Were these temperatures above or below zero? What would they be if the signs were +?

14. Certain railroad stock that sold at 7 above par in January fell to 3 below par in June. Since the value of the stock was reduced 10 points, how may the 3 below par be represented algebraically?

15. Represent by + and − signs the following prices of stock which show the change in 6 months: January, 10 below par; February, 6 below par; March, 1 below par; April, 5 above par; May, 8 above par; June, 2 above par.

16. Augustus Cæsar was ruler of the Roman Empire from -31 to $+14$. Express these dates in other terms.

17. Archimedes, the greatest mathematician of antiquity, was born about the year -287, and was slain by a Roman soldier while studying a geometrical figure that he had drawn in the sand in -212. Express these dates in other terms.

18. If a file of soldiers moves 10 paces to the front, then 6 paces to the rear, it advances how many paces from its first position? If it moves 6 paces to the front, then 10 paces to the rear, it advances $6-10$, or how many paces? If motion forward is positive, what is motion backward?

19. A switch engine, in making up a train, moves forward 100 yards, then backward 150 yards. The two are equivalent to what single motion? Represent by appropriate signs these two motions and the one equivalent motion.

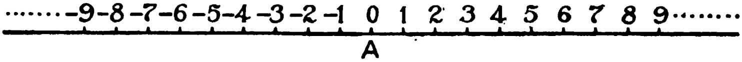
20. If longitude west is positive, longitude east negative, latitude north positive, and latitude south negative, in what country is a city whose longitude is $+106^\circ$ and latitude $+40^\circ$? Longitude -16° and latitude $+48$? Longitude -40° and latitude -15°? Consult a map.

21. The longitude of the city of St. Petersburg, Russia, is given as -31°. This means that it is 31° in what direction from the Prime Meridian? Washington, D.C., is in 77° west longitude. How else may this be expressed?

22. The latitude of New York City is about $+41^\circ$. What will express the latitude of Cape Town, Africa, which is about 34° south of the equator?

28. Numbers Represented by Distances. — In the preceding exercises, it has been seen that many magnitudes may be represented numerically by positive and negative numbers. These include bank deposits and withdrawals, loss and gain in business, income and expense, credits and debts, increases and decreases of physical quantities, temperature above and below zero, dates in history before and after the birth of Christ, stocks above and below par, forces acting in opposite directions, motions in opposite directions, and longitude east and west and latitude north and south. Many others might be suggested.

Since motions in opposite directions tend to neutralize each other, the *distances* moved through in opposite directions may themselves be considered as positive and negative. In representing distances as positive and negative, the signs serve to tell the *directions* in which the distances are measured, as in the case of longitude and latitude. It follows that positive and negative numbers may be represented to the eye by distances measured on



a line. *Positive* numbers are represented by distances measured to the *right* from a starting point A, and *negative* numbers by distances measured to the *left.*

EXERCISES

1. On the above line, how will -12 be represented? $+10$? $-2\frac{1}{2}$? $+3\frac{1}{2}$? -40? $+100$?

2. Draw a horizontal line and mark the middle point *O.* Then mark on it distances representing the following numbers, a unit being represented by a quarter of an inch: $+2$, $+5$, $+8$, $+10$, $+6\frac{1}{2}$, -3, -4, -7, -12, $-4\frac{1}{2}$, $-10\frac{1}{2}$.

3. On a horizontal line represent the following dates (*i.e.* years): -31, beginning of Augustus Cæsar's reign; $+14$, end of Augustus Cæsar's reign; -146, fall of the Roman Empire; $+70$, destruction of Jerusalem by Titus; -275, death of Euclid, who wrote the first great book on geometry; -287, birth of Archimedes, the greatest mathematician of antiquity.

4. On a vertical line, represent the following temperatures: $+5°$; $+10°$; $+15°$; $+20°$; $-5°$; $-10°$; $-15°$; $-20°$.

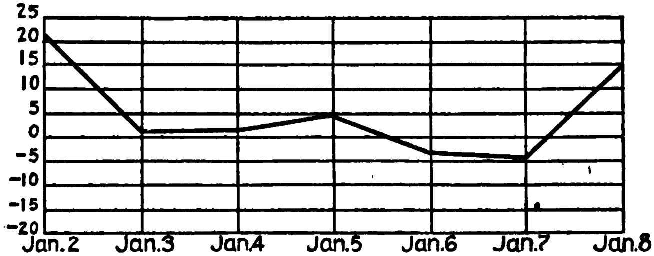
5. The minimum temperatures at St. Paul on the six days beginning Feb. 9, 1910, were $+16°$, $-10°$, $+6°$, $+4°$, $+22°$. Represent these on a line.

6. The longitude of New York is +74°, of Berlin −13°, of Pekin −116°, of Calcutta −88°, of Pittsburg +80°, of St. Louis +90°, of Rome −12°. Draw a horizontal line and mark off on it the longitudes of these places.

29. **Graphs.** — The following table gives the minimum temperature at Chicago on each of the seven consecutive days beginning with Jan. 2, 1910:

DATE	JAN. 2	JAN. 3	JAN. 4	JAN. 5	JAN. 6	JAN. 7	JAN. 8
TEMP.	21°	1°	1°	4°	− 4°	− 5°	14°

Suppose that a sheet of paper is ruled by horizontal and vertical lines, as shown in the following diagram, each vertical line representing the thermometer scale. By marking the temperature for each date on the corresponding vertical line, and joining the consecutive points thus marked by straight lines, the heavy broken line is obtained, showing to the eye the variation from day to day.



Such diagrams as this are called **graphs**. They are much used in practical work to depict to the eye graphically the changes in quantities, such as temperature, death rates, population, imports and exports, cost of living, etc.

NOTE. — Ruled paper, often called "squared paper" or "coördinate paper," may be obtained cheaply for drawing graphs. It is ruled accurately into small squares, and saves much time when used. This paper will be needed in other exercises later.

EXERCISES

1. The following table gives the minimum temperature at St. Paul, Minn., on each of seven consecutive days, beginning Jan. 5, 1910:

DATE	JAN. 5	JAN. 6	JAN. 7	JAN. 8	JAN. 9	JAN. 10	JAN. 11
TEMP.	6°	– 16°	– 12°	– 4°	– 12°	2°	20°

Draw a graph showing the variation in temperature during the period.

2. According to the report of the Weather Bureau, on Jan. 5, 1910, the height of the Ohio River above low-water mark at Pittsburg was 11.3 feet. The amounts in feet by which it rose or fell on the seven consecutive days thereafter were as follows, the negative sign indicating fall: – 2.1, – 2.8, 1.2, 1.3, – 2.2, – 2.7, – 1.4.

Draw a graph showing the rise and fall of the river from day to day. Measure the heights of the river on vertical lines, marking the lowest horizontal line low-water mark.

3. The latitude of a ship going east at noon on each of six consecutive days was as follows: – 1°, equator, + 1°, + 3°, + 1°, – 1°. Draw a graph showing the course of the ship.

4. The highest points reached by certain railroad stock during the 12 months of a year were as follows: 10 above par, 6 above par, 2 below par, 5 below par, par, 2 below par, 3 above par, 7 above par, 10 above par, 15 above par, 18 above par, 12 above par.

Draw a graph showing the variation in price of the stock during the year.

30. Addition of Positive and Negative Numbers. — The addition of positive and negative numbers may be indicated by writing them in succession with their signs.

Thus, to indicate the addition of $+6$, $+2$, -4, $+3$, and -5, we write $+6+2-4+3-5$. The first term being positive, we may omit its sign, giving $6+2-4+3-5$.

Positive and negative numbers to be added may be placed also in columns, with the signs attached, as in the margin.

$$\begin{array}{r} -3 \\ 2 \\ -6 \\ \hline \end{array}$$

Since positive numbers are in nature ordinary arithmetical numbers, their sum is obtained by adding their absolute values, and is positive.

Thus, a bank deposit of $125 and another of $100 together make a deposit of $225. The sum of $+4$ and $+8$ is $+12$.

Also, to subtract each of two numbers in succession is equivalent to subtracting their sum. Hence, two negative numbers, since each is in nature a subtracted number, may be combined by adding their absolute values, and the sum is negative.

Thus, a withdrawal from the bank of $25 and an additional withdrawal of $75 together make a withdrawal of $100. The sum of -6 and -10 is -16.

Therefore,

(1) *To add two numbers with like signs, find the sum of their absolute values, and prefix the sign common.*

In adding two numbers with unlike signs, the one with the greater absolute value may be separated into two parts, one of which just neutralizes the number with the smaller absolute value, leaving as result the other part of the numerically greater number.

Thus, to add $+7$ and -3, by separating $+7$ into $+3$ and $+4$, we have

$$\begin{array}{r} +3+4 \\ -3 \\ \hline 0+4 \end{array} \qquad \text{or} \qquad \begin{array}{r} +7 \\ -3 \\ \hline +4 \end{array}$$

And, to add -9 and $+4$, by separating -9 into -4 and -5, we have

$$\begin{array}{r} -4-5 \\ +4 \\ \hline 0-5 \end{array} \qquad \text{or} \qquad \begin{array}{r} -9 \\ +4 \\ \hline -5 \end{array}$$

Hence, the following rule is evident:

(2) *To add two numbers with unlike signs, find the difference of their absolute values, and prefix the sign of the one with the greater absolute value.*

Evidently three or more positive and negative numbers may be added by adding two at a time, taking them in the order written. In the steps of the addition use rules (1) and (2).

A second method sometimes used when there are both positive and negative numbers to be added is to add the positive and the negative numbers separately, and then combine these two sums.

$$\begin{array}{r} -3\\ +8\\ -6\\ \underline{+4}\\ +3 \end{array}$$

Thus, in this column, adding upwards, the partial sums are -2, $+6$, and $+3$, the result. Or, the sum of -3 and -6 is -9, and that of $+8$ and $+4$ is $+12$, and the sum of these sums is $+3$.

$$\begin{array}{r} \left.\begin{array}{r}-3\\ -6\end{array}\right\} \\ \left.\begin{array}{r}+8\\ \underline{+4}\end{array}\right\} \end{array}$$

Note. — If desirable, the numbers in the preceding illustration might be rewritten in the order shown in the margin. This fundamental principle that numbers to be added may be arranged in any order is known as the **Law of Order in Addition.**

The principle that numbers to be added may be grouped in any manner — by which we find the sum of the -8 and the -6 and of the $+8$ and $+4$ and then the sum of these sums, as here indicated — is known as the **Law of Grouping in Addition.**

EXERCISES

Find the sums of the following:

1. A profit of $12 and a profit of $16.
2. A loss of $10 and a loss of $23.
3. A profit of $15 and a loss of $6.
4. A profit of $18 and a loss of $30.
5. A rise in temperature of 14° and a rise of 6°.
6. A fall in temperature of 8° and a fall of 17°.
7. A rise in temperature of 7° and a fall of 16°.
8. A credit of $250 and a debt of $325.
9. A debt of $136 and a debt of $224.

10.
$$\begin{array}{rrrrrr} +6 & -8 & +9 & -12 & -20 & +8\\ \underline{-2} & \underline{-7} & \underline{+5} & \underline{+17} & \underline{-65} & \underline{-16} \end{array}$$

$$\textbf{11.}\quad \begin{array}{rrrrrr} -25 & -12 & +13 & -1 & +1 & -98 \\ \underline{+32} & \underline{-17} & \underline{+17} & \underline{+14} & \underline{-30} & \underline{+16} \end{array}$$

$$\textbf{12.}\quad \begin{array}{rrrrrr} -14 & +7 & -8 & +3 & -14 & +20 \\ \underline{+21} & \underline{-18} & \underline{+7} & \underline{-12} & \underline{-3} & \underline{+16} \end{array}$$

$$\textbf{13.}\quad \begin{array}{rrrrr} -\$12 & +\$27 & -7\text{ lb.} & +13\text{ lb.} & +14\text{ mi.} \\ \underline{-\$\ 6} & \underline{-\$36} & \underline{+12\text{ lb.}} & \underline{+19\text{ lb.}} & \underline{-18\text{ mi.}} \end{array}$$

$$\textbf{14.}\quad \begin{array}{rrr} -\ 8^\circ\text{ longitude} & +\ 6^\circ\text{ latitude} & 25\text{ ft. forward} \\ \underline{-17^\circ\text{ longitude}} & \underline{-14^\circ\text{ latitude}} & \underline{36\text{ ft. backward}} \end{array}$$

$$\textbf{15.}\ \begin{array}{r} -2 \\ +6 \\ +3 \\ -8 \\ \underline{-4} \end{array}\quad \textbf{16.}\ \begin{array}{r} +12 \\ -7 \\ -3 \\ +6 \\ \underline{+5} \end{array}\quad \textbf{17.}\ \begin{array}{r} -4 \\ -9 \\ -3 \\ -1 \\ \underline{+12} \end{array}\quad \textbf{18.}\ \begin{array}{r} +6 \\ -3 \\ -4 \\ +12 \\ \underline{+7} \end{array}\quad \textbf{19.}\ \begin{array}{r} -5 \\ +8 \\ +14 \\ -3 \\ \underline{+6} \end{array}\quad \textbf{20.}\ \begin{array}{r} +8 \\ -16 \\ -21 \\ -7 \\ \underline{-1} \end{array}$$

$$\textbf{21.}\quad \begin{array}{rrrr} -3.25 & -12.08 & +.62 & -4.125 \\ \underline{+4.09} & \underline{-10.37} & \underline{-6.07} & \underline{+1.450} \end{array}$$

$$\textbf{22.}\quad \begin{array}{rrrr} -3\tfrac{1}{2} & +6\tfrac{3}{8} & -\tfrac{1}{4} & +1\tfrac{5}{6} \\ \underline{+1\tfrac{1}{4}} & \underline{-4\tfrac{1}{2}} & \underline{-5\tfrac{2}{3}} & \underline{+\ \tfrac{3}{4}} \end{array}$$

23. $5, -8, 7, -4, -12, 3.$

24. $-9, -6, -2, 10, -8, 9, -1.$

25. $16, 5, 9, -5, 3, -7, -2, 6.$

26. The Ahmes papyrus, the earliest mathematical book of which there is any record, was written about -1700. The next great mathematical book was Euclid's *Elements*, written about 1400 years later. What was the date that Euclid wrote the *Elements*?

27. Euclid was born about -330. Diophantus, who wrote one of the first books on algebra, was born 750 years later. At what date was Diophantus born?

28. A freight car is running at the rate of 20 feet per second. A brakeman walks to the rear on the top of the car at the rate of 5 feet per second. Express by + and − signs these rates and the rate and direction of his motion with reference to the ground.

29. If the basket of a balloon weighs 280 pounds, the instruments 57 pounds, the sandbags 800 pounds, and the balloon itself − 1400 pounds, what is the total weight of the balloon and contents?

30. A boat using both sails and steam is driven against a current by its steam power at the rate of 10 miles per hour, and by the wind at the rate of 3 miles per hour. The current flows at the rate of 4 miles per hour. Express these rates by + and − signs, and indicate their sum. What is the boat's rate of progress?

31. A work train travels 12 miles north, then 35 miles south, then 8 miles north. By use of + and − signs express these distances, their sum, and hence its final distance and direction from the starting point.

32. The temperature falls 15°, rises 18°, falls 10°, then rises 6°. By use of + and − signs express these changes and their net result.

33. The rise and fall in feet of the Mississippi River at Vicksburg on each of the seven days beginning Jan. 6, 1910, were given by the Weather Bureau as follows: $-.2$, $-.2$, $-.4$, $-.2$, $+.4$, $+.2$, $+.4$. The sign − indicated fall. Find the net change during that period.

34. The *average* of two or more numbers is their sum divided by the number of them.

Find the average of the following temperatures: 8 A.M., $-7°$; 9 A.M., $-6°$; 10 A.M., $-3°$; 11 A.M., $-1°$; 12 Noon, $+1°$; 1 P.M., $+4°$; 2 P.M., $+7°$; 3 P.M., $+8°$; 4 P.M., $+8°$; 5 P.M., $+5°$; 6 P.M., $+3°$; 7 P.M., $+1°$.

35. At Madison, Wis., the minimum temperatures on the seven days beginning Jan. 4, 1910, were as follows: $-12°$; $+10°$; $-14°$; $-18°$; $+6°$; $-6°$; $-10°$. Find the average temperature for the week.

36. At Cheyenne, Wyo., the temperatures for six days, beginning Jan. 3, 1910, were as follows: $-10°$; $-2°$; $0°$; $-2°$; $+14°$; $+10°$. Find the average temperature for the six days.

37. If one place is midway between two other places, its latitude or longitude equals one-half the sum of their latitudes or longitudes.

The latitude of Panama is approximately $+8°$, and of Chicago $+42°$. Find the latitude of Key West, Florida, which is approximately midway between these two.

38. The longitude of New Orleans is $+90°$, and of Pekin, China, $-116°$. Berlin, Germany, is approximately midway between these two places. What is its longitude?

31. Subtraction of Positive and Negative Numbers. — To subtract a number (subtrahend), is to find a number (remainder) which added to it gives the minuend.

Thus, to subtract 6 from 9 is to find a number (3) which added to 6 gives 9.

Hence, the rule for subtracting positive and negative numbers may be obtained from addition.

(1) What number added to $+3$ gives $+5$?

Hence, by subtracting, $\begin{array}{r} +5 \\ \underline{+3} \\ +2 \end{array}$ But, by adding, $\begin{array}{r} +5 \\ \underline{-3} \\ +2 \end{array}$

Hence, to subtract $+3$ from $+5$ is equivalent to adding -3 to $+5$.

(2) What number added to $+3$ gives -5?

Hence, by subtracting, $\begin{array}{r} -5 \\ \underline{+3} \\ -8 \end{array}$ But, by adding, $\begin{array}{r} -5 \\ \underline{-3} \\ -8 \end{array}$

Hence, to subtract $+3$ from -5 is equivalent to adding -3 to -5.

(3) What number added to -3 gives $+5$?

Show, as above, that to subtract -3 from $+5$ is equivalent to adding $+3$ to $+5$.

(4) What number added to -3 gives -5?

Show, as above, that to subtract -3 from -5 is equivalent to adding $+3$ to -5.

From these examples it is evident that to subtract one number (subtrahend) from another (minuend) is equivalent to adding the subtrahend, *with its sign changed*, to the minuend. Therefore,

To subtract, first change the sign of the subtrahend, then proceed as in addition.

Thus, to subtract $\begin{array}{r} -4 \\ \underline{-7} \end{array}$ change to $\begin{array}{r} -4 \\ \underline{+7} \end{array}$ then add.

Note. — The pupil should learn not to rewrite the problem with the sign changed in the subtrahend, but to make the change *mentally*, and add at once.

EXERCISES

1. What number added to -6 gives -4?
2. What number added to 8 gives -2?
3. What number added to -4 gives 6?
4. What number added to 12 gives 7?
5. What number added to -15 gives -8?
6. What number added to -5 gives 0?
7. What added to \$ 7 loss will give \$ 9 gain?
8. What added to a withdrawal of \$ 125 from a bank will give a balance of \$ 150?
9. What added to a rise of 12° temperature will give a net fall of 18°?
10. What added to a fall of 15° temperature will give a net fall of 8°?

Subtract the following:

11. $\begin{array}{r} +7 \\ -6 \\ \hline \end{array} \quad \begin{array}{r} +12 \\ -3 \\ \hline \end{array} \quad \begin{array}{r} +2 \\ -10 \\ \hline \end{array} \quad \begin{array}{r} +20 \\ -8 \\ \hline \end{array} \quad \begin{array}{r} +9 \\ -14 \\ \hline \end{array} \quad \begin{array}{r} +16 \\ -4 \\ \hline \end{array} \quad \begin{array}{r} 0 \\ -6 \\ \hline \end{array}$

12.	-3	-7	-25	-14	-9	-7	0
	$+5$	$+6$	$+15$	$+16$	$+6$	$+8$	$+15$
13.	$+12$	$+9$	$+7$	$+15$	$+21$	$+12$	$+40$
	$+3$	$+5$	$+9$	$+20$	$+16$	$+20$	$+55$
14.	-9	-8	-3	-12	-9	-7	-16
	-6	-4	-5	-15	-4	-10	-17
15.	-3	$+6$	-5	$+10$	-16	$+5$	$+9$
	$+7$	-15	-7	$+6$	$+4$	$+12$	-6

16. What is indicated by $-3-(-2)$? Find the value.

17. What is indicated by $+8-(+12)$? Find the value.

18. Find the value of the following: $5-(-5)$; $-3-(-10)$; $6-(+8)$; $-9-(-12)$; $6-(+12)$; $-7-(+8)$; $9-(-6)$.

19. Subtract:	-6.25	$+.23$	$-.2$ ft.	$+3.5$ pk.	-12.3 lb.
	$+3.85$	-4.17	-1.6 ft.	$+7.4$ pk.	-15.6 lb.
20. Subtract:	$+1\frac{1}{2}$	$+5\frac{3}{8}$	$+\frac{3}{4}$	$-3\frac{1}{3}$	$-16\frac{1}{2}$
	$+2$	$-2\frac{1}{2}$	$-4\frac{1}{4}$	$-7\frac{3}{8}$	$+5\frac{1}{4}$

21. A thermometer registered one day $-15°$, and the next day $-9°$. What was the change in temperature?

22. A thermometer registered one day $+10°$, and the next day $-7°$. What was the change in temperature?

23. Give the range of temperature in the following places (Jan. 1, 1908 to Jan. 1, 1909):

PLACE	EXTREMES		PLACE	EXTREMES	
Montgomery, Ala. . . .	107	− 5	Louisville, Ky.	107	− 20
Little Rock, Ark. . .	106	− 12	Boston, Mass. . . .	102	− 13
Denver, Col.	105	− 29	Duluth, Minn. . . .	99	− 41
Washington, D.C. . .	104	− 15	Havre, Mont. . . .	108	− 55
Boisé, Ida.	111	− 28	New York, N.Y. . .	100	− 6
Des Moines, Ia. . . .	109	− 30	Chicago, Ill. . . .	103	− 23

24. What is the difference in time between the year + 36 and the year − 40?

25. Augustus Cæsar lived from − 60 to + 14. How old was he when he died?

26. Thales, the founder of the first Greek school of mathematics and philosophy, was born in the year − 640 and died in the year − 550. At what age did he die?

27. Pythagoras, who is believed to have given the first proof that in a right triangle the square on the hypotenuse equals the sum of the squares on the other two sides, was born in − 569 and murdered in − 500. At what age was he murdered?

28. Euclid wrote his geometry, called the *Elements*, in about − 300. By + 1300 the book had been carried into western Europe and translated there. How long was this after the book had been written?

29. What is the difference in longitude between Chicago and Berlin, the longitude of the former being approximately + 88° and of the later approximately − 13°?

30. During the month of December, 1909, the maximum temperature at Chicago, as recorded by the Weather Bureau, was + 55°, and the minimum − 7°. What was the amount of variation during the month?

32. Multiplication of Positive and Negative Numbers. — Multiplication as defined in arithmetic is a short form of addition, where the addends are all equal. For example, 3 × $ 25 means $ 25 + $ 25 + $ 25. This definition is only a special case of a more general definition which may be stated in the following way:

To multiply one number (multiplicand) by a second number (multiplier) is to use the first as we use 1 *(unity) to obtain the second.*

Thus, in multiplying 3 by 5, to obtain 5 by using 1, we take

$$1+1+1+1+1=5.$$

Hence, to obtain 5 × 3, we take

$$3+3+3+3+3=15.$$

From this definition are obtained the rules for multiplying positive and negative numbers.

(1) To obtain the multiplier $+3$ we must add three 1's; *i.e.*

$$+3 = 1 + 1 + 1.$$

Hence, to obtain $(+3)(+5)$ we must add three $+5$'s.

That is, $(+3)(+5) = +5 + 5 + 5 = +15$, the product.

And to obtain $(+3)(-5)$ we must add three -5's.

That is, $(+3)(-5) = -5 - 5 - 5 = -15$, the product.

(2) To obtain the multiplier -3 we must subtract three 1's from 0; *i.e.*

$$-3 = 0 - 1 - 1 - 1.$$

Hence, to obtain $(-3)(+5)$ we must subtract three $+5$'s from 0; *i.e.* change their signs and add them.

Hence, $(-3)(+5) = -5 - 5 - 5 = -15$, the product.

And to obtain $(-3)(-5)$ we must subtract three -5's from 0; *i.e.* change their signs and add them.

Hence, $(-3)(-5) = +5 + 5 + 5 = +15$, the product.

From these examples the following principles are evident:

(1) *The product of two numbers with like signs is positive.*

(2) *The product of two numbers with unlike signs is negative.*

(3) *The absolute value of the product of two numbers is the product of the absolute values of the numbers.*

In symbols:

$$(+a)(+b) = +ab.$$
$$(-a)(-b) = +ab.$$
$$(+a)(-b) = -ab.$$
$$(-a)(+b) = -ab.$$

EXERCISES

Give orally the products of the following:

1. $+3, +7$.
2. $-6, -2$.
3. $+4, +5$.
4. $-8, -3$.
5. $-2, -10$.
6. $+7, +3$.
7. $-2, +5$.
8. $-3, +6$.
9. $+5, -7$.
10. $+8, -3$.
11. $-2, -15$.
12. $-7, +4$.

13. $+6, -8$.
14. $-5, -12$.
15. $+12, +4$.
16. $-9, -8$.
17. $+12, -8$.
18. $-7, +12$.
19. $-5, -15$.
20. $+6, +20$.
21. $-10, +14$.
22. $-\frac{1}{2}, -\frac{1}{2}$.
23. $+\frac{3}{4}, -\frac{2}{3}$.
24. $-\frac{5}{6}, +\frac{2}{3}$.
25. $-\frac{3}{5}, -\frac{4}{5}$.
26. $+1\frac{1}{2}, -\frac{2}{3}$.
27. $-2.5, -2.5$.
28. $+1.6, -1.6$.
29. $-.8, +.7$.
30. $-1.3, -1.1$.

31. If the lifting force of a balloon is +, the weight of the ballast is −. Adding weights is indicated by +, and removing weights by −. When 5 weights, each of 20 pounds, are thrown overboard, what is the effect on the upward pull of the balloon? Show that this is equivalent to saying that $(-5)(-20) = +100$.

32. When four weights, each of 15 pounds, are added to a balloon, what is the effect on the upward pull of the balloon? Show that this is equivalent to saying that $(+4)(-15) = -60$.

33. If 10 boxes of goods, each weighing 250 pounds, are removed from a freight boat, what is the effect of the downward pressure of the boat in the water? Show that this is equivalent to saying that $(-10)(+250) = -2500$.

33. The Product of More than Two Factors. — The product of more than two factors is obtained by performing one multiplication at a time, according to the principles in § 32.

Thus, to find $(-2)(-3)(+4)(-5)$, we have

$$(-2)(-3) = +6$$
$$(+6)(+4) = +24$$
$$(+24)(-5) = -120, \text{ the product.}$$

Show that the following rule holds for finding the product of more than two factors:

Find the product of the factors regardless of signs, and prefix + or − according as the number of negative factors is even or odd.

34. Powers of Positive and Negative Numbers. — If the factors are all equal, what is the product called?

The rule in § 33 will serve for finding powers of positive and negative numbers.

Thus,
$$(+5)^2 = (+5)(+5) = +25.$$
$$(+3)^3 = (+3)(+3)(+3) = +27.$$
$$(+2)^4 = (+2)(+2)(+2)(+2) = +16.$$

Evidently, *any power of a positive number is positive.*

Again,
$$(-9)^2 = (-9)(-9) = +81.$$
$$(-5)^3 = (-5)(-5)(-5) = -125.$$
$$(-4)^4 = (-4)(-4)(-4)(-4) = +256.$$
$$(-2)^5 = (-2)(-2)(-2)(-2)(-2) = -32.$$

Evidently, *any even power of a negative number is positive, and any odd power of a negative number is negative.*

EXERCISES

Find the products of the following:

1. $-1, 2, -3$.
2. $-2, -4, -1$.
3. $3, -2, -5, 2$.
4. $2, -5, -8, -1$.
5. $-6, -2, -1, 3$.
6. $4, -5, -2, 3$.
7. $-1, -1, -2, -2$.
8. $2, -7, 4, -2, -1$.
9. $-1, -1, -1, -1, -1$.
10. $-4, 2, -1, -1, -1, -2$.
11. $-1, -2, -3, 4, 1, -5$.
12. $2, 3, -1, -2, 1, -2$.

Find the values of:

13. $(+2)^4$.
14. $(-2)^3$.
15. $(-3)^2$.
16. $(+1)^6$.
17. $(-1)^5$.
18. $(-4)^3$.
19. $(-2)^5$.
20. $(-3)^3$.
21. $(+4)^4$.
22. $(-1)^9$.
23. $(-2)^8$.
24. $(-3)^5$.
25. $(-1)^5(-2)^3$.
26. $(-2)^2(-3)^2$.
27. $(-5)^2(+2)^3$.
28. $(-1)^6(-2)^3$.
29. $(-1)^5(-2)^3(-3)^2$.
30. $(+2)^3(-1)^4(-3)^3$.

If $a = 2$, $b = -3$, $c = -1$, find the value of:

31. b^3.
32. a^2b^2.
33. abc.
34. $2ab^2$.
35. $3ac^3$.
36. $a^2b^2c^2$.
37. b^3c.
38. $a^3 + b^3$.
39. $a^2 + b^2 + c^2 - abc$.

35. Division of Positive and Negative Numbers. — Division is the inverse of multiplication. For example, to divide 24 by 8 is to find a number (3) by which 8 must be multiplied to give 24. Hence, the rules for division of positive and negative numbers come directly from the rules for multiplication.

Since $(+5)(+3) = +15$, then $(+15) \div (+3) = +5$.
Since $(+5)(-3) = -15$, then $(-15) \div (-3) = +5$.
Since $(-5)(+3) = -15$, then $(-15) \div (+3) = -5$.
Since $(-5)(-3) = +15$, then $(+15) \div (-3) = -5$.

Evidently, from these examples we get the same laws of signs in division as in multiplication:

(1) *If the dividend and divisor have like signs, the quotient is positive.*

(2) *If the dividend and divisor have unlike signs, the quotient is negative.*

EXERCISES

Divide:

1. -12 by 3.
2. -20 by -4.
3. $+18$ by -3.
4. -27 by $+9$.
5. $+64$ by -16.
6. -81 by -9.
7. $+42$ by -7.
8. -54 by -6.
9. $+72$ by $+8$.
10. -60 by -6.
11. $+14$ by -14.
12. -20 by -20.
13. $+50$ by -5.
14. -100 by $+25$.
15. $+66$ by -11.
16. -96 by -8.
17. -82 by $+41$.
18. $+63$ by -7.
19. -84 by -12.
20. $+132$ by $+11$.
21. $+\frac{1}{2}$ by $-\frac{1}{2}$.
22. $-\frac{1}{3}$ by $-\frac{2}{3}$.
23. $-\frac{3}{4}$ by $+\frac{1}{2}$.
24. $+\frac{1}{5}$ by $-\frac{2}{5}$.
25. $+\frac{2}{3}$ by $+\frac{3}{4}$.
26. $-\frac{3}{8}$ by $-\frac{1}{4}$.
27. $+6.4$ by $-.8$.
28. -2.1 by $-.3$
29. $-.63$ by $+.7$
30. -1.44 by -1.2

If $a = -2$, $b = 3$, and $c = -4$, find the value of:

31. $\frac{a^2}{c}$.
33. $\frac{2ab^3}{c}$.
35. $\frac{a^5b^3}{c^2}$.
37. $\frac{abc}{6}$.
32. $\frac{b^2c^2}{a}$.
34. $\frac{3ac^2}{2b}$.
36. $\frac{c^3}{a^2b}$.
38. $\frac{a+b+c}{a+b}$.

36. Positive and Negative Numbers in Equations. — In Chapter II we learned how to find the roots of equations by means of *axioms*. We may now consider the members of equations as made up of positive and negative terms.

Let us solve $8x - 7 = 3x + 3$.
To remove -7 from the first member add $+7$ to both members.
Then $8x = 3x + 3 + 7$.
To remove $3x$ from the second member add $-3x$ to both members.
Then $8x - 3x = 3 + 7$.
Uniting terms, $5x = 10$.
Dividing by 5, $x = 2$, the root.

Observe that to remove the *known term* from the first member and the *unknown term* from the second member, we proceed as follows:

(1) *The known term in the first member, with its sign changed, was added to both members.*

(2) *The unknown term in the second member, with its sign changed, was added to both members.*

Note. — The effect of adding $+7$ to both members in the equation above was to make the term -7 disappear from one member and reappear in the other, with its *sign changed*. Adding $-3x$ to both members had a similar effect. Evidently, adding such terms to both members of an equation is equivalent to *moving terms from one member to the other and changing the signs of the terms moved.*

This mechanical process of moving terms from one member to the other by changing their signs is called **transposition**. The terms are said to be **transposed**. Thus, by transposing $+2$ and $-2a$, $a + 2 = 5 - 2a$, becomes $a + 2a = 5 - 2$.

The process of transposition was discovered by the Arabs. About $+830$ an Arab named Mohammed ben Musa Al Khowarazmi wrote a mathematical book with the title *Aljabr Walmuqabalah*. The word Aljabr meant *restora-*

tion or *transposition*, as the term is now called. From this word came the name *algebra*.

The pupil should use the correct phraseology of "adding equals to both members of the equation" until the thing actually done is firmly fixed in mind.

37. Equations with Negative Roots. — The roots of some equations are negative numbers.

EXAMPLE 1. — Solve $7t + 15 = 4t + 3$.

Adding -15 and $-4t$ (transposing), $7t - 4t = 3 - 15$.

Uniting like terms, $3t = -12$.

Dividing by 3, $t = -4$.

EXAMPLE 2. — Solve $3V + 3 = 8 + 4V$.

Adding -3 and $-4V$ (transposing), $3V - 4V = 8 - 3$.

Uniting like terms, $-V = 5$.

Dividing by -1, $V = -5$.

EXERCISES

Solve:

1. $9y = 7y - 28$.
2. $3A - 4 = 2A - 7$.
3. $5s + 20 = 4 - 3s$.
4. $3m + 27 = 6 - 4m$.
5. $5K - 15 = K - 35$.
6. $2t + 3 = 4t + 9$.
7. $7P - 8 = 11P + 12$.
8. $13 = 3x + 40$.
9. $6 - a = 12 + a$.
10. $17 = 19 + M$.
11. $B = 5 + 9B$.
12. $3 - g = 3g + 8$.
13. $W + 7 = 5W + 20$.
14. $\frac{Q}{5} - \frac{Q}{2} = 3$.
15. $8 - \frac{b}{4} = 10$.
16. $x = \frac{x}{2} - 9$.
17. $\frac{R}{2} + \frac{R}{3} = \frac{R}{6} - 4$.
18. $t + \frac{t}{2} + 10 = \frac{t}{4}$.
19. $6 + p + \frac{p}{3} = \frac{2}{3} - 4p$.
20. $5 + \frac{z}{3} - \frac{z}{2} = \frac{13}{2}$.

38. Interpretation of Negative Roots. — In solving a problem by means of an equation, a negative answer may have a *natural interpretation*, or it may indicate that the *problem is impossible*.

EXERCISES

Solve the following problems and interpret the answers:

1. On the second of three days the thermometer rose 20° more than on the first, and on the third day 13° more than on the second. The total rise during the three days was 35°. What was the change of temperature on the first day?

2. In a certain year a merchant, on the average, lost $200 in each of the first 5 months, and made a profit of $450 in each of the next 6 months. His net profit for the year was $1550. Did he have a loss or profit in the last month, and how much?

3. A man gained $1300 during three months. During the second month he gained $650 more than the first month, and during the third month $350 more than the second. Did he gain or lose during the first month, and how much?

4. A farmer sowed 18 acres of wheat more the second day than the first, 12 more the third than the second, and 5 more the fourth than the second. He sowed 59 acres in all. How many acres did he sow each day?

5. In a triangle the sum of the angles is 180°. The second is 12° less than one twelfth of the first, and the third 15° more than the first. How many degrees in each angle?

SUPPLEMENTARY EXERCISES

1. Herodotus, the Greek historian, commonly called the "Father of History," was born in -484 and died in -424. At what age did he die?

2. Livy, one of the most famous of the Roman historians, was born in -59 and lived to be 76 years old. In what year did he die?

3. A ship which leaves port at 42° north latitude goes one day 4° south, the next 3° north, the next 2° south, the next 3° south, and the next 1° south. If sailing south is called $-$ and sailing north $+$, express the sum of these motions, and find the latitude of the ship at the end of the fifth day.

4. A switch engine, making up a freight train, moves forward 30 yards, back 120 yards, forward 150 yards, forward 40 yards, then back 300 yards. Represent these distances by + and − numbers, and find the position of the engine at the end of the work.

5. Find the average of the following temperatures: −10°, −6°, 2°, −4°, 3°, 7°, −4°, 5°, 16°, and −14°.

6. Add $-\frac{1}{2}$, $-\frac{1}{3}$, $+\frac{1}{4}$, $-\frac{1}{6}$, and $+\frac{1}{8}$.

7. Add $+14\frac{2}{5}$, $-8\frac{2}{3}$, $-1\frac{1}{6}$, and $-5\frac{4}{15}$.

8. Find the values of:

$$(-1)^7;\ (-2)^6;\ (-2)^4(-3)^3;\ (-5)(-4)^2(-3)^2.$$

9. If $A=-2$ and $B=-3$, find the value of $(A-B)^3$.

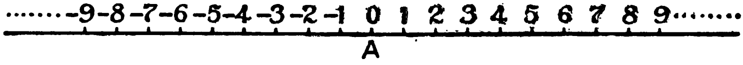
10. If $x=4$ and $y=-2$, find the value of

$$(x^2+xy+y^2)(x^2-xy+y^2).$$

11. Solve $\frac{3n}{2}-\frac{1}{3}=\frac{n}{6}-\frac{1}{2}$.   **12.** Solve $5-\frac{4y}{5}=8y+9\frac{2}{3}$.

13. Positive and negative numbers may be added *graphically* as follows:

Let positive and negative numbers be represented by distances measured to the right and left, respectively, of a point A on a line, as in § 28.



To add a positive number count to the right as many spaces as there are units in the number, and to add a negative number count to the left as many spaces as there are units in the number.

Thus, to add −3 and +7, begin at −3 and count to the right 7 spaces. What is the result? To add −2 and −6, begin at −2 and count to the left 6 spaces. What does it give?

Add graphically the following:

14. -5 and $+8$.

15. $+6$ and -4.

16. $+2$ and $+7$.

17. -4 and -3.

18. $+4$ and -9.

19. -7 and $+12$.

20. -2, -3, and $+4$.

21. $+5$, $+2$, and -3.

22. -6, $+8$, and -4.

23. -1, $+5$, -7, and $+4$.

24. $+4$, $+3$, -12, and $+1$.

25. $+8$, -8, -4, and $+10$.

26. Show why a positive number is subtracted graphically by counting to the left the number of units in the subtrahend, and a negative number is subtracted by counting to the right the number of units in the subtrahend.

27. Subtract graphically the following: -3 from -7; $+2$ from -4; $+6$ from -8; -5 from $+5$; $+1$ from -1; -1 from $+1$; -6 from -6; $+8$ from -2.

28. Viewing multiplication as a special form of repeated addition, where the addends are equal, show graphically that the product of two positive numbers is positive, and that the product of a negative number by a positive number is negative.

CHAPTER IV

ADDITION AND SUBTRACTION OF LITERAL EXPRESSIONS

39. Addition of Monomials. — In § 9 it was shown that similar terms are added or subtracted by adding or subtracting the coefficients, and attaching to the results the common letters with their exponents.

Thus, 3 lb. + 4 lb. = ? $3l + 4l = ?$ 6 sq. yd. + 10 sq. yd. = ? $6y^2 + 10y^2 = ?$ 9 men − 4 men = ? $9m - 4m = ?$ $25x^2y - 14x^2y = ?$

Since positive and negative terms are by nature added and subtracted terms, respectively, they may be added by the same rule: that is,

To add two or more positive or negative terms, add their coefficients according to the rules in § 30, *and attach to the result the common letters with their exponents.*

For example, the sum of $-4a^2b$, $+7a^2b$, $+3a^2b$, and $-9a^2b$ is $-4a^2b + 7a^2b + 3a^2b - 9a^2b = -3a^2b$. The terms may be written in a column and added, as in the margin.

$$\begin{array}{r} -4a^2b \\ +7a^2b \\ +3a^2b \\ -9a^2b \\ \hline -3a^2b \end{array}$$

EXERCISES

Add:

1. $5a, 4a, -2a$.
2. $9n, -12n, -3n$.
3. $10ab, 4ab, -ab$.
4. $6P, -8P, -10P$.
5. $-7m^3n, 4m^3n, -5m^3n$.
6. $12A, -4A, -7A, -9A$.
7. $-3x^4, -x^4, 10x^4, -5x^4$.
8. $PQ, -4PQ, 16PQ, -12PQ$.
9. $-V^2, 9V^2, 7V^2, -8V^2$.
10. $b^2c^3, -4b^2c^3, -b^2c^3, -6b^2c^3$.
11. $5B^2, -8B^2, -12B^2, 17B^2$.
12. $-3xy^5, -7xy^5, xy^5, -2xy^5$.

13.
$$\begin{array}{rrrrr}
-4a & 3x & -10K & 15y^3 & -20zw \\
6a & -7x & -8K & -6y^3 & 13zw \\
-8a & -9x & 12K & y^3 & zw \\
\hline
\end{array}$$

14.
$$\begin{array}{rrrrr}
13xyz & -a^4 & -12pq & 100M & 35V^2 \\
-10xyz & 7a^4 & -10pq & -30M & -10V^2 \\
-6xyz & -4a^4 & 14pq & -4M & -25V^2 \\
-xyz & -a^4 & -pq & M & -V^2 \\
\hline
\end{array}$$

15.
$$\begin{array}{rrrrr}
a^2bc & -s^3t^2 & 2ab^3 & -7T & -10N^6 \\
-4a^2bc & -4s^3t^2 & -3ab^3 & 9T & 4N^6 \\
10a^2bc & 20s^3t^2 & 60ab^3 & -16T & 16N^6 \\
-15a^2bc & -8s^3t^2 & -32ab^3 & -T & -3N^6 \\
-a^2bc & s^3t^2 & -15ab^3 & 12T & -14N^6 \\
\hline
\end{array}$$

16. $-2.05A$, $3.25A$.

17. $0.36x$, $-3.42x$.

18. $-1.2n$, $-3.4n$, $7.1n$.

19. $-0.75P^3$, $-2.15P^3$, P^3.

20. $\frac{1}{2}y$, $-\frac{1}{4}y$, $-\frac{5}{8}y$.

21. $\frac{2}{3}a$, $-\frac{1}{6}a$, $-\frac{1}{2}a$.

22. $3mn$, $-\frac{1}{2}mn$, $2\frac{1}{2}mn$.

23. $-2\frac{1}{3}v^3$, $-v^3$, $3\frac{2}{3}v^3$.

Simplify:

24. $-3x+7x+x-6x$.

25. $12c^4+c^4-8c^4-c^4$.

26. $-5at^2-at^2+10at^2$.

27. $6v'-8v'-10v'+v'$.

Simplify by adding similar terms:

28. $4a-2a-12x+6a+3x$.

Suggestion. — Adding the terms in a and the terms in x separately gives $8a-9x$.

29. $x^2-ab+10ab-6x^2-2ab$.

30. $3m^2-2mn-m^2-mn$.

31. $a^2+2ab+b^2-a^2+2ab-b^2$.

32. $y^3-3y^2z+3yz^2-z^3-2y^3+z^2y-4z^3$.

Add:

33. $4a$ and $-7b$.

Solution. — Since these terms are not similar, their coefficients cannot be added. The addition may be merely *indicated*, giving $4a - 7b$.

Indicating the addition of dissimilar terms is called adding them.

34. $-2a$, $3b$, and $-c$.

35. h^2, $-2hk$, and k^2.

36. a^2b, $-ab^2$, a^3, and $-b^3$.

Note. — The process of adding monomials by adding the coefficients may be used to shorten computations in arithmetical work. For example, $27 \times 54 + 73 \times 54 = 5400$, which is determined without the aid of a pencil.

At sight give the results of:

37. $6 \times 48 + 4 \times 48$.
38. $8 \times 27 + 12 \times 27$.
39. $17 \times 25 + 13 \times 25$.
40. $84 \times 63 + 16 \times 63$.
41. $46 \times 35 - 6 \times 35$.
42. $32 \times 46 + 18 \times 46$.
43. $53 \times 84 - 13 \times 84$.
44. $128 \times 96 - 28 \times 96$.
45. $42 \times 28 + 16 \times 28 + 2 \times 28$.
46. $34 \times 17 + 27 \times 17 - 11 \times 17$.
47. $16 \times 38 - 4 \times 38 + 18 \times 38$.
48. $16\pi + 48\pi + 36\pi$.

40. Addition of Polynomials. — It is evident that two or more polynomials are added if all of their terms are added. Hence, *two or more polynomials may be added by adding their similar terms.*

The process of adding polynomials is similar to that of adding denominate numbers involving two or more units of measure.

For example:

$$\text{Just as}\quad \begin{array}{r} 12\text{ bu. }1\text{ pk. }2\text{ qt.} \\ \underline{5\text{ bu. }2\text{ pk. }5\text{ qt.}} \\ 17\text{ bu. }3\text{ pk. }7\text{ qt.} \end{array} \quad \text{so} \quad \begin{array}{r} 12b + p + 2q \\ \underline{5b + 2p + 5q} \\ 17b + 3p + 7q \end{array}$$

To add polynomials, arrange the similar terms in columns and add each column as in § 39. The sums of the columns, connected by their signs, constitute the sum of the polynomials.

As pointed out in § 30 (See Law of Order in Addition, § 30, NOTE), numbers to be added may be arranged in any order without changing the value of their sum. In polynomials to be added, if the similar terms do not come in the same order in all the polynomials, they should be *rearrarged* so that they do come in the same order.

Thus, to add $2a+3b-4c$, $5c-4a-b$, and $b+5a-2c$, the terms in the last two expressions are rearranged, and the terms in a are put first, the terms in b second, and the terms in c third, as in the margin.

$$\begin{array}{rrr} 2a & +3b & -4c \\ -4a & -b & +5c \\ 5a & +b & -2c \\ \hline 3a & +3b & -c \end{array}$$

EXERCISES

Add:

1. $\begin{array}{r} 3a-2b \\ 4a+5b \\ \hline \end{array}$

2. $\begin{array}{r} -6x+y \\ 4x-7y \\ \hline \end{array}$

3. $\begin{array}{r} 4u-2v+3w \\ -2u+v-6w \\ \hline \end{array}$

4. $\begin{array}{r} -3H+K-12L \\ 8H-4K+3L \\ -2H-5K-L \\ \hline \end{array}$

5. $\begin{array}{r} -8m+6n-p \\ 5m-2n+3p \\ -6m-8n-8p \\ \hline \end{array}$

6. $\begin{array}{r} 4r-5r'-2r'' \\ -2r+3r'-7r'' \\ -9r-r'+4r'' \\ \hline \end{array}$

7. $\begin{array}{r} 3A-B+2C \\ 5A-2B+C \\ -A+3B-4C \\ \hline \end{array}$

8. $\begin{array}{r} 2P+4Q+R-7S \\ -5P-Q+3R+2S \\ 7P-6Q-9R+3S \\ \hline \end{array}$

9. $\begin{array}{r} -20a+3ab-10b \\ 16a-ab+4b \\ -7a-5ab+6b \\ -2a-8ab-12b \\ \hline \end{array}$

10. $2x-y$, $-3x+y$, and $4y-7x$.

11. $5a+3c-b$, $c-a+b$, and $2b-3a+4c$.

12. $33x^2-5xy+2y^2$, $8xy-y^2+6x^2$, and $4y^2-x^2-xy$.

13. $1-2m+3m^2$, $5m-6+4m^2$, and $m^2-10m+5$.

14. Show that the process of adding polynomials by adding the similar terms is used in adding numbers in arithmetic.

Thus, to add 723, 422, and 614, show that each number may be written as a polynomial, as in the margin. In what does this differ from the ordinary process of arithmetic?

$$\begin{array}{rl}
723 = & 7 \times 100 + 2 \times 10 + 3 \\
422 = & 4 \times 100 + 2 \times 10 + 2 \\
\underline{614} = & \underline{6 \times 100 + 1 \times 10 + 4} \\
 & 17 \times 100 + 5 \times 10 + 9 \\
 & \text{or } 1700 \quad + \quad 50 \quad + 9 \\
 & \text{or } \; 1759
\end{array}$$

15. By writing them as polynomials, find by the method in Problem 14 the sums of the following:

$$\begin{array}{r} 831 \\ 614 \\ \underline{743} \end{array} \qquad \begin{array}{r} 123 \\ 502 \\ \underline{954} \end{array} \qquad \begin{array}{r} 302 \\ 824 \\ \underline{553} \end{array} \qquad \begin{array}{r} 915 \\ 201 \\ \underline{672} \end{array}$$

41. Checking Work. — In much of the work of algebra one can easily *check* the results obtained; that is, determine whether or not mistakes have been made in the work.

A good method of checking addition of polynomials consists of assigning particular values to all of the general numbers involved in the polynomials and in the sum, and seeing if *the sum of the values* thus obtained for the polynomials is equal to *the value of the sum* of the polynomials.

Example. — Add and check: $2a-5b+3c$, $3a+4b-2c$, and $5a-2b-4c$.

$$\left.\begin{array}{r}
2a - 5b + 3c = -5 \\
3a + 4b - 2c = 9 \\
\underline{5a - 2b - 4c = -3} \\
10a - 3b - 3c = 1
\end{array}\right\} \text{when } a = 1,\ b = 2,\ c = 1.$$

When $a = 1$, $b = 2$, $c = 1$, the values of the polynomials are -5, 9, and -3, respectively, and the sum of these values is 1. The value of the sum of the polynomials also is 1, which shows that the work is accurate. Any other values of a, b, and c might have been used. Check this work by giving a, b, and c other values.

Note. — This method may be used in checking much of the work in algebra that follows. While other methods of checking work may be practiced, it will be found that this method has the special advantage of helping the pupil to master more thoroughly the meanings of the symbols of algebra. *The pupil should form the habit of carefully checking all of his work in some way.*

EXERCISES

Add and check:

1.
$$\begin{array}{rrr} 2x^2 & -3x & +1 \\ 5x^2 & +7x & -4 \\ -3x^2 & -6x & +5 \\ \hline \end{array}$$

2.
$$\begin{array}{rrrr} a^3 & -3a^2 & +3a & -1 \\ & 4a^2 & -a & +3 \\ -a^3 & +2a^2 & +2a & \\ \hline \end{array}$$

3.
$$\begin{array}{rrr} a^2 & +2ab & +b^2 \\ a^2 & -2ab & +b^2 \\ 2a^2 & & -2b^2 \\ \hline \end{array}$$

4.
$$\begin{array}{rrr} 26m & +10n & +14p \\ -12m & +15n & -20p \\ & -12n & -5p \\ \hline \end{array}$$

5.
$$\begin{array}{rrr} 5u & -6v & +7w \\ -3u & -2v & +9w \\ -8u & -5v & +11w \\ \hline \end{array}$$

6.
$$\begin{array}{rrr} 5r & -6s & +8 \\ -7r & -9s & -5 \\ 4r & +8s & \\ \hline \end{array}$$

7.
$$\begin{array}{rrrr} x^3 & & & -y^3 \\ x^3 & +6x^2y & & \\ & & -6xy^2 & -y^3 \\ \hline \end{array}$$

8.
$$\begin{array}{rrr} -5an & +3bm & +4 \\ -3an & -5bm & -1 \\ an & +bm & +1 \\ 7an & -2bm & -2 \\ \hline \end{array}$$

9. $4s+3t-4,\ 3s-2t+3,\ -s+t+5.$

10. $4n^2-3n+7,\ -2n^2-9n-3,\ 5n^2+4n-9.$

11. $6a^2+5ab-2b^2,\ a^2-ab-8b^2,\ 3a^2+7ab+b^2.$

12. $6x^3-4x^2-3x+8,\ 5x^2+3x+2,\ 8x^3-7x+9,\ x^2-2x.$

13. $m^3-n^3,\ 2m^3-5mn^2+6n^3,\ 3m^2n-7n^3,\ -m^3-m^2n+mn^2-n^3.$

14. $A^6-2A^4+5A^2-3,\ A^6-5A^4-6A^2,\ 9A^4+7,\ -6A^6-A^2-9.$

15. $v^2-vt-t^2,\ 3v^2+4vt-t^2,\ t^2-3vt+v^2,\ 7vt-8t^2.$

16. $4ax-7by-cz,\ ax-2by,\ -3ax+3cz,\ 7by-4cz.$

17. $pq+2qr-pr,\ -2pq-5qr+4pr,\ pq-pr.$

18. $k-2k^3+5k^2-3,\ -3k^2+2k^3+6k-3,\ 2k^2-5,\ 12+3k^2-k.$

19. $-12r+15s-8t,\ 9r-7s-7t,\ -9r+4t,\ -s+5r,\ 6r-7t.$

20. $3v' - 5v'' + v'''$, $3 - 4v'$, $-9v'' + v'''$, $7v' - v''$, $-6v'' + v'''$.

21. $24P + W + 3H$, $5W - 9H$, $6P + 7H$, $-H - 5P$, $2H + 8W - P$.

22. $5.6a - 2.7b$, $3.5a + 4.2b$, $-2.2a + b$, $a - 6.5b$.

23. $.25x^2 - 1.3x + 8.1$, $-4.5x^2 - 1.9x - 4$, $6.3 - 5.8x + 3.5x^2$.

24. $5R^2 - 2.5m$, $-.5R^2 + 4m$, $1.45R^2 + 3.2m$, $-.75R^2 - 9m$.

25. $1 - 2D - 2.6D^2 + 7.5D^3$, $D^3 + 5.5D^2 - 7.2$, $-1.5 + 6.5D + 8D^2$.

26. $\frac{1}{2}x - \frac{1}{3}y$, $\frac{1}{3}x + \frac{1}{2}y$, $x - y$.

27. $\frac{1}{4}m - \frac{1}{2}n$, $2\frac{1}{2}m + \frac{1}{4}n$, $-\frac{1}{2}m + \frac{1}{4}n$.

28. $\frac{2}{3}H + \frac{4}{5}K$, $1\frac{1}{3}H + \frac{1}{2}K$, $2H - 3K$.

42. Subtraction of Monomials. — What rule for subtracting positive and negative numbers was discovered in § 31? Since a monomial is itself a *number*, this rule holds equally in the subtraction of monomials.

To subtract one monomial from another, change the sign of the subtrahend and add the result to the minuend.

The change of sign of the subtrahend should be made *mentally*, the written sign being unchanged.

Thus, to subtract $9x^2y$ from $-18x^2y$, let the written work be as shown in the margin.

$$\begin{array}{r} -18x^2y \\ \underline{9x^2y} \\ -27x^2y \end{array}$$

EXERCISES

Subtract:

$$\begin{array}{lrrrrr} 1. & 2x & -3a & -6M^2 & 8y^3 & -5ab \\ & \underline{-3x} & \underline{-7a} & \underline{4M^2} & \underline{-2y^3} & \underline{-2ab} \end{array}$$

$$\begin{array}{lrrrrr} 2. & 4m^2n & -21xyz & 16A & -6P & -3ab^2 \\ & \underline{-m^2n} & \underline{-4xyz} & \underline{-14A} & \underline{-14P} & \underline{-ab^2} \end{array}$$

$$\begin{array}{lrrrrr} 3. & 2t^3 & -7w^3v & -hk & 6Q & 2abc \\ & \underline{-3t^3} & \underline{2w^3v} & \underline{-4hk} & \underline{-22Q} & \underline{12abc} \end{array}$$

4.
$$\begin{array}{r} 3\,wv^4 \\ -6\,wv^4 \\ \hline \end{array} \qquad \begin{array}{r} 5\,by^6 \\ -8\,by^6 \\ \hline \end{array} \qquad \begin{array}{r} 21\,x^2y^2z \\ -\ \ 3\,x^2y^2z \\ \hline \end{array} \qquad \begin{array}{r} 11\,R^3 \\ -\ \ 3\,R^3 \\ \hline \end{array} \qquad \begin{array}{r} -4\,v' \\ -8\,v' \\ \hline \end{array}$$

5. From $2a$ take $-11a$.

6. From $7x$ take $10x$.

7. From $-6t$ take $-4t$.

8. From $-20P$ take $-9P$.

9. From $32\,at^2$ take $-16\,at^2$.

10. From $-16\pi r^2$ take $-7\pi r^2$.

11. $4x-(-3x)=?$

12. $-3y-(-5y)=?$

13. $5m-(-2m)=?$

14. $16A-(-4A)=?$

43. Subtraction of Polynomials.—It is evident that one polynomial is subtracted from another when all of its terms are subtracted. Consequently for subtraction of polynomials we have the following rule:

Add to the minuend the subtrahend with the sign before each of its terms changed.

Similar terms should be written in columns, as in addition.

The method of checking work used in § 41 may be used also for checking subtraction.

The change of signs should only be made *mentally.*

Example 1.—From $3x^2-2xy+y^2$ take $5y^2+4xy+3y^2$.

$$\left.\begin{array}{rcr} 3x^2-2xy+\ \ y^2 & = & 2 \\ 5x^2+4xy-3y^2 & = & 6 \\ \hline -2x^2-6xy+4y^2 & = & -4 \end{array}\right\}\begin{array}{l} \text{when} \\ x=1 \\ y=1 \end{array}$$

If there is a term in the minuend and no term similar to it in the subtrahend, such a term will be written in a column by itself, and, in subtracting, it will be placed in the answer unchanged. If such a term occur in the subtrahend instead, in subtracting, it will be placed in the answer with its sign changed.

Example 2.—From $4x^4-3x+2x^5+7$ take $x^5-3x^3+2x^2-x$.

$$\left.\begin{array}{rcr} 2x^5+4x^4\qquad\qquad\qquad\ -3x+7 & = & 10 \\ x^5\qquad\quad -3x^3+2x^2-\ \ x\qquad & = & -\ \ 1 \\ \hline x^5+4x^4+3x^3-2x^2-2x+7 & = & 11 \end{array}\right\}\begin{array}{l} \text{when} \\ x=1 \end{array}$$

EXERCISES

Subtract and check:

1. $\begin{array}{r} 4x-5y \\ \underline{x-7y} \end{array}$

2. $\begin{array}{r} C+7D \\ \underline{-2C+3D} \end{array}$

3. $\begin{array}{r} 4T-12h \\ \underline{T-\ \ 6h} \end{array}$

4. $\begin{array}{r} -\ \ m+15a \\ \underline{-3m-\ \ 4a} \end{array}$

5. $\begin{array}{r} 8b-13d \\ \underline{9b+\ \ 8d} \end{array}$

6. $\begin{array}{r} r-2s+5t \\ \underline{r+3s-2t} \end{array}$

7. $\begin{array}{r} 4x+\ \ y-10z \\ \underline{-\ \ x-6y+11z} \end{array}$

8. $\begin{array}{r} -12p-6q+4r \\ \underline{-\ \ 8p+\ \ q-2r} \end{array}$

9. $\begin{array}{r} u+\ \ v-\ \ w \\ \underline{u-2v+3w} \end{array}$

10. $\begin{array}{r} 3f-\ \ g+2h \\ \underline{5f-2g+\ \ h} \end{array}$

11. $\begin{array}{r} 20p-\ \ q+\ \ r \\ \underline{2p+5q-7r} \end{array}$

12. $\begin{array}{r} a^2+2ab+b^2 \\ \underline{a^2-2ab+b^2} \end{array}$

13. $\begin{array}{r} x^3+x^2+x+1 \\ \underline{x^3-x^2+x-1} \end{array}$

14. $\begin{array}{r} F^4+4F^2+1 \\ \underline{3F^4-2F^2-1} \end{array}$

15. $\begin{array}{r} 2ab-3bc+5ac \\ \underline{6ab+7bc-2ac} \end{array}$

16. $\begin{array}{r} 2P+4Q-7R \\ \underline{-5P-\ \ Q+3R} \end{array}$

17. $\begin{array}{r} 3v-4v'+10v'' \\ \underline{-2v-5v'+12v''} \end{array}$

18. $\begin{array}{r} 7W-5S+9H \\ \underline{2W+5S-9H} \end{array}$

19. $\begin{array}{r} a^6-a^4+a^2-1 \\ \underline{2a^6-a^4-a^2-1} \end{array}$

20. $\begin{array}{r} x^3-3x^2+x+5 \\ \underline{2x^2\qquad-5} \end{array}$

21. $\begin{array}{r} a^3+3a^2b+3ab^2+b^3 \\ \underline{a^3-\ \ a^2b+\ \ ab^2-b^3} \end{array}$

22. $\begin{array}{r} 6-2m+9m^2-m^3-m^4 \\ \underline{9+\ \ m-9m^2+m^3-m^4} \end{array}$

23. $\begin{array}{r} a^2+2ab+b^2 \\ \underline{a^2\qquad\quad-b^2} \end{array}$

24. $\begin{array}{r} W^2\qquad\qquad-V^4 \\ \underline{-6WV^2-V^4} \end{array}$

25. $$\begin{array}{r} 4 - t^2 + 7t^4 \\ -t^2 - 4t^4 - 6t^6 \\ \hline \end{array}$$

26. $$\begin{array}{rrr} & 3n^2 - 2n & + 1 \\ n^3 - & n^2 & - 5 \\ \hline \end{array}$$

27. $$\begin{array}{r} 8x - 2x' + 7x'' \\ 9x + 6x' - 9x'' \\ \hline \end{array}$$

28. $$\begin{array}{r} Wt^2 - W^2t - 10 \\ - W^2t + 15 \\ \hline \end{array}$$

29. $$\begin{array}{r} 1.5a + 3.5b \\ .7a - 1.3b \\ \hline \end{array}$$

30. $$\begin{array}{r} \frac{1}{2}m - \frac{1}{4}n \\ \frac{1}{4}m + \frac{1}{8}n \\ \hline \end{array}$$

31 From $6a - 7b - 10c$ take $12a + 4b - 7c$.

32. From $12x^2 - 3x + 1$ take $-2x^2 + 9x - 7$.

33. From $m^2 - 2mn + n^2$ take $4m^2 - mn + 3n^2$.

34. From $2 - 7A^2 + 4A$ take $3A - 5A^2 + 8$.

35. From $2t + 6t^2 + 1$ take $9t^2 + 4t + 3$.

36. From $15v' - 6 - 2v''$ take $-5v'' + 9 - v'$.

37. From $6a - 4a^3 + 7a^2$ take $10 - 3a + a^2 - 8a^3$.

38. From $6P - 5P^2 + 2P^3$ take $P^3 + 2P^2 - 7P + 3$

39. From $D^2 + R^2 + 2DR$ take $5R^2 - 7DR$.

40. From $x^3 - 3x^2y + 3xy^2$ take $x^3 + x^2y - y^3$.

41. From $1.5r - s + 2.8t$ take $.9t - 3r - s$.

42. From $20E - 12.5E^2 + 4.2$ take $6 - 2.4E + 9E^3$.

43. From $5.6ab - 3k$ take $4.5k - 25$.

44. From $\frac{1}{8}x^2 - \frac{1}{2}x - \frac{1}{4}$ take $\frac{2}{3}x^2 - \frac{1}{3}x + \frac{3}{4}$.

45. $m^2 - 5m + 7 - (-2m^2 - 3m + 2) = ?$

46. $W^2 + 24W - 10 - (16 - 5W + W^2) = ?$

47. $a^2 - b^2 - (2ab - b^2) = ?$

48. $6 - (S - 4) = ?$

44. Removal of Signs of Grouping. — When the sign $-$ precedes a sign of grouping which incloses an expression, it indicates that the expression is a *subtracted* expression.

Subtraction is performed by changing the sign of each term of the subtrahend and adding the resulting expression to the minuend. Hence,

A sign of grouping preceded by the sign — may be removed from an expression, if the sign before each term inclosed is changed.

For example, $3R-(5R-2S)=3R-5R+2S.$

And $-(-7y+2y^2)=7y-2y^2.$

Since the sign + expresses addition, or, when convenient, may be considered merely as a sign of distinction, therefore,

A sign of grouping preceded by the sign + may be removed from an expression, without changing the sign of any of the terms inclosed.

For example, $4n+(3n-7)=4n+3n-7.$

And $+(8-2H)=8-2H.$

When signs of grouping are inclosed within other signs of grouping, it is wise for the beginner to *remove the innermost sign first, then the innermost remaining sign, etc.*

Thus,

$$\begin{aligned}&3a-(9a-\{2b-[6a-b]\})\\&=3a-(9a-\{2b-6a+b\}), \text{ removing brackets,}\\&=3a-(9a-2b+6a-b), \text{ removing braces,}\\&=3a-9a+2b-6a+b, \text{ removing parentheses,}\\&=3b-12a.\end{aligned}$$

EXERCISES

Remove the signs of grouping and add similar terms:

1. $2r+(5r-3)$.
2. $-8T-(7-2T)$.
3. $12+\{-w+w^2\}$.
4. $C-[30C-2]$.
5. $27-(F-32)$.
6. $a-(b-2a-5)$.
7. $2s+\overline{s+10}$.
8. $5-(4\pi-1)-2\pi$.
9. $6L+(9-L)+8$.
10. $2M+t-(5M-t)$.
11. $a^2-(b^2-2ab)$.
12. $18y^2-7y+\{3y^2-y\}-y$.
13. $6w-4mn+(3mn-2w)$.
14. $2B^2-(B^2-4AC)$.
15. $(2r-7h)-(r-4h)$.

Suggestion. — Since neither of the signs of grouping incloses the other, the two may be removed at the same time, independently of each other.

16. $(1 - x + x^2) + (3x - 8)$.

17. $(2p - 5q) + (4p + q) + 6q$.

18. $W - (3w - 2.5) - (w - 1.5W)$.

19. $P - \{3Q - 1\} + \{4Q - 6R\}$.

20. $x^2 + \{2x - (x^2 - 3x)\}$.

21. $2v - [t - v + (3t - 4v)]$.

22. $(2D - \{3 - D\} + 1)$.

23. $-(-(-1))$.

24. $-(-(-(-(-(-a)))))$.

25. $2t - (d - (2t - d))$.

26. $s - \frac{1}{2}at + \{2\frac{1}{2}at - (s - at)\} - \frac{1}{2}at$.

27. $m - (5n - (m - (2n - 6m)))$.

28. $a^2 - (2ab - (-(-b^2))) - (a^2 + (b^2 - 2ab))$.

29. Solve $6y - (4 - 2y) = 12$.

30. Solve $V + (3 + 5V) - 7 = 0$.

31. Solve $12.5k - (10 - 2k) = 19 - (k - 2)$.

32. Solve $3z - (8 - z) = z - (2z - 10)$.

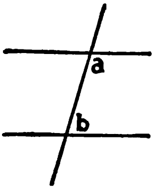
33. Solve $u = 8 - (3u - 14 - \{6 - u\})$.

34. The sum of two numbers is 20 and their difference 6. What are the numbers?

SUGGESTION. — Let n = the larger number. Then $20 - n$ = the smaller.

35. The sum of two numbers is 7 and their difference 17. What are the numbers?

36. The sum of the angles a and b is 180° and their difference 30°. How many degrees in each?



37. My living expenses for two years amounted to \$2500. The expenses during the second year were \$280 more than during the first year. Find the expenses for each year.

45. Insertion of Signs of Grouping. — By reversing the processes in § 44, it follows that *terms of a polynomial may be inclosed*

within a sign of grouping, when this sign of grouping is preceded by the sign +, without changing the signs of the terms; and, when preceded by the sign —, by changing the signs of the terms.

Thus, to inclose the last three terms of $a^2 + 2bc - b^2 - c^2$ in parentheses preceded by the sign +, we have $a^2 + (2bc - b^2 - c^2)$. If preceded by the sign —, it becomes $a^2 - (b^2 - 2bc + c^2)$.

EXERCISES

Inclose the last three terms within a sign of grouping preceded by the sign +:

1. $A + B - C - D$.
2. $xy + yz - zw + wx$.
3. $5 - 2n - 9n^2 + n^3$.
4. $r - 4s + 4s^2 + 1$.
5. $m^2 + 6pq - q^2 - 9p^2$.
6. $x^2 + 2xy + y^2 - z^2 + 2zw - w^2$.
7. $16 - 8P + P^2 + 12M - 9M^2 - 4$.
8. $k + 7 + 10k - 1 - 25k^2$.
9. $18 - B^2 - b^2 - Bb$.
10. $1 + 6rr' - r^2 - 9r'^2$.

11–20. Inclose the last three terms in each of the above expressions within a sign of grouping preceded by the sign —.

21. In $an - 2t - 3n + bt$, place the terms involving n in parentheses preceded by the sign + and those involving t in parentheses preceded by —.

22. In $10P - 2W + mP - nW$, place the terms involving P in parentheses preceded by + and those involving W in parentheses preceded by —.

SUPPLEMENTARY EXERCISES

Simplify:

1. $5\sqrt{n} - 8\sqrt{n} + 6\sqrt{n} + \sqrt{n} - 10\sqrt{n}$.
2. $\sqrt{a+b} - 4\sqrt{a+b} + 6\sqrt{a+b} - 9\sqrt{a+b}$.
3. $12(x - y) - 3(x - y) - 16(x - y) + (x - y)$.
4. $-9(s - at^2) - 6(s - at^2) + (s - at^2) + 7(s - at^2)$.
5. $(2\pi R + 7)^2 + 14(2\pi R + 7)^2 - 8(2\pi R + 7)^2 + (2\pi R + 7)^2$.
6. $-20\sqrt[3]{8 + \frac{1}{2}Wg^2} + 6\sqrt[3]{8 + \frac{1}{2}Wg^2} + 18\sqrt[3]{8 + \frac{1}{2}Wg^2} - 9\sqrt[3]{8 + \frac{1}{2}Wg^2}$

Simplify by combining similar terms:

7. $16 - 2\sqrt{A} + 5\sqrt{A} - 24 - 7\sqrt{A}$.

8. $D + \sqrt{w} + \sqrt{t} - 4\sqrt{w} - 5D + 7\sqrt{w} - 6\sqrt{t}$.

9. $5(W - V) - 8W^2 + (W - V) + 17W^2 - 4(W - V)$.

10. $x^2 - 2xy + y^2 - 3x^2 - 6xy - 3y^2 + 4xy$.

11. $r_1^2 + 4r_1r_2 + r_2^2 - 2r_1^2 + 8r_1r_2 - 8r_2^2$.

Simplify by writing with a polynomial coefficient:

12. $ax + bx + cx$.

SOLUTION. — These terms are similar in x, and hence are added by adding the coefficients of x. Since the coefficients a, b, and c cannot be added into one term, but their sum can only be indicated, we get $(a + b + c)x$.

13. $my + ny - py$.

14. $aR^2 - bR^2 - cR^2 + dR^2$.

15. $Mt^3 + 2Nt^3 - 3Pt^3 + Qt^3$.

16. $5D^2 + \pi D^2 - hD^2 - 7D^2$.

17. $4a(x - y) - 3b(x - y) + 5(x - y)$.

18. Add $nA + mB + pC$ and $xA - yB + zC$.

19. From $aU - bV - cW$ subtract $gU + hV - kW$.

Remove signs of grouping and combine like terms:

20. $a + [2a - (3a - 2b)] - (4b - 3a)$.

21. $[m - (2m - n)] - [-3n + (4m - n)]$.

22. $(A - 5B) - \{3A - [4B + (6A - B) - (4B - 2A)]\}$.

In the following remove parentheses, leaving brackets, and simplify the results as far as possible:

23. $[(x - 2y) + (x + y)][(x - 2y) - (x + y)]$.

24. $[r - (s - t)][r + (s - t)]$.

25. $[5A + (3 + A)][5A - (3 + A)]$.

26. $[(4r - 3) + (3r' - 4)][(4r - 3) - (3r' - 4)]$.

27. $[(7a - 2b) - (c + 3d)][(7a - 2b) + (c + 3d)]$

CHAPTER V

MULTIPLICATION AND DIVISION OF LITERAL EXPRESSIONS

46. Order of Factors. — Is there any difference between the values of 3×4 and 4×3? Of $2 \times 3 \times 4$ and $3 \times 4 \times 2$ or $4 \times 2 \times 3$ or $2 \times 4 \times 3$? Does it change the value of the product to change the order of the factors?

In general, has ab the same value as ba? Has xyz the same value as xzy or yzx? Has $2 \times a \times 3 \times b$ the same value as $2 \times 3 \times a \times b$?

Numbers to be multiplied may be taken in any order without changing the value of the product.

NOTE. — This fundamental principle is known as the **Law of Order in Multiplication.**

47. Grouping of Factors. — In $3 \times (2 \times 4)$ the parentheses indicate what? Is there any difference between the values of $3 \times 2 \times 4$ and $3 \times (2 \times 4)$? Of $2 \times 3 \times 4 \times 5$ and $(2 \times 3) \times (4 \times 5)$? Does it change the value of the product to group the factors so that the multiplications are performed in different ways?

In general, has abc the same value as $a(bc)$? Has $x^3x^2y^2y^4$ the same value as $(x^3x^2)(y^2y^4)$?

Numbers to be multiplied may be grouped in any way without changing the value of the product.

NOTE. — This fundamental principle is known as the **Law of Grouping in Multiplication.**

48. Law of Exponents in Multiplication. — How else is AA written? How else is $nnnn$ written? In x^3 what does the exponent show? $a^5 = aaaaa$. $a^6 = ?$ $R^7 = ?$

Multiplying a^3 by a^2, we get

$$a^2 \times a^3 = aa \times aaa \text{ or } aaaaa = a^5.$$

Similarly, $$m^4 \times m^2 = mmmmmm = m^6.$$

$$v^2 \times v^3 \times v^5 = vvvvvvvvvv = v^{10}.$$

$$D^4 \times D^5 = D^{4+5} = D^9.$$

In multiplying powers of the same base, the exponent of the product is obtained by adding the exponents of the factors.

In symbols, the above principle is expressed by

$$a^m \times a^n = a^{m+n}.$$

49. Multiplication of Monomials. — By use of the principles stated in the preceding sections the product of any two or more monomials may be found.

EXAMPLE 1. — To multiply $2x^2y$ by $3x^3y^3$, we have

$$\begin{aligned}(2x^2y)(3x^3y^3) &= (2 \times 3)(x^2x^3)(yy^3)\\ &= 6x^5y^4.\end{aligned}$$

EXAMPLE 2. — The product of $-5M^2N^5$ and $-4M^4N^2$ is

$$\begin{aligned}(-5M^2N^5)(-4M^4N^2) &= (-5)(-4)(M^2M^4)(N^5N^2)\\ &= 20M^6N^7.\end{aligned}$$

To multiply monomials, find the product of the numerical coefficients, using the laws of signs; then annex to the result the products of the like literal factors, using the law of exponents.

In performing multiplication, the work is often arranged in column form, as here shown.

$$\begin{array}{r} -7v^2t^4 \\ 8v^7t \\ \hline -56v^9t^5 \end{array}$$

EXERCISES

Find the product of:

1. a^3 and a^4.
2. x^2 and x^7.
3. b^3 and b^6.
4. M^9 and M^2.
5. r^6 and r^5.
6. V and V^4.
7. L^9 and L^2.
8. t^{10} and t^3.
9. y and y^6.
10. n^8 and n^{12}.
11. W and W^8.
12. T^2 and T^{11}.

13. d^4 and d^9.
14. s^{12} and s^9.
15. U^6 and U^8.
16. z and z^{12}.
17. A^2, A^3, and A^4.
18. C^3, C^4, and C^6.
19. p^5, p^2, and p^2.
20. x, x^2, and x^4.
21. n, n^2, n^8, and n^5
22. B, B^2, B^6, and B^9.
23. $2x^4$, $3x^3$, and $2x^5$.
24. $4m$, $2m^4$, and m^7.
25. $-2A$, $-3A^5$, and $4A^9$.
26. $3R^3$, $-7R$, and $-2R^8$.
27. $-5y$, $-9y^3$, and $-2y^{10}$.
28. B, $-6B^7$, and $4B^6$.
29. c, $8c^2$, $3c^7$, and $2c^4$.
30. $-8d^2$, $-6d$, d, and $2d^8$.
31. x^7y^6 and x^6y^7.
32. $-2a^3n^8$ and $5an^2$.
33. $-4AC$ and $3A^3B$.
34. $7s$, $-2sr$, and $6s^3r^2$.
35. $-9x^2y^2z^2$ and $-4x^3z^4$.
36. $3D^2W$, $-2D^2W$, and $8D^2W$.
37. $-6M^3N^3$, $9M^2N$, and $-5N^8P$.
38. $16x^4z^2$, $-2x^5z^2$, and $7y^3z^2$.
39. $-2m^3n^5$, $8m^2p^4$, and $-5n^8p^2$.
40. a^5, $-2a^3b^2$, $7a^4c$, and $-3bc^3$.
41. $\frac{1}{2}x^2y$, $-\frac{1}{2}x^3y^2$, and x^2y^3.
42. $-\frac{1}{3}HD$, $\frac{1}{2}H^2D$, and $3D^3$.
43. $\frac{3}{4}rs^2$, $-2r^2s^2$, and $\frac{1}{2}r^3s^4$.
44. $\frac{7}{8}pq^8$, $\frac{3}{7}p^6q^7$, and $2p^2q^5$.
45. V, $\frac{1}{6}V$, $-4V^2$, and $\frac{1}{3}V^3$.
46. $-\frac{3}{8}c^2d^{10}$, $\frac{4}{5}c^8d^2$, and $c^{10}d^6$.
47. $\frac{1}{10}Wt^3$, $5W^2t^2$, and $4Wt^4$.
48. $-MV^2$, $-\frac{1}{2}MV^2$, and $-6MV^2$.
49. 3, $-5Qt$, $8Q^2t$, and $-\frac{1}{4}Q^3t^2$.

50. Multiplication of a Polynomial by a Monomial. — The multiplication of a polynomial by a monomial is similar to the multiplication of denominate numbers containing two or more units of measure.

$$\text{Just as}\quad \begin{array}{lll} 9\text{ yd.} & 2\text{ ft.} & 6\text{ in.} \\ & & 4 \\ \hline 36\text{ yd.} & 8\text{ ft.} & 24\text{ in.} \end{array} \qquad \text{so} \qquad \begin{array}{r} 9y + 2f + 6i \\ 4 \\ \hline 36y + 8f + 24i \end{array}$$

In general, since the whole of a quantity is multiplied by a number when each of its parts is multiplied by that number, we have the following rule:

To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial, and add the partial products obtained.

Note. — This fundamental principle, that a polynomial is multiplied by multiplying each of its terms separately and adding the partial products obtained, is known as the **Law of Distribution.** It is expressed in symbols by

$$n(a+b)=na+nb.$$

Example. — The multiplication of $a^2 - 2ab + b^2$ by $3a^3b^3$ may be indicated either in the form $3a^3b^3(a^2 - 2ab + b^2) = 3a^5b^3 - 6a^4b^4 + 3a^3b^5$ or as in the margin.

$$\begin{array}{l} a^2 - 2ab + b^2 \\ \underline{3a^3b^3} \\ 3a^5b^3 - 6a^4b^4 + 3a^3b^5 \end{array}$$

It is always more convenient in algebra to perform multiplications from left to right. This may also be done in arithmetic, but for convenience in "carrying" it is usually more convenient in arithmetic to begin at the right.

EXERCISES

Multiply:

1. $x+y$ by 5.
2. $a-b$ by 3.
3. $n-4$ by 7.
4 $P+1$ by 6
5. $2A-3$ by 4.
6. $6a+5b$ by 2.
7. $8M-9N$ by 3.
8. $-5E+7K$ by 10.
9. $4x-9y$ by x.
10. $2a-7b+c$ by 6.
11. x^2-2x+1 by 4.
12. $a^2+2ab+b^2$ by ab.
13. $3-8m+4m^2$ by $5m$.
14. $7a^3+12b^2$ by $-2a^2b$.
15. $15Rt-4t^2$ by $4R^3$.
16 $4x^3+7y^3$ by $2x^2y^2$.
17. $3s^3-5s^2+9s$ by $4s^2$.
18. B^2-4AC by $9AC$.
19. $5-2v+7v^2$ by $-8v^3$.
20. L^4-2L^2+3 by $4L^2$.
21. $3n-4+12p$ by $-2n^2p^3$.

22. W^3-6W^2+2W-3 by $-5W^4$.

23. $2c^4+15-4c^3$ by $-3c^2$.

24. $A^2+2AB+B^2$ by $8A^3B^2$.

25. $ab+bc+ca$ by abc.

26. $2D^2-3DW+W^2$ by $-9D^2W^2$.

27. A^3-B^3 by $2AB$.

28. $3x^4-6x^3+x^2+8x-1$ by $8x^5$.

29. $7ab+3bc-4cd$ by $2ac$.

30. $5(4-a-a^2)$.

31. $4x(3x+8y)$.

32. $7t(t^2-at)$.

33. $p^2(5p+3q)$.

34. $A(A^2-5AB+6B^2)$.

35. $-2m^2(3-9m-5m^2)$.

36. $3y^5(x^2-3xy+7y^2)$.

37. $-8a^2b^2(-15a^3b+ab^3-b^4)$.

38. $4m^3n^2(m^3-m^2n+3mn^2-n^3)$.

39. $\frac{1}{2}(4s^2-6s+2)$.

40. $-\frac{1}{3}W(\frac{1}{4}W^2-\frac{1}{3}WV+\frac{1}{6}V^2)$.

41. $$\begin{array}{l}\frac{1}{2}a^2+\frac{1}{3}ab+\frac{1}{2}b^2\\ \underline{\frac{1}{2}ab\qquad\qquad\qquad}\end{array}$$

42. $$\begin{array}{l}-\frac{2}{3}x^3+\frac{1}{2}x^2y-\frac{1}{3}xy^2+\frac{1}{4}y^3\\ \underline{-\frac{1}{3}x^2y^2\qquad\qquad\qquad\qquad}\end{array}$$

43. $$\begin{array}{r}1.2m^2-2.5mn+3.4n^2\\ \underline{-1.5m^3\qquad\qquad\qquad\quad}\end{array}$$

44. $$\begin{array}{l}2.15p^3-.05p^2+1.2p-4\\ \underline{.5p^2\qquad\qquad\qquad\qquad}\end{array}$$

Simplify by removing signs of grouping and uniting similar terms:

45. $2(2a-3b)-3a$.

46. $5(A-B)+2(B-A)$.

47. $7(s^2-2st+t^2)+3(2s^2+st-4t^2)$.

48. $a(ab-b^2)+b(a^2+ab)$.

49. $3P(5-2P)+4(3P+P^2)$.

50. $\frac{1}{2}S(2S^2-\frac{1}{4}S)+\frac{1}{4}(2S^2+S)$.

51. $4(2A-3B)-2(4A+B)$.

NOTE. — In this expression the term $-2(4A+B)$ may be looked upon in two ways, either as indicating that $4A+B$ is to be multiplied by 2 and the result subtracted from the part that precedes, or as indicating that $4A+B$ is to be multiplied by -2 and the result added algebraically to the part that precedes. In either case the terms within the parentheses $(4A+B)$ will have their signs changed in the process.

52. $6(c-2k)-4(2c-3k)$.

53. $6P-4Q-3(7P+2Q)$.

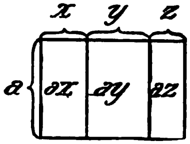
54. $m(n^2-mn)-n(mn+m^2)$.

55. $-5(10r+3t)-2(2t-r)$.

56. $-3(x^2y-xy^2)-(xy^2-x^2y)$.

57. $-3(v^3+2v^2w-2w^3)-2(-2v^3+3v^2w-w^3)$.

58. This rectangle is made up of three parts whose common width is a and whose lengths are x, y, and z, respectively. Show that the areas of the parts are ax, ay, and az, respectively. What does $a(x+y+z)$ represent? Hence, show that $a(x+y+z)=ax+ay+az$, which illustrates the *Law of Distribution.*



59. By a rectangle show in the same way that $2x(a+2b+c)=2ax+4bx+2cx$.

60. Show that the process of multiplying a polynomial by multiplying each of its terms and adding the partial products obtained is used in arithmetic in the multiplication of a number of two or more figures by a number of one figure.

Thus, to multiply 432 by 3, we see that 432 is in nature a polynomial, and may be written as the sum of three terms. 432 is multiplied by 3 by multiplying each of these terms by 3 and adding the results.

$$\begin{array}{rcrcrcr} 432 & = & 400 & + & 30 & + & 2 \\ \underline{3} & & \underline{3} & & \underline{3} & & \underline{3} \\ 1296 & = & 1200 & + & 90 & + & 6 \end{array}$$

NOTE. — The ancients employed this principle in multiplication. Thus, by the Greeks the method of calculating 7×326 was equivalent (in modern notation) to that shown here. Because of the principle of *place value* in our notation, we perform our multiplications mechanically, without stopping to think of this Law of Distribution underlying the work.

$$\begin{aligned} 7 \times 326 &= 7(300+20+6) \\ &= 2100+140+42 \\ &= 2282 \end{aligned}$$

61. By the use of a rectangle as in Ex. 58, illustrate the *Law of Distribution* in finding 3×268

51. Multiplication of a Polynomial by a Polynomial. — The process of multiplying one polynomial by another is similar to the process in arithmetic of multiplying one number of two or more figures by another number of two or more figures. Just as in arithmetic the multiplicand is multiplied by the number represented by each figure of the multiplier, and all the partial products added, so in multiplication of polynomials in algebra the multiplicand is multiplied by each term of the multiplier, and all the partial products added.

For example:

$$\begin{array}{lrl l lr}
 & 25 & & & 2x+5y & \\
 & \underline{34} & & & \underline{3x+4y} & \\
\text{Just as} & 100 & = 4 \times 25 & \text{so} & 6x^2+15xy & =3x(2x+5y) \\
 & \underline{750} & = 30 \times 25 & & \underline{\qquad\quad 8xy+20y^2} & =4y(2x+5y) \\
 & 850 & = 34 \times 25 & & 6x^2+23xy+20y^2 & =(3x+4y)(2x+5y)
\end{array}$$

Each term of $2x+5y$ is multiplied by $3x$ and then by $4y$, and the similar terms of the partial products are written in columns and added.

NOTE. — See that pupils clearly understand that we use this process abridged in arithmetical multiplication. Then have them see that algebraic multiplication is the same.

It is easily seen that this process is only an application of the *Law of Distribution* noted in § 50. Hence,

To obtain the product of two polynomials, multiply the multiplicand by each term of the multiplier, and add the resulting partial products.

The work in multiplication can be checked by assigning particular values to the general numbers involved, as in addition and subtraction, and seeing if the product of the particular values of the factors equals the particular value of the product.

EXAMPLE 1. — Multiply $2a-3b+5c$ by $3a+b-2c$.

$$\begin{array}{rrrrrrl}
2a & -3b & +5c & & & & =4 \\
3a & +b & -2c & & & & =\underline{2} \\
\hline
6a^2 & -9ab & +15ac & & & & \\
 & 2ab & & -3b^2 & +5bc & & \\
 & & -4ac & & +6bc & -10c^2 & \\
\hline
6a^2 & -7ab & +11ac & -3b^2 & +11bc & -10c^2 & =8
\end{array}$$

When $a = 1$, $b = 1$, and $c = 1$, the multiplicand $= 4$, multiplier $= 2$, and product $= 8$, as it should.

Observe that the terms of the first partial product are written in a line by themselves, those of the second in a line by themselves, etc., and that the new kinds of terms, when obtained, are spaced out so that only similar terms will fall in columns.

Note. — A polynomial is said to be *arranged according to the powers of a letter* when the exponents of that letter either increase or decrease in the successive terms as we pass from left to right. Thus, $x^4 - 2x^3 + x^2 + x - 5$ is arranged according to the *descending* powers of x; while $2y^4 + xy^3 + x^2y^2 + x^3y + x^6$ is arranged according to the *ascending* powers of x. How is the latter arranged as to y?

It will be found an advantage in multiplication of polynomials to arrange, if possible, both the multiplicand and the multiplier alike according to the powers of some letter, before multiplying.

Example 2. — Multiply $A^2 - 2 + 2A$ by $A - 6 + A^2$.

Arranging both trinomials according to the descending powers of A, we have

$$\left.\begin{array}{rrrrrr}
A^2 + 2A - 2 & & & & & = 1 \\
\underline{A^2 + A - 6} & & & & & = \underline{-4} \\
A^4 & + 2A^3 & - 2A^2 & & & \\
 & A^3 & + 2A^2 & - 2A & & \\
 & & -6A^2 & - 12A & + 12 & \\
\hline
A^4 & + 3A^2 & - 6A^2 & - 14A & + 12 & = -4
\end{array}\right\}\begin{array}{l}\text{when} \\ A = 1\end{array}$$

Note. — Substituting 1 for each of the letters in the two preceding examples tests the coefficients and signs only. Why? To test the exponents other values than 1 must be assigned to the letters.

EXERCISES

Multiply and test:

1. $n + 2$ by $n + 2$.
2. $A - 3$ by $A + 2$.
3. $x - 4$ by $x - 3$.
4. $2a + 3$ by $3a - 2$.
5. $5z + 6$ by $z + 4$.
6. $4t - 7$ by $4t + 3$.
7. $8b + 1$ by $3b + 1$
8. $3R + 1$ by $4R + 1$.
9. $2V + 5$ by $3V + 7$.
10. $5y - 8$ by $6y - 1$.
11. $4D + 2$ by $-3D + 5$.
12. $2m + 6$ by $2m - 6$.

13. $a - 10$ by $a + 10$.
14. $2W + 3$ by $2W + 3$.
15. $3 + 4x$ by $2 + 3x$.
16. $7 - S$ by $3 + S$.
17. $5 - 4b$ by $5 - 4b$.
18. $2a + 3b$ by $5a - 2b$.
19. $6p + 4t$ by $p - 2t$.
20. $5B + C$ by $5B + C$.
21. $2k - 3c$ by $2k - 3c$.
22. $x^2 - 2x + 1$ by $x + 2$.
23. $A^2 - 3A + 4$ by $A - 5$.
24. $v^2 + vw + w^2$ by $v + w$.
25. $M^2 - N^2$ by $M^2 + N^2$.
26. $2x - 3y + 1$ by $3x + 2y - 1$
27. $x^2 - 4 + 3x$ by $2 + 5x$.
28. $6 - 5R$ by $4R + 7$.
29. $a^3 - a^2 + 4a - 6$ by $2a^2 - 3a + 2$.
30. $3n^2 - n + 4$ by $4n - 3$.
31. $1 - 2S + 3S^2$ by $4S + 5$.
32. $2 + 3T + T^2$ by $2 + 3T + T^2$.
33. $4a + 6b + 10c$ by $2a - 3b + 4c$.
34. $R - 5S + 3W$ by $3R + 2W - S$.
35. $\frac{1}{2}a + \frac{1}{3}$ by $\frac{1}{2}a + \frac{1}{3}$.
36. $2n + 6p - 5r$ by $2p - 3n$.
37. $a - 1$ by $1 + 2a + a^2$.
38. $3b^3 - 5b^2 + b - 4$ by $3 - 2b$.
39. $D - 2D^2 + 4$ by $8 + 2D - D^2$.
40. $2w^3 - 5w^2 + 4w - 1$ by $3 - 2w^2 - w$.
41. $(x + 5)(2x - 5)$.
42. $(2A + 4)(5A - 1)$.
43. $(1 - 4t)(1 + 4t)$.
44. $(B^2 - 4AC)(B^2 + 4AC)$.
45. $(x^2 - 2x + 1)(x - 1)$.
46. $(3 - v)(v^2 + 6 - v)$.
47. $(a^3 + b^3)(a^3 - b^3)$.
48. $(n^2 - 4)(n^4 + 6n^2 + 5)$.
49. $(a - 3t)(2a^2 + at - t^2)$
50. $(\frac{1}{4}x + \frac{1}{5}y)(\frac{1}{4}x - \frac{1}{5}y)$.

Find the product of:

51. $x - 1$, $x + 2$, and $x + 1$.

Suggestion. — Multiply $x - 1$ by $x + 2$, and that product by $x + 1$.

52. $a+3,\ a-2,\ 2a+1.$

53. $2B-1,\ B+2,\ 3B-2.$

54. $3V-4,\ 3V-4,\ 1+V.$

55. $m+n,\ m-n,\ m^2+n^2.$

56. $2+d,\ 2-d,\ 4+d^2.$

57. $2p-3t,\ 3p+2t,\ t+p.$

58. $S^2+S+1,\ S-1,\ S^3+1.$

59. $x^3-z^3,\ x^3+z^3,\ x^6+z^6.$

60. $(A+1)(A+2)(A+3)(A+4).$

Remove signs of grouping and simplify by combining terms:

61. $(t+2)(t+3)+(2t-1)(t-4).$

62. $(2+F)(3F-1)+F(5F-2).$

63. $b^2-2b+(1-b)(1+b).$

64. $(x+w)(x+w)+(x-w)(x-w).$

65. $(a+2)(a+5)-(2a+1)(a-2).$

Suggestion. — The product of $(2a+1)(a-2)$ is to be subtracted from $(a+2)(a+5)$. What operation must be performed first?

66. $(A+B)(A+B)-(A-B)(A-B).$

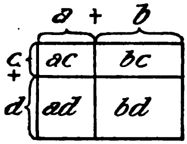
67. $2(y^2-3y+1)-(y+4)(y-1).$

68. $(v+t)(v-2t)-(v-t)(v+2t).$

69. $n^2-3n+4-(2-n)(3n+2).$

70. $(2a+3b)(2a+3b)-(2a-3b)(2a-3b).$

71. Show by the accompanying diagram that $(a+b)(c+d)=ac+bc+ad+bd.$



72. Show by a diagram, as in Problem 71, that $(x+5)(x+7)=x^2+12x+35.$

73. Show by a diagram, as in Problem 71, what the product of $(a+b+c)(x+y+z)$ is.

74. Show how the *Law of Distribution* is used in multiplying the following:

$$\begin{array}{r}234\\ \underline{42}\end{array}\qquad\begin{array}{r}506\\ \underline{63}\end{array}\qquad\begin{array}{r}427\\ \underline{135}\end{array}$$

75. A rectangular field is 16 rods longer than it is wide. A second field is 5 rods shorter and 2 rods wider than the first. If w represents the width of the first, express the area of the second.

76. A rectangle is 8 inches longer than it is wide. If its length is increased by 4 inches, and its width decreased by 2 inches, represent by use of one letter its resulting area.

52. Law of Exponents in Division. — Since division is the inverse of multiplication, the law of exponents in division is obtained from the law in multiplication.

Since $a^3 \times a^2 = a^5$, $a^5 \div a^3 = a^2$.
Since $n^4 \times n^6 = n^{10}$, $n^{10} \div n^4 = n^6$.

Tell how the exponent of each quotient is obtained from the exponents of the dividend and divisor.

By use of cancellation we arrive at the same conclusion.

For example, $\frac{x^7}{x^3} = \frac{\cancel{x}\cancel{x}\cancel{x}xxxx}{\cancel{x}\cancel{x}\cancel{x}} = xxxx = x^4.$

In dividing powers of the same base, the exponent of the quotient is obtained by subtracting the exponent of the divisor from the exponent of the dividend.

In symbols this rule is expressed by

$$a^m \div a^n = a^{m-n}.$$

53. Division of Monomials. — By reversing the rule for multiplying monomials the rule for division is obtained.

Since $2a \times 3b = 6ab$, $6ab \div 2a = 3b$.
Since $(7x^3)(-3x^5) = -21x^8$, $(-21x^8) \div (7x^3) = -3x^5$.

In dividing monomials, divide the coefficient of the dividend by that of the divisor, using the laws of signs; then divide the literal factors by subtracting the exponents of the letters in the divisor from the exponents of the like letters in the dividend.

Thus, $(6x^4y^6) \div (-2xy^4) = -3x^3y^2.$

Evidently, if a letter has the same exponent in the dividend and divisor, this letter will not appear in the quotient. Why?

Thus, $$(-24\,x^3y^6) \div (-4\,x^3y^4) = 6\,y^2.$$

Here x does not appear in the quotient.

This division may be indicated also in the form

$$\frac{-24\,x^3y^6}{-4\,x^3y^4} = 6\,y^2, \qquad \text{or} \qquad \begin{array}{r} 6\,y^2 \\ -4\,x^3y^4\overline{)-24\,x^3y^6} \end{array}.$$

EXERCISES

Divide:

1. a^4 by a^3.
2. N^6 by N^4.
3. R^3 by R^2.
4. y^7 by y^3.
5. t^8 by t^2.
6. K^{10} by K^6.
7. v^5 by v^3.
8. x^6 by x^5.
9. s^9 by s^3.
10. P^4 by P^3.
11. x^{18} by x^9.
12. A^{11} by A^7.
13. m^{12} by m^{10}.
14. D^4 by D^4.
15. b^7 by b^5.
16. z^5 by z.
17. a^3b^3 by a^2b^2.
18. x^4y^2 by x^2y^2.
19. m^8n^3 by m^4n^2.
20. $a^2b^4c^6$ by $a^2b^2c^2$.
21. $6\,x^6$ by $2\,x^2$.
22. $8\,Rv^3$ by $2\,Rv^2$.
23. $4\,\pi R^2$ by $2\,\pi R$.
24. $18\,Wg^2$ by $9\,W$.
25. $9\,a^8$ by $-3\,a^3$.
26. $15\,t^7$ by $-5\,t^3$.
27. $-24\,P^4n$ by $-6\,P$.
28. $-56\,M^5N^3$ by $7\,N^2$.
29. $36\,a^5b^2c^2$ by $-9\,a^2b^3c^2$.
30. $4\,x^{10}y^8$ by $-x^6y^7$.
31. $-6\,Vt^2$ by $2\,Vt^2$.
32. $18\,D^4v$ by $6\,D^2$.
33. $-50\,x^3d^4$ by $2\,x^2$.
34. $\frac{1}{2}\,A^4$ by $\frac{1}{2}\,A^3$.
35. $6\,z^7t$ by $4\,z^4t$.
36. $-2\,m^{12}n$ by $3\,m^8n$.
37. $\frac{1}{3}\,Qt^3$ by $2\,Qt^2$.
38. $\frac{1}{4}\,x^6y^6z^6$ by $-\frac{1}{2}\,x^4y^4$.
39. $7\,F^2n^4$ by $-\frac{1}{2}\,n^4$.
40. $-xD^5$ by $-\frac{1}{8}\,xD^3$.

54. Division of a Polynomial by a Monomial. — The division of a polynomial by a monomial is similar to the division of denominate numbers containing two or more units of measure.

Just as $$\begin{array}{r} 5\text{ ft.}\quad 3\text{ in.} \\ 3\overline{)15\text{ ft.}\quad 9\text{ in.}} \end{array} \qquad \text{so} \qquad \begin{array}{r} 5f+3i \\ 3\overline{)15f+9i} \end{array}.$$

In general, since *divisor* × *quotient* = *dividend*, the quotient must be such that the product of its terms by the divisor will give the terms of the dividend. Hence the rule:

To divide a polynomial by a monomial, divide each term of the dividend by the divisor, and add the partial quotients.

Example. — Divide $4x^3y - 8x^2y^2 + 6xy^3$ by $-2xy$.

We divide each term of the dividend by $-2xy$ and set the results down in a line with their signs, which indicates their sum. The work may be written in either the form

$$(4x^3y - 8x^2y^2 + 6xy^3) \div (-2xy) = -2x^2 + 4xy - 3y^2$$

or

$$\begin{array}{r} -2x^2 + 4xy - 3y^2. \\ -2xy\overline{)4x^3y - 8x^2y^2 + 6xy^3} \end{array}$$

EXERCISES

Divide:

1. $4m - 6n$ by 2.
2. $12D + 9F$ by 3.
3. $16x + 24$ by 4.
4. $21a - 28b$ by 7.
5. $36 + 42y$ by 6.
6. $-24P + 72Q$ by -12.
7. $81ab - 18ac$ by $9a$.
8. $x + 2x^2$ by x.
9. $A^3 - 3A^2$ by A^2.
10. $30m^3 + 40m^2n$ by $5m^2$.
11. $2pq - 4pt + 6pv$ by $2p$.
12. $-6W^4g + 18W^3g^2 - 12W^2g^3$ by $6W^2g$.
13. $x^6 - x^3$ by x^3.
14. $a^4 - a^3b + a^2b^2$ by a^2.
15. $27t^5 - 18t^3$ by $-9t^3$.
16. $-x - y + z$ by -1.
17. $-A + 2B - C$ by -1.
18. $4w^4v^2 + 8w^3v^3 - 12w^2v^4$ by $4w^2v^2$.
19. $12p^6t - 8p^4t^2 + 24p^2$ by $4p^2$.
20. $-21x^8 - 35x^6 - 14x^4$ by $-7x^4$.
21. $(8ab - 4bc - 20bd) \div 4b$.
22. $(z^2xy - 2zyx^2 + 3zy^3) \div zy$.
23. $(2P^2t^5 - 3P^3t^2 + P^4t^3) \div P^2t^2$.

24. $(-a^3 - ta^2 + t^2a) \div (-a)$.

25. $(M^2N^4 - 9MN^3) \div (-MN^3)$.

26. $(a^2K - 7b^2K + 2cK) \div (-K)$.

27. $(\frac{1}{2}n^2 + \frac{1}{4}n) \div \frac{1}{2}n$.

28. $(-\frac{3}{8}xy - \frac{1}{4}x^2y) \div (-\frac{1}{8}xy)$.

29. $(\frac{1}{3}Wg - \frac{1}{6}Wg^2) \div (\frac{1}{3}Wg)$.

30. $(.5t + t^2 - 1.5t^3) \div (.5t)$.

55. Division of One Polynomial by Another. — Let us divide $x^3 + x^4 + 7x^2 - 6x + 8$ by $x^2 + 2x + 8$.

First, arrange both dividend and divisor according to the descending powers of x (see § 51). The work may be arranged as below.

$$\begin{array}{rrrrrr}
 & x^2 & -x & +1 & & \\ \cline{2-6}
x^2+2x+8) & x^4 & +x^3 & +7x^2 & -6x & +8 \\
 & x^4 & +2x^3 & +8x^2 & & \\ \cline{2-6}
 & & -x^3 & -x^2 & -6x & +8 \\
 & & -x^3 & -2x^2 & -8x & \\ \cline{3-6}
 & & & x^2 & +2x & +8 \\
 & & & x^2 & +2x & +8 \\ \cline{4-6}
\end{array}$$

Explanation. — The product of the term of highest power in x in the quotient and the term of highest power in the divisor must give the term of highest power in the dividend. Hence, the highest term of the quotient is obtained by dividing the highest term of the dividend x^4 by the highest term of the divisor x^2. This gives x^2, the first term of the quotient.

Multiply the whole divisor by the term of the quotient just found. This gives $x^4 + 2x^3 + 8x^2$, which is placed below the dividend.

The dividend is the product of the divisor by the whole quotient. Hence, subtracting the product $x^4 + 2x^3 + 8x^2$ from the dividend, the remainder $-x^3 - x^2 - 6x + 8$ must be the product of the divisor by the part of the quotient to be found.

Therefore, the product of the next highest term of the quotient by the highest term of the divisor must equal the highest term of the remainder. Hence, dividing $-x^3$ of the remainder by x^2 of the divisor gives $-x$, the second term of the quotient.

Multiply the whole divisor by the new term, $-x$; subtract the product from the remainder. This leaves $x^2 + 2x + 8$.

Evidently the third term of the quotient will be obtained from this second remainder just as the second term was obtained from the first remainder.

By continuing this process, all of the terms of the quotient may be found.

If in any problem the divisor is an *exact divisor* of the dividend, the work may be carried on until a remainder zero is found. Otherwise the work may be continued until a remainder is obtained in which the highest term is of lower power than the highest term of the divisor. This is a *true remainder.*

Hence the rule for dividing one polynomial by another:

1. *Arrange both the dividend and divisor according to the descending or ascending powers of some letter.*
2. *Divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient.*
3. *Multiply the whole divisor by this term of the quotient, and subtract the result from the dividend.*
4. *Treat the remainder as a new dividend (having the terms arranged as before), and repeat the process, continuing until either the remainder zero, or a true remainder, is found.*

Example 1. — Divide $a^2 - 11a + 30$ by $a - 5$.

$$\begin{array}{r|l}
 & a - 6 \\
\hline
a - 5 & a^2 - 11a + 30 \\
 & \underline{a^2 - 5a \phantom{{}+30}} \\
 & - 6a + 30 \\
 & \underline{ - 6a + 30}
\end{array}$$

Check. — When $a = 1$, dividend $= 20$, divisor $= -4$, and quotient $= -5$, as it should.

Example 2. — Divide $x^4 + y^4$ by $x + y$.

$$\begin{array}{r|lllll}
 & x^3 & - x^2y & + xy^2 & - y^3 & \\
\hline
x + y & x^4 & & & & + y^4 \\
 & x^4 & + x^3y & & & \\
\cline{2-6}
 & & - x^3y & & & + y^4 \\
 & & - x^3y & - x^2y^2 & & \\
\cline{3-6}
 & & & x^2y^2 & & + y^4 \\
 & & & x^2y^2 & + xy^3 & \\
\cline{4-6}
 & & & & - xy^3 & + y^4 \\
 & & & & - xy^3 & - y^4 \\
\cline{5-6}
 & & & & & 2y^4
\end{array}$$

In this example there is a true remainder $2y^4$ obtained. By using the fractional form to indicate the division of the remainder, as in arithmetic, the entire quotient in this problem may be expressed thus:

$$\frac{x^4+y^4}{x+y} = x^3 - x^2y + xy^2 - y^3 + \frac{2y^4}{x+y}.$$

Check. — When $x = 2$ and $y = 1$, dividend $= 17$, divisor $= 3$, quotient $= 5\frac{2}{3}$, as it should.

Note. — Observe that the terms in the dividend, divisor, and all the products and remainders in the above example are arranged in descending powers of x. Similar terms should be kept in columns in the work. For this purpose it is often necessary to leave vacant spaces between the terms, as in this example.

EXERCISES

Divide and check:

1. $a^2 + 3a + 2$ by $a + 1$.
2. $N^2 + 3N - 40$ by $N + 8$.
3. $x^2 - 5x + 6$ by $x - 2$.
4. $P^2 + 7P + 12$ by $P + 3$.
5. $t^2 - 6t + 8$ by $t - 4$.
6. $y^2 - 9y + 20$ by $y - 4$.
7. $H^2 - H - 30$ by $H - 8$.
8. $b^2 + b - 20$ by $b + 5$.
9. $m^2 + 3m - 40$ by $m - 10$.
10. $2x^2 + 3x + 1$ by $2x + 1$.
11. $3x^2 + 16x - 12$ by $x + 6$.
12. $12K^2 - K - 6$ by $4K - 6$.
13. $6V^2 + 29V + 35$ by $3V + 7$.
14. $4A^2 - 9$ by $2A + 3$.
15. $25R^2 - 4$ by $5R - 2$.
16. $4p^2 - 1$ by $2p + 1$.
17. $1 - 9a^2$ by $1 - 3a$.
18. $16 - 25D^2$ by $4 + 5D$.
19. $a^2 - b^2$ by $a - b$.
20. $x^2 - 4y^2$ by $x + 2y$.
21. $W^2 - 25g^2$ by $W - 5g$.
22. $m^2 + 5mn + 4n^2$ by $m + 4n$.
23. $P^2 - 9PQ + 20Q^2$ by $P - 5Q$.
24. $x^2 + 21 + 10x$ by $3 + x$.
25. $1 + 2a + a^2$ by $a + 1$.
26. $15y + 36 + y^2$ by $y + 4$.
27. $12E^3 - 30E^2 + 2E - 5$ by $2E - 5$.
28. $2r^3 - 3r^2 + 3 + 4r$ by $3 + 2r$.
29. $B^6 - 1$ by $B - 1$.
30. $y^3 - z^3$ by $y + z$.
31. $W^4 - V^4$ by $W^2 + V^2$.
32. $H^6 - t^6$ by $H^3 + t^3$.
33. $144R^2 - 1$ by $12R + 1$.

34. $b^3 + 3b^2 + 3b + 1$ by $b^2 + 2b + 1$.

35. $x^8 + x^4 + 1$ by $x^4 - x^2 + 1$.

36. $n^6 - a^6$ by $n^2 - a^2$.

37. $2y^4 - 9y^3 - 14y + 17y^2$ by $y^2 - 2y$.

38. $8d^3 + 27t^3$ by $2d - 3t$.

39. $4q^4 - 9q^2 + 20q - 25$ by $2q^2 + 3q - 5$.

40. $A^3 - 2A^2 + 5$ by $A - 5$.

41. $g^4 - 3g^2k + 6k^2$ by $g^2 - 3k$.

42. $a^4 + n^4 + a^2n^2$ by $n^2 + a^2 - an$.

43. $16h^4 - 96h^3 + 81 - 216h + 216h^2$ by $9 + 4h^2 - 12h$.

44. A rectangular field whose area is $x^3 + 3x^2y + 3xy^2 + y^3$ has a length of $x^2 + 2xy + y^2$. What is its width?

45. Show that the problem $682 \div 31$ may be written in the form $(6t^2 + 8t + 2) \div (3t + 1)$. Find the quotient in this form, and compare the answer with that obtained by the arithmetical process.

46. In the problem $882 \div 42$ express the given numbers as polynomials and find the quotient. Compare the answer found with that obtained by the arithmetical process.

47. Divide $1 + x$ by $1 - x$. How many terms are there in the quotient?

48. Divide $1 + 2n$ by $1 + n$, carrying the quotient to four terms.

49. Divide $1 + y$ by $1 - y$, carrying the quotient to four terms.

SUPPLEMENTARY EXERCISES

Find the product of:

1. a^3, $-2a^2$, and $-3a$.
2. x^n and x^{2n}.
3. y^m, y^{3m}, and y^{4m}.
4. $3t^a$, $-4t^{a+1}$, and $-2t^{a-1}$.
5. $-5R^{a+b}$, $-2R^{a-b}$, and R^{3a}.
6. a^nb^{n+1} and $a^{n+1}b^n$.
7. $-8W^2v$ and $3W^dv^d$.
8. $5z^{a-5}$, $-2z^{a+7}$, and z^{a-3}.
9. $a^n \cdot a^n \cdot a^n \cdot a^n \cdot a^n$.
10. $x^m \cdot x^m \cdot x^m \cdot x^m \cdots$ to n factors.

Divide:

11. w^{17} by w^6.

12. M^a by M^b.

13. y^{5m} by y^{2m}.

14. B^{a+5} by B^a.

15. $12x^{4n}$ by $-6x^n$.

16. $-63A^{3n}$ by $-7A^5$.

17. $48x^{a+4}b^{a-4}$ by $-6x^4b^{a-4}$.

18. $-80v^mt^n$ by $16v^{m-5}t^{n-1}$.

Multiply:

19. $a^n+b^n+c^n$ by a^n.

20. $x^{n-3}-y^n+z^{n+3}$ by x^3y^3.

21. x^a-y^b by x^a-y^b.

22. P^n+Q^n by $P+Q^n$.

23. a^x+b^y by a^x-b^y.

24. x^n-y^n, x^n+y^n, and $x^{2n}+y^{2n}$.

25. V^t+1, V^t-1, $V^{2t}+1$, $V^{4t}+1$.

Divide:

26. $a^{2v}-a^{3v}$ by a^v.

27. $10a^{n+2}-4a^{n+3}$ by $2a^2$.

28. $M^c+M^{c+2}+M^{c+4}$ by M^c.

29. $y^{n+1}x^{2n+1}+y^{2n+1}x^{n+1}$ by $x^{n+1}y^{n+1}$.

30. x^6-y^6 by x^2-y^2.

31. $W^{4n}-V^{4n}$ by $W^{2n}-V^{2n}$.

32. $a^{3m}+b^{3m}$ by a^m+b^m.

33. $S^{4a}-T^{4a}$ by S^a-T^a.

34. $x^3+3x^2y+3xy^2+y^3+z^3$ by $x+y+z$.

Suggestion. — Arrange the dividend, divisor, all partial products, and remainders according to the descending powers of x, without reference to the other letters involved. If two or more terms in any case contain the same power of x, it does not matter which precedes.

35. $a^3+b^3-c^3+3abc$ by $a+b-c$.

36. $m^3+n^3+p^3-3mnp$ by $m+n+p$.

37. $u^2+v^2+w^2+2uv+2uw+2vw$ by $u+v+w$.

38. $4a^2+4ab+b^2-12ac-6bc+9c^2$ by $2a+b-3c$.

CHAPTER VI

LINEAR EQUATIONS: PROBLEMS

56. In Chapters II and III we found how to solve problems by means of equations containing one unknown number. With a knowledge of the principles discovered in Chapters IV and V at command, we are now able to consider additional points connected with the solutions of problems, including the solutions of problems by means of equations containing more than one unknown number.

57. Degree of an Equation. — The **degree of a term** is the sum of the exponents of its literal factors. If there is only one literal factor, its exponent is the degree of the term.

For example:

x is of the *first degree*,
$5x^2$ is of the *second degree*,
$2a^2b$ is of the *third degree*,
$3w^2v^2$ is of the *fourth degree*.

The degree is expressed sometimes with respect to some one letter.

Thus, $4An^2t^3$ is of the first degree with respect to A, of the second degree with respect to n, and of the third degree with respect to t.

The **degree of an equation** is the degree of its term of highest degree with respect to the unknown number or numbers.

Thus,

$n + 3 = 4n$ is of the *first degree*,
$x^2 - 3x = 8y$ is of the *second degree*,
$R^3 + R^2 - 6 = 0$ is of the *third degree*,
$y^2z^2 - yz = 2y + 3$ is of the *fourth degree*.

An equation of the first degree is called a **linear** equation.

One of the second degree is called a **quadratic** equation.

One of the third degree is called a **cubic** equation.

58. Linear Equations with One Unknown Number. — The process of solving a linear equation with one unknown number that does not contain a sign of grouping was given in § 15 and § 36. In solving linear equations that contain signs of grouping, the latter should first be removed by use of the principles in Chapters IV and V.

Example 1. — Solve $P-(6-2P)=9(P-1)$.

Here the first parentheses are removed by changing the signs of the terms inclosed, and the second by performing the indicated multiplication, giving:

$$\begin{aligned} P-6+2P&=9P-9,\\ P+2P-9P&=6-9,\\ -6P&=-3,\\ P&=\tfrac{1}{2}. \end{aligned}$$

Check. — When $P=\frac{1}{2}$ the equation becomes

$$\tfrac{1}{2}-(6-1)=9(\tfrac{1}{2}-1),$$

or

$$-4\tfrac{1}{2}=-4\tfrac{1}{2}.$$

Example 2. — Solve $(n-2)(n-3)=(n-4)(n-5)$.

The parentheses are removed by multiplication, giving:

$$\begin{aligned} n^2-5n+6&=n^2-9n+20,\\ n^2-n^2-5n+9n&=20-6,\\ 4n&=14,\\ n&=3\tfrac{1}{2}. \end{aligned}$$

Check. — When $n=3\frac{1}{2}$ the equation becomes

$$(3\tfrac{1}{2}-2)(3\tfrac{1}{2}-3)=(3\tfrac{1}{2}-4)(3\tfrac{1}{2}-5),$$

or

$$\tfrac{3}{4}=\tfrac{3}{4}.$$

EXERCISES

Solve and check:

1. $3(a+1)=12+4(a-1)$.
2. $3(x-2)=2(x-3)$.
3. $5(2-4y)=4(1-3y)$.
4. $5(4-3R)=7(3-4R)$.
5. $2t-(5t+5)=7$.
6. $3(v+1)+5(v-1)=0$.
7. $3(d-14)=7(d-18)$.
8. $2(W-1)+3(W-2)+4(W-3)=0$
9. $2P-5(20-P)-6=0$.
10. $5(2D+1)-7=3(2D-7)+51$.

11. $2(m-2)+3(m-3)=20-4(m-4).$

12. $5k+6(k+1)=7(k+2)+8(k+3).$

13. $2c-3(c-4)+20=2c+17.$

14. $85-3(2V+7)=6V+4(4V+2).$

15. $4(5+4E)-5(6+4E)+100=2E+36.$

16. $12T-(4T-7)=3T-(9T-28).$

17. $(5-n)(1+n)=(2-n)(4+n).$

18. $(K-1)(K-2)=(K+3)(K-4).$

19. $(s-1)(s-4)=2s+(s-2)(s-3).$

20. $(5Y+7)(3Y-8)=(5Y+4)(3Y-5).$

21. $(4A-7)(4A-7)=(2A-5)(8A+3).$

22. $(5-3u)(4u+3)+1=(3u+7)(1-4u).$

23. $(Q+4)(Q+8)-Q(Q+4)=128.$

24. $(4-3w)(5+4w)=(8+2w)(1-6w)-82.$

25. $(a-4)(a+4)=a^2-8a.$

26. $2(z+2)(z-4)=z(2z+1)-21.$

59. Problems solved by Linear Equations. — The following example illustrates a type of problems solved by means of linear equations involving signs of grouping.

Example. — A man made two investments in railroad stocks, amounting together to $15,000. On the first he gained 14%, and on the second he lost 5%. His net profits from the two investments amounted to $1055. How many dollars in each investment?

Let a = amount of first investment, in dollars.
Then, $15000 - a$ = amount of second investment.
$.14a$ = gain on first investment.
$.05(15000-a)$ = loss on second investment.

$$\text{Hence, } .14a - .05(15000-a) = 1055.$$
$$.14a - 750 + .05a = 1055.$$
$$.19a = 1805.$$
$$a = 9500.$$
$$15{,}000 - a = 5500.$$

Therefore the first investment was $9500, and the second, $5500.

For the steps in the process of expressing a problem by means of an equation involving one unknown letter, as illustrated in the above example, review § 16.

EXERCISES

1. The sum of two numbers is 25, and twice the larger exceeds four times the smaller by 2. Find the numbers.

2. Separate 34 into two parts such that twice the greater shall be less by 12 than 5 times the smaller.

3. The sum of two numbers is 64. Three times the less is 12 more than twice the greater. Find the numbers.

4. Separate 86 into two parts such that three times the greater shall exceed 5 times the smaller by 2.

5. There are two consecutive odd numbers whose product exceeds the square of the less by 34. Find the numbers.

6. When the square of a whole number is increased by 29 the result is found equal to the product of the two next larger consecutive whole numbers. Find the three consecutive numbers.

7. I bought 4 lb. of coffee and 5 lb. of tea for $5.40. The tea cost 45 cents a pound more than the coffee. Find the cost of each per pound.

8. A newsboy sold 51 papers for 70 cents, of which some were one-cent papers and the others two-cent papers. How many papers of each kind were there?

9. A newsboy sold two-cent evening papers and *Saturday Evening Posts* (price 5 cents), receiving in all 92 cents. The total number of papers and *Posts* together was 34. How many were there of each?

10. Some boys had a refreshment booth at which they sold lemonade and cider. The price of lemonade was 2 cents a glass and of cider 5 cents a glass. In an afternoon they sold 43 glasses, and in counting their money found that they had $1.31. How many glasses of each did they sell?

11. Two tanks contained equal amounts of oil. But after 75 gallons had been taken from one, and 50 gallons added to the other, one contained twice as much as the other. How many gallons did each contain originally?

12. What amount must be subtracted from each of the numbers 12, 14, 18, 10, so that the product of the first two remainders shall equal the product of the last two?

13. Two electric lights have together a strength of 128 candle-power. One has twice as much and 16 candle-power more than the other. How many candle-power has each?

14. Two grades of coffee costing the dealer 28 cents and 36 cents per pound are to be mixed so that the mixture shall cost 30 cents per pound. What parts must he take to make 50 pounds of the mixture?

15. How can a merchant mix 10 pounds of tea, one kind costing 50 cents and the other 65 cents per pound, so that the mixture shall cost 60 cents per pound?

16. Two grades of spice worth 25 cents and 45 cents a pound are to be mixed so that the mixture can be sold for 50 cents a pound and at a profit of 25 %. What parts must be taken to make 8 pounds of the mixture?

17. A grocer received a shipment of 80 dozen eggs. Part of these were sold at 28 cents per dozen and the rest at 30 cents per dozen. The total receipts from the sale of the eggs amounted to $23.12. How many dozens were sold at each price?

18. Listerine contains 25 % alcohol. When used to spray the throat, it should be diluted by adding water to it. How much water must be added to 100 parts of listerine so that the mixture contains only 15 % alcohol?

19. If a medicine contains 40 % alcohol, how much of other ingredients must be added to 10 quarts of it so that the mixture shall contain only 25 % alcohol?

20. How many quarts of water must be mixed with 40 quarts of alcohol, 80 % pure, to make a mixture 75 % pure?

21. How many quarts of water must be added to 60 quarts of alcohol, 85 % pure, to make a mixture 75 % pure?

22. How many gallons of cream containing 28 % butter fat and milk containing 4 % butter fat must be mixed to make 10 gallons of cream containing 25 % butter fat?

23. How many gallons of cream containing 30 % butter fat and milk containing 4.5 % butter fat must be mixed to make 20 gallons of cream containing 20 % butter fat?

24. In an alloy of gold and silver weighing 60 ounces, there are 5 ounces of gold. How much silver must be added in order that 10 ounces of the new alloy shall contain only $\frac{10}{17}$ ounce of gold?

25. In an alloy of copper and tin weighing 36 ounces, there are 12 ounces of copper. How many ounces of copper must be added to this in order that the new alloy may have 4 ounces of copper to every 10 ounces of the alloy?

26. A man has two investments together amounting to $1800. On the first investment he gets 5 % annually, and on the other he gets 6 % annually. His annual income from the two investments is $95. Find the amount of each investment.

27. $5000 is invested in two places, part at 4 % and the rest at 5 %. The annual income from the two investments is $222. How much is each investment?

28. One sum of money exceeds another by $1400. The first at 8 % and the second at 6 % give an annual income of $280. Find the value of each.

29. There are 51 coins in a money drawer, consisting of nickels and dimes. The total value is $3.35. How many coins of each kind are there?

30. The value of 29 coins, consisting of quarters and dimes, is $4.55. Find the number of each.

31. A is 42 years of age, and B is 12. In how many years will A be only twice as old as B?

32. Eight years ago a man was just 16 times as old as his son, and now he is only 4 times as old. What are their present ages?

33. Find where to cut a board which is 50 inches long into two parts whose difference is 26 inches.

34. A bar of iron 60 inches long is to be cut into two parts such that twice one part is equal to 5 times the other. Find where to cut it.

35. A belt runs over a pulley 48 inches in diameter, making 180 revolutions a minute. It is desired to reduce the size of the pulley so that by making 216 revolutions a minute, the belt will move with the same speed. By how much must the diameter of the pulley be reduced?

36. A rectangular field is 6 rods longer than it is wide; and if the length and breadth were each 4 rods more, the area would be 120 square rods more than it is. What are the dimensions of the field?

37. A square court has the same area as a rectangular court whose length is 18 feet greater and width 9 feet less than the side of the square court. Find the side of the square court.

38. A tennis court is 6 feet more than 2 times as long as it is wide, and its area exceeds by 216 square feet twice the area of the square whose side is its width. Find the dimensions of the court.

60. Business Problems. — Certain kinds of business problems are easily solved by use of linear equations.

EXERCISES

1. A university has an endowment which invested at 4 % yields an annual income of $140,000. What is the endowment?

SUGGESTION. — If d dollars is the endowment, $.04\,d = 140{,}000$.

2. What sum of money invested at 6 % will yield $510 simple interest per year?

3. A certain investment at $4\frac{1}{2}$ % simple interest yields $3240 in 6 years. What is the investment?

4. A certain sum of money was invested at 5 % simple interest. In 3 years the principal and interest together amounted to $ 6900. What was the amount of the investment?

5. Five years ago I invested a certain sum of money at 6 % simple interest. It now amounts to $ 1560. How much did I invest?

6. A number of years ago I invested $ 1500 at 5 % simple interest. The amount at present is $ 2025. How many years ago was the investment made?

7. A commission merchant charged $ 10.80 for selling a car load of fruit, and remitted $ 349.20. What was his rate of commission?

8. Merchants sometimes mark goods to sell at an advance, and then allow a discount on the marked price. At what advance must a merchant mark goods costing $ 12 in order that he may sell them at a reduction of 20 % from the marked price, and yet make a profit of 25 % on the goods?

SUGGESTION. — If $a =$ the advance, in dollars, the marked price will be $12 + a$, and the selling price 80 % of this, or $.80(12 + a)$. Hence, $.80(12+a) = 12 + .25 \times 12$.

9. At what advance must a merchant mark an article costing him $ 3.50 in order that he may sell it at a reduction of 10 % from the marked price, and yet make a profit of 30 % on it?

10. A clothing merchant puts on sale a lot of boys' suits costing him $ 8 each, and advertises them to sell at a reduction of 25 % from the marked price. At what advance must he first mark them to sell in order that he may make this reduction and still make a profit of 20 % on the cost of the suits?

11. A milliner advertises a lot of hats that cost her $ 2.85 each for special sale at "$\frac{1}{3}$ off." At what advance must she first mark them in order that by allowing this discount she may make a profit of 75 cents on each hat?

12. The proprietor of a china store marks for sale a lot of dishes that cost him $ 16.50. At what advance must he mark them so that he may then mark them down 15 % and make a profit of 20 %?

13. A ladies' tailoring establishment marked a suit at an advance of 50 %, then reduced the selling price by 20 % of the marked price, and got $16.80 for it. What was the cost of the suit?

14. A druggist had marked perfume to sell at a profit of 40 %. He then advertised it for special sale at 42 cents an ounce, which was a reduction of 14$\frac{2}{7}$ % from the marked price. What did it cost the druggist per ounce?

15. Men's suits that had been marked to sell at a gain of 25 % were damaged and disposed of at a fire sale at half price, which was a loss of $ 7.50 on each suit. What was the cost of the suits?

61. Problems involving Motion. — When an object moves, the distance that it goes in a unit of time, as a second or an hour, is called its **rate** of motion, **speed**, or **velocity**.

For example: If a train runs 60 miles in 2 hours, its velocity is *30 miles an hour.*

If the velocity does not change throughout the motion, the motion is said to be *uniform*. In case a body moves with uniform motion, the distance that it goes in any length of time is obtained by multiplying the velocity by the number of units of time; that is,

distance = time × velocity.

EXERCISES

1. An express train runs with a velocity of 45 miles an hour. How far does it go in 4 hours?

2. The "Rocky Mountain Limited," on the Rock Island Road, runs from Omaha to Denver, a distance of 580 miles, in 14 hours and 48 minutes. What is the average speed of the train, making no allowances for stops?

3. Sound travels through the air 1080 feet a second. If a flash of lightning is a mile away, how long after the flash should one hear the thunder?

4. A train leaves a station and runs at a speed of 40 miles an hour. Two hours later a second train leaves the same station and runs over the same track at the rate of 55 miles an hour. In how many hours will the second train pass the first?

SUGGESTION.— Let t = number of hours after second train starts until it passes the first. Then, since the first train has a start of two hours, $55t = 40(t + 2)$.

5. A messenger, going 6 miles per hour, has been gone for two hours when it is found that the message is wrong. A second messenger, riding at an average of 10 miles per hour, is sent to overtake him. In how many hours will the second messenger overtake the first?

6. Two motor cyclists start at the same time from points 280 miles apart, and travel towards each other. One rides 24 miles per hour, but is delayed 3 hours on the road because of a break in his machine. The other rides 20 miles per hour uninterruptedly. In how many hours will they meet?

7. Two pedestrians started at the same time from points $44\frac{5}{8}$ miles apart, one walking at the rate of $2\frac{1}{2}$ miles an hour, and the other at the rate of $2\frac{3}{4}$ miles an hour. When and where did they meet?

8. Two automobile parties started from the same place, one going north at 20 miles an hour and the other south at 18 miles an hour. In what time will they be 120 miles apart?

9. A motor cyclist rode 75 miles in 4 hours. Part of the distance was on a country road at a speed of 20 miles an hour, and the rest within the city limits at 10 miles an hour. Find how many hours of his ride were in the country.

10. A train runs from Joliet, Ill., to Chicago, 39.6 miles, in 1.1 hours. The run from Joliet to Englewood is made at 40 miles an hour, and the run from Englewood to Chicago at 24 miles an hour. Find the time required to make the run from Englewood to Chicago. Find the distance from Englewood to Chicago.

11. A train running from Chicago to Denver at the average speed of 40 miles an hour takes three hours longer to make the run than one running at 45 miles an hour. Find the distance from Chicago to Denver.

Suggestion. — First find the time required at a speed of 40 miles an hour.

12. An ocean liner going 20 knots an hour leaves New York when a freighter going 6 knots an hour is already 90 knots out. How long will it take the liner to overtake the freighter?

13. A travels 8 hours at a rate of 3 miles per hour less than the rate of B. B travels an equal distance in 6 hours. What is the rate of each?

14. Two trains approach each other, leaving stations 149 miles apart at the same time. One goes 10 miles per hour faster than the other, and they meet in 2 hours. What is the rate of each?

15. The longest steel bridge in the world is said to be one of the Spokane, Portland, & Seattle Railroad, over the Columbia River. Including approaches, it is 2 miles long. It would take a freight train 1144 feet long, running 15 miles per hour (1320 feet per minute), just 3 minutes to cross the ten main spans. Find the length of the ten main spans.

16. A man shooting at a target heard the bullet strike the target $3\frac{3}{5}$ seconds after he fired. The bullet was known to travel 1375 feet a second; and sound travels approximately 1100 feet a second. How long after he fired did the bullet hit the target? How far away was the target?

17. A bullet going 1650 feet per second is heard to strike a target 2 seconds after it is fired. How long after it is fired does it hit the target? How far away is the target?

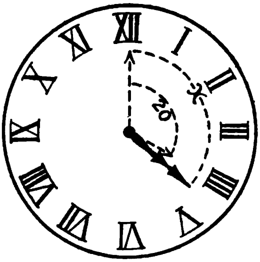
18. Find the time between 4 and 5 o'clock when the hands of a clock are together.

Suggestion. — Let $x =$ the number of minute spaces which the minute hand has traveled from 4 o'clock on until it overtook the hour hand. Then $\frac{1}{12}x$ will be the number of minute spaces which the hour hand has traveled meanwhile. Why? The difference is 20. Why?

19. Find the time between 7 and 8 o'clock when the hands of a clock are together.

20. At what time between 6 and 7 o'clock is the minute hand 15 minute spaces behind the hour hand?

SUGGESTION. — Starting at 6 o'clock, how many minute spaces has the minute hand had to gain on the hour hand?

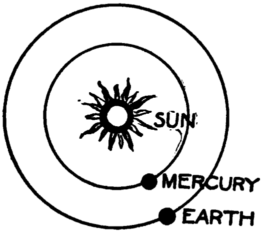


21. Find the time between 4 and 5 o'clock when the hands of a clock are directly opposite each other.

22. At what time between 8 and 9 o'clock are the hands of a clock directly opposite each other?

23. How long is it from the time that the hour and minute hands of a clock are together until they are together again?

SUGGESTION. — Let $t =$ the number of minutes required. Since the minute hand makes one revolution in 60 minutes, it makes $\frac{1}{60}$ of a revolution in one minute, and $\frac{1}{60}t$ revolutions in t minutes. Similarly, the hour hand makes $\frac{1}{720}$ of a revolution in one minute, and $\frac{1}{720}t$ revolutions in t minutes. Hence, $\frac{1}{60}t - \frac{1}{720}t = 1$.



24. The earth makes a circuit around the sun in 12 months, and Mercury makes a circuit around the sun in 3 months. If the earth and Mercury are "in conjunction," as in the figure, how long will it be until they are in conjunction again?

25. Venus makes a circuit around the sun in 225 days, and Mars in 687 days. How many days elapse between two consecutive conjunctions of these two planets?

62. Problems on Specific Gravities of Substances. — A cubic foot of steel weighs 7.8 times as much as a cubic foot of water. This number, 7.8, is called the **specific gravity** of steel. In general, *The ratio of the weight of a given volume of any solid or liquid substance to the weight of an equal volume of water at the freezing point of temperature is the specific gravity of the substance.*

What does it mean to say that the specific gravity of alcohol is .79?

A cubic centimeter of distilled water at the freezing point weighs just 1 gram. Since the specific gravity of pure gold is 19.36, 1 cubic centimeter of gold weighs 19.36 grams; 2 cubic centimeters weigh 2×19.36 grams; 3 cubic centimeters weigh 3×19.36 grams; etc. Evidently,

The weight of an object in grams equals the product of its volume in cubic centimeters and its specific gravity.

Thus, if the specific gravity of copper is 8.9, 1 cubic centimeter of copper weighs 8.9 grams. Hence, 10 cubic centimeters of copper weigh 10×8.9 grams, or 89 grams.

Certain problems in the specific gravities of mixtures of substances are solved by use of linear equations.

EXERCISES

1. The specific gravity of cast iron is 7.2. Find the weight of 10 cubic centimeters of cast iron. Of 200 cubic centimeters.

2. The specific gravity of glass is 2.89. Find the weight of 100 cubic centimeters of glass. Of 25 cubic centimeters.

3. Brass is made of copper and zinc, and its specific gravity is 8.4. How many cubic centimeters of copper, of which the specific gravity is 8.9, must be used with 200 cubic centimeters of zinc, of which the specific gravity is 6.9, to make brass?

SUGGESTION. — The volume of the brass is the sum of the volumes of the copper and zinc, and the weight of the brass is the sum of the weights of the copper and zinc. Let v = volume of copper in cubic centimeters. Then, weight of copper $= 8.9v$, weight of zinc $= 6.9 \times 200$, and weight of brass $= 8.4(v + 200)$. Hence, $8.9v + 6.9 \times 200 = 8.4(v + 200)$.

4. How much zinc must be combined with 250 cubic centimeters of copper to form brass?

5. How many cubic centimeters of water (specific gravity 1) must be mixed with 500 cubic centimeters of alcohol (specific gravity .79) so that the specific gravity of the mixture shall be .9?

6. A piece of ice containing 1,000,000 cubic centimeters, specific gravity .92, floats in water. How many cubic centimeters in an oak beam, specific gravity 1.17, that may be placed upon the ice without making it sink?

SUGGESTION. — The specific gravity of the combination of ice and oak must equal 1, the specific gravity of water.

7. How much steel, specific gravity 7.8, must be attached to a piece of white pine, specific gravity .42, containing 10,000 cubic centimeters, in order that the specific gravity of the combined materials may be 2?

8. In Problem 7, how much steel must be used so that the combined materials will just float in water?

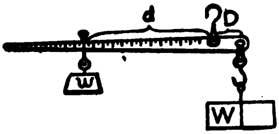
9. A piece of glass containing 850 cubic centimeters, specific gravity 2.89, is made to float by attaching cork to it. How much cork, specific gravity .24, must be used?

10. My watch case is made of 14-karat gold, specific gravity 14.88. How much pure gold, specific gravity 19.36, must be combined with 20 cubic centimeters of nickel, specific gravity 8.57, to make 14-karat gold?

11. When 100 cubic centimeters of mercury and 10 cubic centimeters of gold (specific gravity 19.36) are combined, it is found that the specific gravity of the combined materials is 14.1. Find the specific gravity of mercury.

12. It is found that when 90 cubic centimeters of copper (specific gravity 8.9) and 150 cubic centimeters of tin are combined, the specific gravity of the combination is 7.9. What is the specific gravity of tin?

63. Problems involving the Lever. — In every form of *lever*, such as the steelyard, crowbar, nutcracker, teeter board, etc., it is found that the two weights or forces exerted balance when they are placed at such distances from the point of support or fulcrum that *the products of the weights or forces by their distances from the fulcrum are equal.*



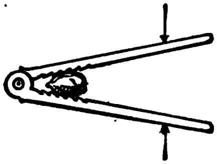
Thus, in case of the steelyard, $wd = WD$.

Some problems on the use of the lever may be solved by linear equations.

EXERCISES

1. On a steelyard the distance D is 2 inches and the weight w is $\frac{1}{2}$ pound. If d is 8 inches, what is the weight W?

2. A nutcracker is held 4 inches from the hinge or fulcrum, and a nut is placed 1 inch from the hinge. A squeezing force of 3 pounds is required to crack the nut. What is its resistance?



3. Two children play at teeter. One weighs 80 pounds, and sits 5 feet from the point of support of the teeter board. If the board balances when the other sits 6 feet from the point of support, what is the weight of the second child?

4. Two boys play at teeter. One weighs 100 pounds, and sits 6 feet from the point of support. The other weighs 120 pounds. How far from the point of support must he sit in order to make the board balance?

5. A board 12 feet long is to be used as a teeter board. If the people weigh 95 pounds and 110 pounds, respectively, and sit at the ends of the board, find the point at which it must be supported in order to balance.

6. A workman lifts a stone weighing 420 pounds by means of a crowbar. If the fulcrum is placed 6 inches from the point at

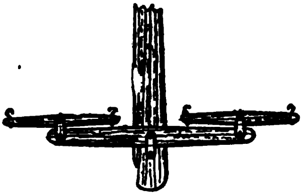
which the crowbar is applied to the stone, how far from the fulcrum must the workman grasp the crowbar to lift the stone with a force of 100 pounds?

7. In Problem 6, if the man grasps the crowbar at a point 48 inches from the point where the crowbar is applied to the stone, where must the fulcrum be placed in order that he may lift the stone with a pressure of 70 pounds?

8. Weights of 240 pounds and 300 pounds, respectively, are applied at the ends of a beam 40 inches long. At what point must the beam be supported in order to balance?

9. A and B carry an object suspended from a horizontal rod 60 inches long held between them. If one is to lift only three fourths as much as the other, at what point of the rod must the object be suspended?

10. A man has a team of which one horse weighs 1200 pounds and the other 1500 pounds. If their draft power is proportional to their weight, how must he divide the 50-inch doubletree in order to justly distribute the load?



11. A farmer has a team of which one horse weighs 1400 pounds and the other 1600 pounds. If their draft power is proportional to their weight, where must he place the clevis on the 4-foot doubletree so as to justly distribute the load?

64. Problems on the Decimal Number System. — In any number such as 4258, the 8 is called the *ones' digit*, the 5 the *tens' digit*, and 2 the *hundreds' digit*, etc.

Any number of two or more digits in our decimal system of notation is in nature a polynomial. Thus, $483 = 400 + 80 + 3$, or $100 \times 4 + 10 \times 8 + 3$. Similarly, any number whose hundreds', tens', and ones' digits are respectively h, t, and u, may be written in the polynomial form $100h + 10t + u$. In writing a number in the polynomial form, each digit must be multiplied by 10, 100, 1000, etc., according to the position that it occupies in the number.

EXERCISES

1. Write in the polynomial form the following numbers: 25; 347; 4196.

2. Write the number whose ones' digit is a, tens' digit b, hundreds' digit c, and thousands' digit d.

3. In a number of two digits, the sum of the digits is 9. If the digits are interchanged in position, the new number obtained exceeds the given number by 27. Find the number.

SUGGESTION. — If $x =$ the tens' digit, $9 - x =$ the ones' digit.

Hence, $10x + 9 - x =$ the number.

Hence, $10(9 - x) + x =$ the number with the digits interchanged.

Hence, $10(9 - x) + x = 10x + 9 - x + 27$.

4. A number is composed of two digits whose sum is 12. This number exceeds by 36 the number obtained by interchanging the digits. Find the number.

5. In a certain number of two digits, the tens' digit exceeds the ones' digit by 2. The sum of the given number and that obtained by interchanging the digits is 154. What is the number?

6. A number consists of two digits of which the ones' digit is 3 more than the tens' digit, and the number is 3 more than 4 times the sum of its digits. What is the number?

7. What is the number of which the tens' digit is 2 more than the ones' digit, and which is 7 times the sum of the digits?

8. In a number of two digits the ones' digit is 3 less than the tens' digit, and the number is 20 less than twice that obtained by interchanging the digits. What is the number?

9. In a number of three digits, the tens' digit is 1 more than the ones' digit, and the hundreds' digit is 1 more than the tens' digit. The number is 21 more than 50 times the sum of its digits. Find the number.

65. Linear Equations with Two Unknown Numbers. — Problems in which the values of more than one unknown quantity are to be found are often most easily solved by use of equations containing more than one unknown number.

EXAMPLE. — A merchant sold 10 suits for $162. He received $15 for each of one kind and $18 for each of the other kind. How many were there of each kind?

Let $n =$ the number sold at $15 each, and $m =$ the number sold at $18 each.

Then $\begin{cases} n + m = 10, & (1) \\ 15n + 18m = 162. & (2) \end{cases}$

Multiplying (1) by 15, $15n + 15m = 150.$ (3)

Subtracting (3) from (2), $3m = 12.$ (4)

Hence, $m = 4.$ (5)

Replacing m by 4 in (1), $n + 4 = 10.$ (6)

Hence, $n = 6.$ (7)

Therefore, 6 suits were sold at $15 each and 4 suits at $18 each.

66. Simultaneous Equations. — In the example in § 65, we have seen two equations concerning two unknown numbers whose values satisfy both equations.

Thus, in the example, when $n = 6$ and $m = 4$, equation (1) becomes $6 + 4 = 10$, and equation (2) becomes $90 + 72 = 162$.

Two or more equations containing two or more unknown numbers which satisfy all of the equations are called **simultaneous**.

67. Systems of Simultaneous Equations. — Two or more simultaneous equations are said to constitute a **system**. To **solve a system** of equations is to find the *sets of values* of the unknown numbers which satisfy all of the equations.

Thus, in the system of equations in § 65, the set of values $n = 6$ and $m = 4$ constitute a *solution*.

68. Elimination. — The solution of a system of two linear equations is most easily found by so combining the two equations as to obtain a new equation in which one of the unknown numbers does not appear. This process is called **elimination**. A simple method of elimination is shown in the example in § 65. Compare the steps in the process of solving the following system with those in the example in § 65.

Example. — Solve the system $\begin{cases} A+4B=12, & (1) \\ 2A-B=6. & (2) \end{cases}$

Let us first eliminate B.

Multiplying (2) by 4, $8A-4B=24.$ (3)

Adding (3) to (1), $9A=36.$ (4)

Hence, $A=4.$ (5)

The value of B may now be found in like manner by eliminating A between equations (1) and (2), but is more easily found by replacing A by its value 4 in equation (1) or equation (2).

From (1), when $A=4$, $4+4B=12.$ (6)

Whence, $B=2.$ (7)

See if the solution $A=4$, $B=2$ satisfies both equations of the system.

A study of this example and that in § 65 will reveal the following rule:

To eliminate one of the unknown numbers, multiply the members of each equation, if necessary, by such a number as will make the numerical coefficients of that unknown number the same in both of the resulting equations. Add or subtract (according to signs) the corresponding members of the resulting equations.

After finding the value of one unknown number, substitute its value in either of the given equations and solve the resulting equation for the value of the other unknown number.

Note. — Other methods of elimination will be discussed in Chapter XII.

EXERCISES

Solve:

1. $\begin{cases} a+b=7, \\ a-b=3. \end{cases}$

2. $\begin{cases} 5M+4N=22, \\ 3M+N=9. \end{cases}$

3. $\begin{cases} 6E-d=10, \\ 7E-2d=15. \end{cases}$

4. $\begin{cases} 4x-3y=1, \\ 3x-4y=6. \end{cases}$

5. $\begin{cases} 2n+5k=15, \\ 3n-4k=11. \end{cases}$

6. $\begin{cases} P+4R=12, \\ 2P-R=6. \end{cases}$

7. $\begin{cases} 7z + c = 42, \\ 3z - c = 8. \end{cases}$

8. $\begin{cases} 6T + 7H = -33, \\ 5T - 4H = 2. \end{cases}$

9. $\begin{cases} 5W - 2V - 35 = 0, \\ 4W - V - 25 = 0. \end{cases}$

10. $\begin{cases} 2m - n - 5 = 0, \\ 5m - 2n = 14. \end{cases}$

11. $\begin{cases} 5t - 2a = 1, \\ 8t = 5a - 11. \end{cases}$

12. $\begin{cases} 10G - 11v + 27 = 0, \\ 10v - 11G = 36. \end{cases}$

69. Problems solved by Use of Systems of Equations. — In a problem in which the values of *two* unknown quantities are sought it always will be found that the problem expresses or implies *two* facts or suppositions. Represent each unknown number by a letter. Then each of the facts or suppositions expressed in the problem gives an equation containing the unknown numbers. These two equations constitute a system, which may be solved by the method of § 68.

Many of the problems in the exercises in the earlier part of this chapter might have been solved by use of two equations containing two unknown numbers.

EXAMPLE. — I have $1.50 in dimes and nickels. There are 25 coins in all. How many dimes and how many nickels are there?

Let d = number of dimes, and n = number of nickels.

Then,
$$\begin{cases} d + n = 25, & (1) \\ 10d + 5n = 150. & (2) \end{cases}$$

Note that the fact that there are 25 coins in all gives equation (1), and the fact that their total value is $1.50 gives equation (2).

Solving this system, $d = 5$, and $n = 20$.

Hence, there are 5 dimes and 20 nickels.

EXERCISES

Solve by using two equations with two unknown numbers:

1. A man invested $5600, part in bonds that paid 4%, and the remainder in a business enterprise that yielded 8% on the investment. The total yearly earnings from the two investments amounted to $328. Find the amount of each investment.

SUGGESTION. — Let a = amount of investment in bonds, and b = amount of investment in business enterprise.

Then,
$$\begin{cases} a + b = 5600, \\ .04\,a + .08\,b = 328. \end{cases}$$

2. A man had \$2150 to invest. Part of this sum he loaned at 6 % interest, and the remainder he invested in stock of a building and loan association that paid 7 % dividends. From the two investments his annual income was \$142.50. What was the amount of each investment?

3. A money drawer contains \$3.50 in dimes and nickels. There are 50 coins in all. How many coins of each kind are there?

4. In playing teeter, two boys use a board 12 feet long. One boy weighs 80 pounds and the other 110 pounds. At what point must the board be supported to balance?

5. A teamster has a team of which one horse weighs 1250 pounds and the other 1500 pounds. Assuming that draft power is proportional to weight, how should he divide the 50-inch doubletree in order to properly distribute the load?

6. A merchant mixes 28-cent coffee and 36-cent coffee to sell at 30 cents a pound. What quantities of the two grades of coffee should he take to make 40 pounds of the mixture?

7. The specific gravity of copper is 8.9, that of zinc 6.9, and that of brass 8.4. Find the number of cubic centimeters of copper and the number of cubic centimeters of zinc that must be combined to produce 600 cubic centimeters of brass.

SUGGESTION. — Let c cubic centimeters copper and z cubic centimeters zinc be the amounts required.

Then,
$$\begin{cases} c + z = 600, \\ 8.9\,c + 6.9\,z = 8.4\,(c + z). \end{cases}$$

8. How much gold, specific gravity 19.36, and how much nickel, specific gravity 8.54, must be combined to make 10 cubic centimeters of an alloy of which the specific gravity is 12?

9. In a number of two digits, the sum of the digits is 10, and when the digits are interchanged, the number is increased by 36. Find the number.

10. A belt runs over two pulleys. The circumference of one pulley is 2 feet more than the circumference of the other. One pulley makes 3 revolutions while the other makes 2. Find the circumference of each pulley.

11. When the Panama Canal was finished, the distance from New York to San Francisco by boat was reduced by 7796 miles. The distance by the old route exceeded twice the distance by the Panama route by 2502 miles. Find the distance by the old route and the distance by the Panama route.

12. The distance from New York to Yokohama, Japan, was reduced by 2952 miles when the Panama Canal was completed. If the distance by the old route were twice as great, and the distance by the Panama Canal three times as great, the latter would exceed the former by 4184 miles. Find the distance by each route.

13. The ship *Mauretania*, built in 1907, is 583 feet longer than the ship *Britannica*; built in 1840. The *Mauretania* is 169 feet more than three times as long as the *Britannica*. Find the length of each.

14. In 1847 the *Deutschland*, the first ship of the Hamburg-American Line, arrived in New York. The present *Deutschland* carries 15,283 tons more than the first *Deutschland*. The tonnage of the present *Deutschland* exceeds by 226 tons 22 times the tonnage of the old *Deutschland*. Find the tonnage of each.

15. It is stated that the daily ration for a laboring man should contain 4 ounces each of fat and protein. In pork the per cent of fat is 26 and the per cent of protein is 13; and in beans the per cent of fat is 2 and the per cent of protein is 22. How many ounces each of pork and beans are required to make a ration for one day?

SUPPLEMENTARY EXERCISES

NOTE. — As we have seen, many practical formulæ contain two or more unknown numbers. It often is found necessary to solve such formulæ for one of the unknown numbers *in terms of the others.* Thus, the circumference of a circle is computed by the formula $C = 2\pi R$. If this is solved for R in terms of C, we get $R = \frac{C}{2\pi}$. Such formulæ are solved by the same processes as equations with one unknown number.

1. In the simple interest formula, $I = prt$, solve for t in terms of I, p, and r.

2. The distance an object moves in t seconds at v feet per second is $s = tv$. Solve for t.

3. The area of a triangle is expressed by $A = \frac{1}{2}BH$. Solve for H.

4. The area of a trapezoid is expressed by $A = \frac{1}{2}H(B + B')$. Solve for B'.

5. The relation between the readings on the Fahrenheit and Centigrade thermometers for any temperature is expressed by $C = \frac{5}{9}(F - 32)$. Solve for F.

6. The formula for finding the horse power of a steam engine is $\text{H. P.} = \frac{plan}{33000}$, where p = the mean effective pressure in pounds per square inch, l = the length of stroke in feet, a = area of piston in square inches, n = twice the number of revolutions. Solve for l, the length of stroke.

7. In a polygon with n sides the sum of all the angles, in degrees, is given by $s = 180(n - 2)$. Solve for n.

8. Solve $2ax - b = cy$ for x.

9. Solve $A(n - 1) + A = n$ for A

10. Solve $(V - S)(V + 2S) = V^2 + S^2$ for V.

11. Solve $2(2t - 1) + 3 = a(t + 2)$ for t.

12. Solve $R(R + k) - k(1 - R) = R^2 + 1$ for R

In each of the following systems solve for x and y:

13. $\begin{cases} 8x + y = 60a, \\ 7x - 10y = 9a. \end{cases}$

15. $\begin{cases} 3x + 7by = 16b, \\ 2x + 5by = 13b. \end{cases}$

14. $\begin{cases} x + y = 30n, \\ 3x - 2y = 25n. \end{cases}$

16. $\begin{cases} 5mx - 2ny = 63mn, \\ 2mx + ny = 18mn. \end{cases}$

17. A square has the same area as a rectangle whose length is 3 inches greater and width 2 inches less than the side of the square. Find the dimensions of the square and of the rectangle.

18. In a certain family each daughter has as many sisters as brothers; but each son has twice as many sisters as brothers. How many children are in the family?

19. The leader in a "guessing game" tells each of the others to add 6 years to his age, multiply the result by 4, subtract 24 from the product, and to the remainder add his age. When the results are announced, the leader tells at once the age of each individual. If an individual gives 80 as the result, what is his age?

20. Find the longitude of a place in the Central Time belt at which it is observed that the local or sun time is 15 minutes faster than standard time.

21. Find the longitude of a place in the New York Time belt whose local time is 10 minutes slower than standard time.

22. A crop of 60 bushels per acre of corn takes from one acre of soil 58 pounds less of phosphoric acid than of nitrogen, and 23 pounds less of potash than of nitrogen. Five times the amount of phosphoric acid exceeds twice the amount of potash by 8 pounds. How many pounds of each of these fertilizers are taken from an acre of soil?

23. Fifteen pounds of tin weigh 13 pounds in water, and 15 pounds of zinc weigh 13.5 pounds in water. How much tin and how much zinc in an alloy which weighs 56 pounds in air and 49 pounds in water?

NOTE. — The method given in this problem of finding the proportional parts of metals in an alloy by first weighing the pure metals and the alloy each in air, then in water, is said to have been first discovered and used by Archimedes, about 220 B.C., in determining for King Hieron of Syracuse whether a crown, claimed by the maker to be pure gold, was not alloyed with silver.

24. Two seconds after a marksman fires he hears the bullet hit the target, which is 440 yards distant. If sound travels through the air at a velocity of 1100 feet a second, find the average velocity of the bullet.

25. Two weights balance when one is 12 inches and the other 15 inches from the point of support. If the first is replaced by a weight 6 pounds greater, the second must be moved 3 inches farther from the point of support to balance it. Find the weights.

26. A crew that can row 6 miles an hour down a stream can row 2 miles an hour up the stream. Find the speed of the current and the speed at which the crew can row in still water.

27. A local train 820 feet long and an express train 500 feet long run on parallel tracks. When running in the same direction it requires 88 seconds for the express to pass the local, but when running in opposite directions it requires only 17.6 seconds for them to pass. Find the rates of the two trains.

CHAPTER VII

SPECIAL PRODUCTS AND QUOTIENTS

70. Special Rules. — The products or quotients of expressions of many forms, called **type forms**, may be written down at sight by special rules, without performing in detail the complete processes of multiplication or division. By discovering and learning these rules, much time and labor in multiplications and divisions may be saved. They reveal also many "short cuts" in the multiplication of arithmetical numbers that are valuable to know, some of the most important of which are given in this chapter. The need for use of these special rules presents itself so often in algebra that they should be thoroughly mastered before proceeding farther.

71. Squares and Cubes of Monomials. — What is the product of $a^3 \times a^3$? Of $5\,m^4n^5 \times 5\,m^4n^5$? Of $(-3\,xy^2z^6)(-3\,xy^2z^6)$?

Since $(7\,M^4)^2$ means $7\,M^4 \times 7\,M^4$, what is its value? Find the value of $(9\,P^2Q^5)^2$. Of $(-4\,h^3k^8)^2$.

From these examples, the following rule is evident:

To square a monomial, square the numerical coefficient, and multiply each exponent by 2.

Since $(2\,w^2)^3$ means $2\,w^2 \times 2\,w^2 \times 2\,w^2$, what is its value? Find the value of $(3\,H^3t^4)^3$. Of $(-4\,A^2B^4C^6)^3$.

These examples reveal the following rule:

To cube a monomial, cube the numerical coefficient, and multiply each exponent by 3.

Similar rules may be discovered for raising monomials to the fourth power, the fifth power, etc.

EXERCISES

Give at sight the values of the indicated powers:

1. $(a^4)^2$.
2. $(R^6)^2$.
3. $(w^{10})^2$.
4. $(k^5)^2$.
5. $(p^7)^2$.
6. $(n^{12})^2$.
7. $(B^8)^2$.
8. $(V^8)^2$.
9. $(x^9)^2$.
10. $(A^{15})^2$.
11. $(L^4)^3$.
12. $(t^5)^3$.
13. $(S^8)^3$.
14. $(x^6)^3$.
15. $(r^7)^3$.
16. $(Q^{12})^3$.
17. $(v^{10})^3$.
18. $(N^{15})^3$.
19. $(p^{20})^3$.
20. $(b^{16})^3$.
21. $(x^4y^2)^2$.
22. $(ab^5)^2$.
23. $(y^2z^6)^2$.
24. $({r_1}^2{r_2}^2{r_3}^2)^2$.
25. $(W^6g^2)^2$.
26. $(H^4p^7)^2$.
27. $(a^2t^3)^3$.
28. $(V^4y^4)^3$.
29. $(A^3n^5)^3$.
30. $(w^6n^4)^3$.
31. $(a^4s^7v^3)^3$.
32. $(3\,x^2y^5)^2$.
33. $(5\,MN^4)^2$.
34. $(2\,a^2b^4c^6)^2$.
35. $(8\,w^{10}v^2)^2$.
36. $(11\,AN^8)^2$.
37. $(15\,xy^4z^7)^2$.
38. $(6\,v^3t^4)^2$.
39. $(9\,S^5H^2)^2$.
40. $(16\,m^2n^8p^3)^2$.
41. $(3\,D^3W^6)^3$.
42. $(5\,x^4T)^3$.
43. $(2\,a^3n^2b^4)^3$.
44. $(4\,p^2t^4y)^3$.
45. $(6\,V^3n^2g^2)^3$.
46. $(-2\,A^4)^2$.
47. $(-3\,m^3n^7)^2$.
48. $(-8\,E^5H^6)^2$.
49. $(-2\,a^9b^9c^9)^2$.
50. $(-10\,RS^4V^4)^2$.
51. $(-M^6W^6)^2$.
52. $(-3\,a^5b^4c^3)^2$.
53. $(-x^2y^2)^3$.
54. $(-2\,a^2t^4)^3$.
55. $(-3\,B^6T^2)^3$.
56. $(-4\,H^2g^7v^9)^3$.
57. $(-2\,P^5V^2R)^3$.
58. $(-3\,L^2T^4)^3$.
59. $(-2\,w^2v^2)^3$.
60. $(-3\,a^3)^2(2\,a^2)^3$.
61. $(2\,n^4)^3(-3\,n^5)^2$.

72. Square Roots and Cube Roots of Monomials. — What is the number which squared gives 9? 25? a^2?

The number whose square is a given number is called the **square root** of the given number.

Thus, since $7^2 = 49$, the square root of 49 is 7.

Similarly, the number whose cube is a given number is called the **cube root** of the given number.

Thus, since $(2\,v^2)^3 = 8\,v^6$, the cube root of $8\,v^6$ is $2\,v^2$.

The numbers whose fourth powers, fifth powers, etc., are a given number are called the **fourth root, fifth root**, etc., of the given number.

A root of a number is indicated by placing before it the sign $\surd$, called the **radical sign.** Usually a vinculum is attached to the radical sign, to show how far its effect is to extend. To indicate what root it is, except in the case of a square root, a little number called the **index** is written above the radical sign.

For example, the square root of n is written $\sqrt{n}$; the cube root of n is written $\sqrt[3]{n}$; the fourth root of n is written $\sqrt[4]{n}$; etc. In the case of square root the index 2 is *understood* and left off.

In most simple practical work, one needs only a knowledge of square roots and cube roots.

Since $(+3)^2=9$ and $(-3)^2=9$, $\sqrt{9}$ is either $+3$ or -3. Similarly, since $(+6)^2=36$ and $(-6)^2=36$, $\sqrt{36}$ is either $+6$ or -6. In general,

Every positive number has two square roots, which have the same absolute value, one positive and the other negative.

The two square roots of a positive number are often written together, with a **double sign** $\pm$.

Thus, $\sqrt{81}=\pm 9$; $\sqrt{144}=\pm 12$.

Since no positive or negative numbers squared can give a negative number,

A negative number has no square root that can be expressed as a positive or negative number.

NOTE. — What to do in a problem where the square root of a negative number is required to be found will be discussed later.

Since $(+2)^3=8$ and $(-2)^3=-8$, $\sqrt[3]{8}=2$ and $\sqrt[3]{-8}=-2$. In general,

Any number has only one cube root that can be expressed as a positive or negative number. This cube root of a positive number is positive, and of a negative number it is negative.

NOTE. — It will be shown later that every number has, in addition to the one positive or negative cube root, two other cube roots that cannot be expressed as simple positive or negative numbers. In this chapter only the positive or negative cube root of a number will be considered.

Since the square root of a monomial, when squared, must give that monomial, it follows from § 71 that:

To find the square roots of a monomial, take the square roots of the numerical coefficient and divide the exponent of each letter by 2.

Similarly, it follows, from § 71, that:

To find the cube root of a monomial, take the cube root of the numerical coefficient, and divide the exponent of each letter by 3.

Thus, $\sqrt{64\,a^6b^4} = \pm 8\,a^3b^2$; $\sqrt{100\,x^8y^2} = \pm 10\,x^4y$. And $\sqrt[3]{8\,w^6t^9} = 2\,w^2t^3$; $\sqrt[3]{-27\,m^{12}n^6} = -3\,m^4n^2$.

EXERCISES

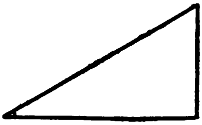
Give at sight the square roots of:

1. 1, 4, 16, 81, 121, 49, 144, 36.
2. a^4, W^8, x^6, P^{60}, n^{16}, A^{12}, y^{10}.
3. M^{14}, D^{22}, S^{18}, k^{32}, $9\,x^6$, $16\,a^4$.
4. $49\,B^{10}$, $25\,v^8$, $100\,R^{12}$, $36\,a^4b^6$, $81\,x^8y^2$.
5. $16\,W^6V^{12}$, $9\,m^8n^2$, $25\,d^6t^4$, $121\,h^4p^4$, $225\,r_1{}^4r_2{}^8$.
6. $144\,a^6t^{10}$, $49\,w^{14}d^8$, $169\,x^4y^6z^2$, $625\,R^4m^{16}$.
7. $196\,a^{10}b^2c^4$, $256\,H^8T^8a^2$, $441\,A^{14}x^{20}y^6$, $400\,m^{40}n^{80}$.

Give at sight the cube roots of:

8. 1, 8, 27, 64, 125, 216, 512, 1000.
9. -1, -8, -27, -64, -125, -1000.
10. A^6, n^{12}, t^3, W^{15}, x^{21}, P^{36}, r^{18}.
11. v^{21}, y^9, M^{24}, a^{33}, $-D^9$, $-z^{12}$, $-L^6$.
12. $-w^{18}$, $-T^{36}$, $-8\,x^6$, $-27\,b^{18}$, $-125\,P^9$.
13. $64\,R^6$, $27\,v^6t^9$, $-8\,m^{12}n^3$, $-125\,r_1{}^6r_2{}^6$, $-A^{12}t^9d^3$.
14. $-p^3q^6r^9$, $216\,M^9a^{15}$, $-27\,x^{12}y^9z^6$, $343\,V^9V^6$, $-8\,F^3D^{12}$.

15. The area of a square is $49 d^2$. What is the length of one side?



16. The base and altitude of a right triangle are $8x$ and $6x$, respectively. Find the length of the hypotenuse.

Suggestion. — The square of the hypotenuse equals the sum of the squares of the base and altitude.

17. The volume of a cube is $64 E^3$. Find the length of one edge.

18. The diameter D, in inches, necessary of a steel shaft, upon which are fastened pulleys that drive machines in a factory, in order to impart H horse power to the machinery when making N revolutions a minute, is computed by the formula $D = \sqrt[3]{\frac{80 \times H}{N}}$.

Find the diameter of the shaft necessary to impart 80 horse power when making 100 revolutions a minute.

73. Equations solved by finding Square Roots. — Many equations may be solved by extracting roots. The following illustrate the solutions of equations of the second degree by finding square roots.

Example 1. — Solve $P^2 = 9$.

Since $P^2 = 9,\ P = \sqrt{9}$

$= \pm 3$.

Example 2. — Solve $y^2 - 49 = 0$.

Transposing, $y^2 = 49$.

Hence, $y = \sqrt{49}$

$= \pm 7$.

EXERCISES

Solve:

1. $x^2 = 25$.
2. $t^2 = 81$.
3. $A^2 = 36$.
4. $n^2 = 100$.
5. $y^2 - 16 = 0$.
6. $R^2 - 64 = 0$.
7. $B^2 - 121 = 0$.
8. $F^2 - 225 = 0$.
9. $2a^2 = 8$.
10. $5M^2 = 45$.
11. $3p^2 - 48 = 0$.
12. $7D^2 - 28 = 0$.
13. $49 = v^2$.
14. $144 = S^2$.
15. $75 = 3W^2$.

16. The time required for an object starting from rest to fall a given distance is found from the equation $16\,t^2 = s$, where t is the time in seconds and s is the distance in feet.

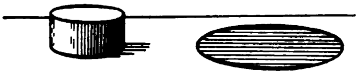
Find the time required for an object to fall 64 feet.

SUGGESTION. — $16\,t^2 = 64$. Solve for t. Has the negative answer any meaning?

17. In the following table are given the distances that an object falls. Complete the table by finding, by the formula in Problem 16, the time required in each case.

DISTANCE	16 FT.	144 FT.	576 FT.	1600 FT.
Time				

18. In the manufacture of lids for metal boxes and cans, a circular piece of metal, called the "blank," is cut from a flat sheet of it, and then stamped into the required shape by means of a machine called a "die." In computing the size of the blank to be cut for a lid of given size, men assume that the area of the blank is equal to the total area of the lid.



Find the radius of the blank necessary to make a lid whose area is 12.5664 sq. in.

SUGGESTION. — If R is the radius, $3.1416\,R^2 = 12.5664$.

19. The area of the surface of the lid of a lard bucket is 28.2744 square inches. Find the radius of the blank from which it is made.

74. Product of the Sum and Difference of Two Terms. — Find, as in Chapter V, the products: $(a-4)(a+4)$; $(P+5)(P-5)$; $(x^2-7)(x^2+7)$; $(2\,R^3+3)(2\,R^3-3)$.

How many terms in each product? Why not more? Why is the last term negative in each product?

In general, where a and b are any two terms, it is found that:

$$(a+b)(a-b)=a^2-b^2.$$

Hence,

To find the product of the sum and the difference of the same two terms, take the difference between their squares.

Example. — $(4w^3+3)(4w^3-3)=(4w^3)^2-3^2$
$=16w^6-9.$

EXERCISES

Give at sight the products of:

1. $n+3$ and $n-3$.
2. $A+6$ and $A-6$.
3. $v+2$ and $v-2$.
4. $x+5$ and $x-5$.
5. $k-4$ and $k+4$.
6. $n-7$ and $n+7$.
7. $w+v$ and $w-v$.
8. $2a+3$ and $2a-3$.
9. $5P+1$ and $5P-1$.
10. $4S-7$ and $4S+7$.
11. $10-3R$ and $10+3R$.
12. $1-x^2$ and $1+x^2$.
13. $6+2v$ and $6-2v$.
14. $2x+5z$ and $2x-5z$.
15. $9M+3T$ and $9M-3T$.
16. $5a-4b$ and $5a+4b$.
17. $(6W+7V)(6W-7V)$.
18. $(12H-5a)(12H+5a)$.
19. $(a^2-9)(a^2+9)$.
20. $(x^3-4)(x^3+4)$.
21. $(2A^2-1)(2A^2+1)$.
22. $(7s^2-4t^2)(7s^2+4t^2)$.
23. $(1-6x^4)(1+6x^4)$.
24. $(a^3-b^3)(a^3+b^3)$.
25. $(x^4-y^4)(x^4+y^4)$.
26. $(P^6+Q^6)(P^6-Q^6)$.
27. $(4-A)(A+4)$.
28. $(1-6x)(6x+1)$.
29. $(2M+7N)(-7N+2M)$.
30. $(12+5t^2)(-12+5t^2)$.
31. $(-10D^3+9)(10D^3+9)$.
32. $(a+1)(a-1)(a^2+1)$.
33. $(2-W)(2+W)(4+W^2)$.
34. $(1-x)(1+x)(1+x^2)(1+x^4)$.

Give at sight the quotients of the following:

35. $(a^2-b^2)\div(a+b)$.
36. $(x^2-y^2)\div(x-y)$.
37. $(D^2-1)\div(D-1)$.
38. $(16-R^2)\div(4-R)$.

39. $(9h^2-25t^2)\div(3h+5t)$.

40. $(81A^4-1)\div(9A^2-1)$.

41. $(y^4-36)\div(y^2+6)$.

42. $(144n^6-1)\div(12n^3-1)$.

43. $(64S^4-25)\div(5+8S^2)$.

44. $(36x^2-49y^4)\div(6x-7y^2)$.

Solve without the aid of pencil:

45. $n^2+n-6=(n-2)(n+2)$.

46. $x^2+2x+3=(x+3)(x-3)$.

47. $4A^2-A+1=(2A+7)(2A-7)$.

48. $9R(R+2)=(3R+6)(3R-6)$.

75. Products of Arithmetical Numbers by the Rule in § 74. — The product of two arithmetical numbers, of which one is less than a multiple of 10 and the other exceeds this multiple by the same amount, may be given at sight by the rule in § 74.

Thus, $58\times 62=(60-2)(60+2)=60^2-2^2=3600-4=3596$.

EXERCISES

Find mentally the products of the following:

1. 19×21.
2. 28×32.
3. 22×18.
4. 39×41.
5. 43×37.
6. 34×26.
7. 48×52.
8. 53×47.
9. 69×71.
10. 72×68.
11. 63×57.
12. 77×83.
13. 85×75.
14. 88×92.
15. 99×101.
16. 97×103.
17. 109×111.
18. 118×122.

19. Find the cost of 22 boxes of berries at 18 cents a box.

20. Find the cost of 17 dozen eggs at 23 cents a dozen.

21. Find the cost of 28 pounds of butter at 32 cents a pound.

22. Buckwheat weighs 52 pounds a bushel. Find the weight of 48 bushels.

23. Onions weigh 57 pounds a bushel. Find the weight of 63 bushels.

24. A field is 77 rods wide and 83 rods long. Find its area.

76. Product of Two Binomials with One Common Term.

$$
\begin{array}{l}
x+7\\
x+5\\ \hline
x^2+\ \ 7x\\
\quad +\ \ 5x+35\\ \hline
x^2+12x+35
\end{array}
\qquad
\begin{array}{l}
x-7\\
x-5\\ \hline
x^2-\ \ 7x\\
\quad -\ \ 5x+35\\ \hline
x^2-12x+35
\end{array}
\qquad
\begin{array}{l}
x+7\\
x-5\\ \hline
x^2+7x\\
\quad -5x-35\\ \hline
x^2+2x-35
\end{array}
\qquad
\begin{array}{l}
x-7\\
x+5\\ \hline
x^2-7x\\
\quad +5x-35\\ \hline
x^2-2x-35
\end{array}
$$

Tell how the first term of each of the above products is obtained. How is the coefficient in the second term of each obtained? How is the third term of each obtained?

In general,

$$(x+a)(x+b)=x^2+(a+b)x+ab.$$

That is,

To find the product of two binomials having a common term, take the square of the common term, plus the algebraic sum of the unlike terms times the common term, plus the algebraic product of the unlike terms.

EXAMPLE 1.— $(n+2)(n+3)=n^2+(2+3)n+2\cdot 3$
$\qquad\qquad\qquad\qquad\quad = n^2+5n+6.$

EXAMPLE 2.— $(t-7)(t+4)=t^2+(-7+4)t+(-7\cdot 4)$
$\qquad\qquad\qquad\qquad\quad = t^2-3t-28.$

EXERCISES

Give at sight the products of:

1. $(a+4)(a+5)$.
2. $(x+1)(x+3)$.
3. $(M+2)(M+6)$.
4. $(y+7)(y+4)$.
5. $(n+6)(n+1)$.
6. $(v+3)(v+8)$.
7. $(t+4)(t-9)$.
8. $(A+7)(A+3)$.
9. $(D+5)(D+5)$.
10. $(m+7)(m+8)$.
11. $(K+1)(K+10)$.
12. $(x+8)(x+6)$.
13. $(a-5)(a-3)$.
14. $(W-8)(W-2)$.
15. $(R-6)(R-1)$.
16. $(t-3)(t-12)$.
17. $(B-16)(B-3)$.
18. $(N-9)(N-7)$.
19. $(d-15)(d-2)$.
20. $(S-6)(S-12)$.
21. $(v-4)(v-10)$.
22. $(d-16)(d-10)$.
23. $(a+5)(a-3)$.
24. $(x+12)(x-8)$.
25. $(n+8)(n-4)$.
26. $(V-3)(V+9)$.
27. $(t-5)(t+8)$.

28. $(b-12)(b+7)$.

29. $(x+3)(x-15)$.

30. $(y-1)(y+6)$.

31. $(H+4)(H-9)$.

32. $(R-15)(R+10)$.

33. $(A+14)(A-4)$.

34. $(m+16)(m-12)$.

35. $(u-11)(u+9)$.

36. $(C+18)(C-7)$.

37. $(x^2+2)(x^2+5)$.

38. $(a^3-1)(a^3+9)$.

39. $(P^2-6)(P^2-8)$.

40. $(y^4+7)(y^4-11)$.

41. $(c^2-10)(c^2+4)$.

42. $(2a+5)(2a+3)$.

43. $(3x+7)(3x+2)$.

44. $(6t-1)(6t-4)$.

45. $(8x+3)(8x-2)$.

46. $(4W-2)(4W+5)$.

47. $(5y+7)(5y-2)$.

48. $(2c-3)(2c-12)$.

49. $(8v+16)(8v-9)$.

50. $(2x^2+5)(2x^2+3)$.

51. $(7A^3-2)(7A^3+6)$.

52. $(12W^2-20)(12W^2+16)$.

53. $(2a+3b)(2a+4b)$.

54. $(5x-2y)(5x-6y)$.

55. $(6k+3t)(6k-5t)$.

56. $(4m+9n)(4m-3n)$.

57. $(9D-3v)(9D+8v)$.

58. $(7v+2w)(7v-9w)$.

59. $(3a^2+4b^2)(3a^2+12b^2)$.

60. $(5t^2-6s^2)(5t^2+4s^2)$.

61. $(8B^2-4AC)(8B^2+AC)$.

62. $(4c^2-22d^2)(4c^2+5d^2)$.

63. $(1-5a)(1+3a)$.

Find mentally the indicated quotients:

64. $(x^2+5x+6)\div(x+2)$.

65. $(n^2-4n+3)\div(n-1)$.

66. $(A^2-2A-15)\div(A+3)$.

67. $(v^2-7v+12)\div(v-4)$.

68. $(T^2-T-12)\div(T-4)$.

69. $(s^2+4s-60)\div(s+10)$.

70. $(W^2+W-72)\div(W-8)$.

71. $(y^2+9y+20)\div(y+4)$.

72. $(r^2-16r+63)\div(r-9)$.

73. $(m^2-3m-54)\div(m+6)$.

Solve mentally:

74. $(x+3)(x+4)=x^2+33$.

75. $(v+5)(v-3)=(v+2)(v-1)$.

76. $(2n+7)(2n-6)=4(n^2+2)$.

77. Products of Arithmetical Numbers by the Rule of § 76. — The products of many arithmetical numbers may be obtained, without the aid of pencil, by the rule of § 76.

Thus,
$$\begin{aligned} 87 \times 92 = (90 - 3)(90 + 2) &= 90^2 - 1 \times 90 - 6 \\ &= 8100 - 90 - 6 \\ &= 8004. \end{aligned}$$

EXERCISES

Find mentally the following indicated products:

1. 32×33.
2. 51×52.
3. 42×43.
4. 64×66.
5. 39×42.
6. 43×38.
7. 48×53.
8. 76×74.
9. 87×83.
10. 98×92.
11. 89×96.
12. 87×91.
13. 112×113.
14. 198×199.
15. 251×253.

16. Find the cost of 34 dozen eggs at 32 cents a dozen.

17. If a train goes 42 miles an hour, how far will it go in 48 hours?

18. Find the cost of 34 yards of cloth at 36 cents a yard.

19. How many square feet in a lot 92 feet wide and 98 feet long?

20. Oats weigh 32 pounds to the bushel. Find the weight of 35 bushels.

78. Square of a Binomial. — What operation is indicated by $(n + 6)^2$? Find, by multiplication, the values of:

$$(x + 3)^2; \qquad (A - 5)^2; \qquad (m + n)^2; \qquad (m - n)^2.$$

What two terms in the square are positive in each of the above cases? When is the second term of the square negative?

It is seen that for any values of a and b,

$$\boldsymbol{(a + b)^2 = a^2 + 2\,ab + b^2} \text{ and } \boldsymbol{(a - b)^2 = a^2 - 2\,ab + b^2}.$$

These identities give the following rule:

To square a binomial take the square of the first term, plus (or minus) two times the product of the terms, plus the square of the second term.

EXAMPLE. — $(2a^2 - 3b)^2 = (2a^2)^2 - 2(2a^2)(3b) + (3b)^2$
$= 4a^4 - 12a^2b + 9b^2.$

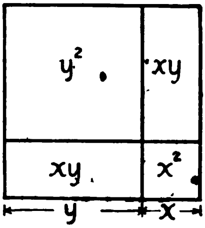
EXERCISES

Find the squares of the following:

1. $n + 3$.
2. $x + 5$.
3. $P + 2$.
4. $a - 4$.
5. $W - 6$.
6. $t - 3$.
7. $m + n$.
8. $x - y$.
9. $2a + 3b$.
10. $5r - 4s$.
11. $3C + 8D$.
12. $7w - 3v$.
13. $5B + 9T$.
14. $10b - c$.
15. $9F + 8C$.
16. $4K - 7L$.
17. $12x + y$.
18. $A - 15B$.
19. $11W + 3a$.
20. $4R - 9n$.
21. $5A + 16K$.
22. $x^2 + 1$.
23. $a^2 - 3$.
24. $n^2 + 5$.
25. $t^2 - 10$.
26. $x^2 - y^2$.
27. $A^2 + 2B^2$.
28. $3t^2 - 4s^2$.
29. $5w^3 + 2v^3$.
30. $y^4 + 1$.
31. $4k^2 - t$.
32. $x^6 + 5$.
33. $12 - 7a^2$.
34. $9 + 15R^3$.
35. $2p^3 - q^6$.
36. $1 - 8d^3$.
37. $B^2 - 4AC$.
38. $x^2y^2 + 5$.
39. $A^2 - 4A$.

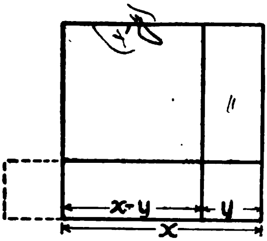
40. Show geometrically, by the accompanying diagram, that

$$(x + y)^2 = x^2 + 2xy + y^2.$$



41. Show geometrically, by this diagram, that

$$(x-y)^2=x^2-2xy+y^2.$$



Solve:

42. $(t+4)^2=t(t+3)+36.$

43. $(2V+3)^2=(2V+1)(2V+3).$

44. $(4k-7)^2=(4k-5)(4k+3).$

45. $(u+2)^2-u^2=u-5.$

79. Squares of Arithmetical Numbers. — By the method of § 78, the squares of many arithmetical numbers are easily found without the aid of pencil.

Thus, $62^2=(60+2)^2=3600+240+4=3844,$

and $79^2=(80-1)^2=6400-160+1=6241.$

EXERCISES

Find mentally the squares of:

1. 41.
2. 59.
3. 72.
4. 38.
5. 68.
6. 89.
7. 78.
8. 83.
9. 92.
10. 99.
11. 98.
12. 101.
13. 109.
14. 199.
15. 148.
16. 498.
17. 249.
18. 301.
19. 502.
20. 999.

80. Square of a Polynomial. — By actual multiplication, it is found that:

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc.$$

In general,

The square of any polynomial equals the sum of the squares of all of its terms, plus two times the algebraic product of each term into all of the terms following it.

Example. — $(2n^2-3n+5)^2=(2n^2)^2+(-3n)^2+(5)^2+2(2n^2)(-3n)$

$$+2(2n^2)(5)+2(-3n)(5)$$

$$=4n^4+9n^2+25-12n^3+20n^2-30n$$

$$=4n^4-12n^3+29n^2-30n+25.$$

EXERCISES

Write down the squares of the following polynomials:

1. $a - b + c$.
2. $2x + 3y + z$.
3. $m - 2n - p$.
4. $A + 3B - C$.
5. $2u - 6v + w$.
6. $A^2 + 2A + 1$.
7. $x^2 - 2x + 3$.
8. $r^2 + s^2 - t^2$.
9. $a + b + c + d$.
10. $x - y + z - w$.
11. $a^3 - a^2 - a + 1$.
12. $1 + 2P - 3P^2 + 4P^6$

81. Quotients of the Form $a^3 - b^3$ Divided by $a - b$.—By actual division,

$$(a^3 - b^3) \div (a - b) = a^2 + ab + b^2.$$

That is,

If the difference between the cubes of two terms is divided by the difference between those terms, the quotient equals the square of the first term, plus the product of the two terms, plus the square of the second term.

Example.—$\frac{8a^3 - 27b^3}{2a - 3b} = (2a)^2 + (2a)(3b) + (3b)^2$

$$= 4a^2 + 6ab + 9b^2.$$

EXERCISES

Find mentally the values of the indicated quotients:

1. $\frac{x^3 - y^3}{x - y}$.
2. $\frac{m^3 - n^3}{m - n}$.
3. $\frac{w^3 - v^3}{w - v}$.
4. $\frac{x^3 - 1}{x - 1}$.
5. $\frac{1 - P^3}{1 - P}$.
6. $\frac{D^3 - 8}{D - 2}$.
7. $\frac{s^3 - 27}{s - 3}$.
8. $\frac{B^3 - 64}{B - 4}$.
9. $\frac{125 - y^3}{5 - y}$.
10. $\frac{a^3 - 8b^3}{a - 2b}$.
11. $\frac{27t^3 - 1}{3t - 1}$.
12. $\frac{d^3 - 64s^3}{d - 4s}$.
13. $\frac{8w^3 - 27v^3}{2w - 3v}$.
14. $\frac{125M^3 - N^3}{5M - N}$.
15. $\frac{64A^3 - 125B^3}{4A - 5B}$

16. $\frac{1000\,r^3-1}{10\,r-1}$.

17. $\frac{27\,b^3-216\,c^3}{3\,b-6\,c}$.

18. $\frac{64\,H^3-343\,D^3}{4\,H-7\,D}$.

19. $\frac{x^3y^3-1}{xy-1}$.

20. $\frac{8-a^3b^3}{2-ab}$.

21. $\frac{a^6-1}{a^2-1}$.

22. $\frac{m^6-n^6}{m^2-n^2}$.

23. $\frac{1-t^9}{1-t^3}$.

24. $\frac{8\,a^3-27\,b^6}{2\,a-3\,b^2}$.

25. $\frac{64\,W^6-g^3}{4\,W^2-g}$.

26. $\frac{M^9-N^9}{M^3-N^3}$.

27. $\frac{1-8\,y^{12}}{1-2\,y^4}$.

Find mentally the following indicated products:

28. $(a-b)(a^2+ab+b^2)$.

29. $(m-n)(m^2+mn+n^2)$.

30. $(W-V)(W^2+WV+V^2)$.

31. $(x-1)(x^2+x+1)$.

32. $(2\,a-1)(4\,a^2+2\,a+1)$.

33. $(3\,x-2\,y)(9\,x^2+6\,xy+4\,y^2)$.

34. $(4\,s-5\,t)(16\,s^2+20\,st+25\,t^2)$.

35. $(x^2-1)(x^4+x^2+1)$.

36. $(1-2\,D^2)(1+2\,D^2+4\,D^4)$.

82. Quotients of the Form a^3+b^3 Divided by $a+b$. — By actual division,

$$(a^3+b^3)\div(a+b)=a^2-ab+b^2.$$

That is,

If the sum of the cubes of two terms is divided by the sum of these terms, the quotient equals the square of the first term, minus the product of the two terms, plus the square of the second term.

Example. — $\frac{125\,S^3+8\,t^3}{5\,S+2\,t}=(5\,S)^2-(5\,S)(2\,t)+(2\,t)^2$

$$=25\,S^2-10\,St+4\,t^2.$$

EXERCISES

Find mentally the values of the indicated quotients:

1. $\dfrac{x^3+y^3}{x+y}.$
2. $\dfrac{W^3+g^3}{W+g}.$
3. $\dfrac{m^3+n^3}{m+n}.$
4. $\dfrac{a^3+1}{a+1}.$
5. $\dfrac{y^3+8}{y+2}.$
6. $\dfrac{27+t^3}{3+t}.$
7. $\dfrac{64+N^3}{4+N}.$
8. $\dfrac{x^3+125}{x+5}.$
9. $\dfrac{1+8\,b^3}{1+2\,b}.$
10. $\dfrac{w^3+27\,v^3}{w+3\,v}.$
11. $\dfrac{P^3+125\,Q^3}{P+5\,Q}.$
12. $\dfrac{8\,m^3+n^3}{2\,m+n}.$
13. $\dfrac{27\,D^3+t^3}{3\,D+t}.$
14. $\dfrac{64\,x^3+125\,y^3}{4\,x+5\,y}.$
15. $\dfrac{1000\,K^3+a^3}{10\,K+a}.$
16. $\dfrac{216\,m^3+27\,n^3}{6\,m+3\,n}.$
17. $\dfrac{b^3+343\,d^3}{b+7\,d}.$
18. $\dfrac{8\,R^3+125\,c^3}{2\,R+5\,c}.$
19. $\dfrac{x^6+1}{x^2+1}.$
20. $\dfrac{1+a^9}{1+a^3}.$
21. $\dfrac{W^6+B^3}{W^2+B}.$
22. $\dfrac{8\,y^6+27\,z^6}{2\,y^2+3\,z^2}.$
23. $\dfrac{125\,P^6+1}{5\,P^2+1}.$
24. $\dfrac{216+a^3t^6}{6+at^2}.$
25. $\dfrac{343+8\,v^9t^6}{7+2\,v^3t^2}.$
26. $\dfrac{H^9+K^9}{H^3+K^3}.$
27. $\dfrac{27+u^{12}}{3+u^4}.$

Find mentally the following indicated products:

28. $(m+n)(m^2-mn+n^2).$
29. $(x+1)(x^2-x+1).$
30. $(1+a)(1-a+a^2).$
31. $(W+V)(W^2-WV+V^2).$
32. $(3\,P+1)(9\,P^2-3\,P+1).$
33. $(4\,S+1)(16\,S^2-4\,S+1).$
34. $(a+5\,b)(a^2-5\,ab+25\,b^2)$
35. $(1+x^2)(1-x^2+x^4).$
36. $(y^2+z^3)(y^4-y^2z^3+z^6).$
37. $(2+3\,A)(4-6\,A+9\,A^2).$

SUPPLEMENTARY EXERCISES

Give the squares of:

1. a^n; S^{2n}; $-x^{3n}$; A^nB^n; y^{n+1}; $-t^{n-1}$; a^{2n+3}.
2. $3X^a$; $5X^{a+1}Y^{a-1}$; $-12a^{2x}b^{3x}$; $-7V^{2r-1}W^{2r+1}$.

Give the square roots of:

3. X^{2n}; Y^{6a}; $P^{4n}Q^{10n}$; t^{2n+2}; v^{4x+6}.
4. $16D^{6a+8}$; $25X^{8n-2}Y^{8n+2}$; $81a^{4n}t^{2n+4}$.

Give the cubes of:

5. X^n; a^{2t}; $-P^{3v}$; W^nV^n; t^{n+1}; t^{n-1}; $-y^{3n+2}$.
6. $2A^n$; $3x^{v+1}y^{v-1}$; $-4M^{4a}N^{2a}$; $-5c^{2v-1}d^{2v+1}$.

Give the cube roots of:

7. w^{3n}; Q^{6t}; $r_1{}^{12p}$; $-A^{6d}$; $-8x^{3n}y^{6n}$.
8. $27x^{6n}t^{9m}$; $-64N^{3t}g^{12t}$; $-125a^{9n}t^{18n}$.

Give at sight the products of:

9. $(x^n+1)(x^n-1)$.
10. $(V^t+3)(V^t-3)$.
11. $(a^{n+1}+2)(a^{n+1}-2)$.
12. $(x^a+y^b)(x^a-y^b)$.
13. $(A^n+3)(A^n+7)$
14. $(D^{2t}+6)(D^{2t}-4)$.
15. $(N^x-1)(N^x+8)$.
16. $(3y^{2n}+6)(3y^{2n}-2)$.

Give the squares of:

17. a^n+1.
18. x^n-y^n.
19. V^t+T^v.
20. $2y^{3a}+3w^{2a}$.
21. $5H^{4t}-d^{4t}$.
22. $3p^{2d}+q^{2c}$.

The products of polynomials often may be obtained mentally by grouping their terms so as to form binomials.

Find the products of the following by writing the factors as binomials, and using the rule in § 74.

23. $(a+b+c)(a+b-c)$.

Solution.—
$$\begin{aligned}(a+b+c)(a+b-c) &= ([a+b]+c)([a+b]-c)\\ &= [a+b]^2-c^2\\ &= a^2+2ab+b^2-c^2.\end{aligned}$$

24. $(x+y-1)(x+y+1)$.

25. $(a^2+a+1)(a^2+a-1)$.

26. $(m-n+p)(m-n-p)$.

27. $(b-c+6)(b-c-6)$.

28. $(S+2t-3)(S+2t+3)$.

29. $(a-5b-c)(a-5b+c)$.

30. $(N+3v-5w)(N+3v+5w)$.

31. $(a-2b+c)(a+2b+c)$.

Suggestion.— Group a and c.

32. $(P-Q+R)(P+Q+R)$.

33. $(x^2+x+1)(x^2-x+1)$.

34. $(a^2-ab+b^2)(a^2+ab+b^2)$.

35. $(a+b-c)(a-b+c)$.

Solution. — Since b and c both have different signs in the two factors, group them. Then

$$\begin{aligned}(a+b-c)(a-b+c) &= (a+[b-c])(a-[b-c])\\ &= a^2-[b-c]^2\\ &= a^2-[b^2-2bc+c^2]\\ &= a^2-b^2+2bc-c^2.\end{aligned}$$

36. $(4n-3x-y)(4n+3x+y)$.

37. $(2u+3v-w)(2u-3v+w)$.

38. $(3P-4T+V)(3P+4T-V)$.

39. $(a+b+c+d)(a+b-c-d)$.

Suggestion. — Group a and b into one term, and c and d into one term. Thus, $([a+b]+[c+d])([a+b]-[c+d])$.

40. $(x-2y-t+z)(x-2y+t-z)$.

41. $(R+2S-2T+U)(R+2S+2T-U)$.

42. $(x^3-x^2-x+1)(x^3-x^2+x-1)$.

43. $(1+a-a^2-a^3)(1+a+a^2+a^3)$.

Find the products of the following by writing the factors as binomials and proceeding as in § 76:

44. $(a+b+3)(a+b-5)$.

SOLUTION. — $(a+b+3)(a+b-5) = ([a+b]+3)([a+b]-5)$
$= [a+b]^2 - 2[a+b] - 15$
$= a^2 + 2ab + b^2 - 2a - 2b - 15.$

45. $(x-y+2)(x-y+3)$.

46. $(p+q+10)(p+q-16)$.

47. $(2W-3V+8)(2W-3V-3)$.

48. $(x^2+x+6)(x^2+x-2)$.

49. $(t^2-t+10)(t^2-t-14)$.

50. $(n^2-1+n)(n^2-2+n)$.

CHAPTER VIII

FACTORS. MULTIPLES. EQUATIONS SOLVED BY FACTORING

83. Factoring. — What are the factors of 6? Of 10? Of 12? Of mn? Of $v(v+t)$?

The process of finding the factors of a given expression is called **factoring.**

If a factor of a given expression does not itself have factors, it is called a **prime factor** of the expression. In general, *to factor an expression means to find its prime factors.*

Facility in the factoring of many expressions depends upon an ability to recognize the special forms of the expressions and upon a knowledge of the rules in Chapter VII.

84. Polynomial with a Monomial Factor. — If each term of a given polynomial is divisible by the same monomial, the polynomial is divisible by the monomial, and hence the monomial is a factor.

For example, in $18a^4 - 9a^3 + 27a^2$, each term is divisible by $9a^2$.

$$\begin{array}{r} 2a^2 - a + 3 \\ 9a^2 \overline{)\,18a^4 - 9a^3 + 27a^2} \end{array}$$

Hence, since the dividend always equals the product of the divisor and quotient,

$$18a^4 - 9a^3 + 27a^2 = 9a^2(2a^2 - a + 3).$$

That is, the factors of $18a^4 - 9a^3 + 27a^2$ are $9a^2$ and $2a^2 - a + 3$.

Hence the rule for finding the factors of a polynomial with a monomial factor:

Find, by inspection, the monomial of highest power that will divide each term of the polynomial. Divide the polynomial by this monomial. The divisor and quotient are the monomial and polynomial factors, respectively, of the given polynomial.

The division should always be performed *mentally*, and the work written as shown in the following example:

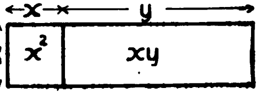
Example.—Factor $2nx^2-4ny^2+6nz^2$.

$$2nx^2-4ny^2+6nz^2=2n(x^2-2y^2+3z^2).$$

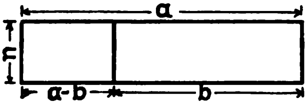
EXERCISES

Factor:

1. $2a+2b$.
2. $3x-6y$.
3. $4B^2-16AC$.
4. $5nR+10nQ$.
5. $3dx^2-3dy^2$.
6. A^3-4A^2.
7. $\pi R^2+\pi R'^2$.
8. x^6+5x^4.
9. $n^2m^2-m^4$.
10. $2\pi R^2+2\pi RH$.
11. $s^6t^2+s^2t^6$.
12. $5w^3-10w^2v$.
13. $18x^5-9x^3$.
14. $c^4n+2c^2n^2$.
15. $7P^4+21P^3$.
16. x^3-x^2+x.
17. $4V_1^2+6V_1V_2+4V_1V_3$.
18. $H^6-6H^5b+2H^4b^2$.
19. $v^2t+2vt^2+vt^3$.
20. $8n^3m^3+6n^2m^4+4nm^5$.
21. $3a^4b-6a^3b^2+3a^2b^3$.
22. $2\pi R^2+2\pi r^2+2\pi Rr$.
23. $a^4b^5c^2+a^2b^4c^6+a^5b^2c^4$.
24. $24w^5d^3-12w^3d+42w^6d^2$.
25. $6v^8+8v^{10}-16v^6$.
26. $-x^3-x^2-x$.
27. $-a^2-ab-b^2$.
28. $4x^6-6x^4y^2+8x^3y^3-12x^2y^4$.
29. $-10M^8+12M^6-16M^4+8M^2$.
30. $56a^3y^5-14a^4y^4+28a^5y^3-105a^6y^2$.



31. Show by this diagram that $x^2+xy=x(x+y)$.



32. Show by a diagram that

$$na-nb=n(a-b).$$

33. Many arithmetical computations may be shortened by applying the process of factoring in § 84. Thus,

$$4\times 7+6\times 7=10\times 7, \text{ or } 70.$$

Find by this method the values of the following: $17\times 45+13\times 45$; $38\times 92+42\times 92$; $64\times 575+85\times 575-24\times 575$; $9\times 3.1416+16\times 3.1416$; $2\times 16\times 8+2\times 24\times 8+16\times 24$; $365\times 784-15\times 784+50\times 784$.

34. If, in two numbers of two digits each, the tens' digits are equal and the sum of the ones' digits is 10, the numbers may be expressed by $10t+u$ and $10t+10-u$, respectively, in which t is the tens' digit, u the ones' digit of one, and $10-u$ the ones' digit of the other.

Show that their product is $100t^2+100t+10u-u^2$.

Show that this may be written $100t(t+1)+u(10-u)$.

Hence, *to multiply two numbers of which the tens' digits are equal, and the sum of the ones' digits is* 10, *take the product of the number of tens by a number one greater for the hundreds and annex the product of the ones.*

Thus, $74\times 76=5624$; $69\times 61=4209$; $43\times 47=2021$.

35. Give at sight the products of the following: 23×27; 16×14; 26×24; 32×38; 46×44; 56×54; 68×62; 77×73; 88×82; 93×97; 81×89; 84×86; 79×71; 75×75; 91×99; 94×96.

36. State a rule for squaring any number ending in 5 that follows as a special case of the rule in Exercise 34.

37. Give at sight the squares of the following:

25; 45; 35; 65; 85; 55; 95; 75; 105; 115.

85. Polynomials Factored by Grouping Terms. — Some polynomials that do not have monomial factors may be factored by the method of § 84, after first grouping the terms.

EXAMPLE 1. — Factor $a^2+ab+ac+bc$.

Grouping the first two terms and factoring them, then the last two, gives $a(a+b)+c(a+b)$.

Now the factor $a+b$ is common to the two terms. Dividing by $a+b$ gives a quotient $a+c$.

Hence the factors are $a+b$ and $a+c$.

That is, $$a^2+ab+ac+bc=(a+b)(a+c).$$

EXAMPLE 2. — Factor $mt+nv-nt-mv$.

Grouping the first and fourth terms, and the second and third terms,

$$\begin{aligned} mt+nv-nt-mv &= m(t-v)-n(t-v) \\ &= (t-v)(m-n). \end{aligned}$$

Notice that we factor out $-n$ from the second group, rather than $+n$, so that the expressions left in parentheses will be the same.

These examples illustrate the rule:

Group the terms of the polynomial to be factored so that a monomial factor may be divided out from each group, and that the expressions left in parentheses are the same in all of the groups. Then divide by the expression in parentheses, writing the divisor as one factor and the quotient as the other.

If in Example 2 we should group the second and fourth terms, factoring out v, and group the first and third terms, factoring out t, we would get the factors $(v-t)(n-m)$. It is evident that these factors differ from those in the solution above only in their signs. So in other types of problems in factoring, *two sets of factors* may be obtained, differing only in sign. For practical purposes only one set of factors is ever needed, and the other may be ignored.

NOTE. — The student should check his work in every problem in factoring by some method. In most cases, this is easily accomplished by multiplying mentally the factors obtained, to see if the product equals the given expression to be factored. A second method, convenient in many cases, consists of assigning special values to all letters involved and comparing the values of the factors with the value of the given expression. Thus, if in Example 1 above we make $a = 1$, $b = 2$, $c = 3$, then the expression to be factored becomes 12 and the factors 3 and 4, respectively.

EXERCISES

Factor:

1. $3(x+y)+n(x+y)$.
2. $a(P-Q)+b(P-Q)$.
3. $v(a+b)-t(a+b)$.
4. $2x(m-n)-3y(m-n)$.
5. $5w(h-k)+u(h-k)$.
6. $a(x-y)+b(y-x)$.

SUGGESTION. — This may be written $a(x-y)-b(x-y)$.

7. $s(A-5)+t(5-A)$.
8. $p(r-R)-q(R-r)$.
9. $B^2(N-M)+4AC(M-N)$
10. $x(w-v)+(v-w)$.
11. $ac+ad+bc+bd$.
12. $nm+ny+mx+xy$.
13. $vt+wt+2v+2w$.
14. $Wg-Wh+gk-hk$.
15. $mp+np-mq-nq$.
16. $as-bs-at+bt$.
17. $vu-vR+uS-RS$.
18. $mn+mt-ns-st$.

19. $x^2+xy+xz+yz$.

20. $p^2+pr-pq-qr$.

21. $M^2-MN+MP-NP$.

22. $2v+v^2-2u-vu$.

23. $3rs-2sx+3ry-2xy$.

24. $2t-tm-2m+4$.

25. $h^2k^2-m^2k^2-h^2+m^2$.

26. $W^2x-V^2x-W^2y+V^2y$.

27. $FC+GC+2F+2G$.

28. $pq-3p-2q+6$.

29. $R^2-R-Rr+r$.

30. $2a^2-cx^2-a^2c+2x^2$.

86. Trinomials that are Perfect Squares.—If a binomial is squared, as in § 78, the result is always a trinomial. A trinomial which is the square of a binomial is called a **perfect square.**

Since any binomial must be either of the form $a+b$ or $a-b$, and since $(a+b)^2=a^2+2ab+b^2$ and $(a-b)^2=a^2-2ab+b^2$, evidently any trinomial which is a perfect square must be either of the form $a^2+2ab+b^2$ or $a^2-2ab+b^2$. In either case, the term $2ab$ is twice the product of the absolute values of $\sqrt{a^2}$ and $\sqrt{b^2}$. Hence,

A trinomial is a perfect square if one term is twice the product of the absolute values of the square roots of the other two terms.

In $9x^2-24xy+16y^2$, what are the square roots of $9x^2$ and $16y^2$? What is twice the product of their absolute values? Is the trinomial a perfect square?

Since the binomial factors of a trinomial which is a perfect square are equal, *to factor such a trinomial is equivalent to expressing it as the square of a binomial.*

EXAMPLE 1.—Factor $25m^2-40m+16$.

$$\sqrt{25m^2}=\pm 5m, \text{ and } \sqrt{16}=\pm 4.$$

Since twice the product of these roots must give $-40m$, the positive root of one term and the negative root of the other must be chosen, which can be done in two ways.

Hence, $25m^2-40m+16=(5m-4)^2$ or $(-5m+4)^2$.

Check.—When $m=2$, the trinomial is 36 and the factors 6 and 6, or -6 and -6.

EXAMPLE 2.—Factor $64v^2+112vt+49t^2$.

$$\sqrt{64v^2}=\pm 8v, \text{ and } \sqrt{49t^2}=\pm 7t.$$

Since twice the product of these roots must give $+112vt$, they must be taken with like signs, which can be done in two ways.

Hence, $64v^2 + 112vt + 49t^2 = (8v + 7t)^2$ or $(-8v - 7t)^2$.

It is seen here, as in § 85, that there are two sets of factors in each problem. In practice only one of these is used, viz. the set in which the first term is positive.

To find a binomial of which a given trinomial is a square, take as the terms the square roots of the two square terms of the trinomial with such signs that their product gives the sign of the remaining term.

EXERCISES

Determine which of the following are perfect squares:

1. $x^2 + 6x + 9$.
2. $n^2 + 16 + 8n$.
3. $1 - 4A + 4A^2$.
4. $25t^2 + 12t + 1$.
5. $x^2 + ax + a^2$.
6. $9v^2 - 6v + 4$.
7. $M + 4MN + 4N^2$.
8. $r^2 - 10rs + 25s^2$.
9. $16H^2 - 12Ht + 9t^2$.
10. $49x^2 + 4y^2 - 20$.
11. $24W^4 - 10W^2 + 1$.
12. $a^4 - 2a^2b^2 - b^4$.
13. $m^2 + 18ma + 81a^2$.
14. $b^2 - 12bc + 36c^2$.
15. $9p^2 + 38pq + 49q^2$.
16. $D^4 + 1 + 2D^2$.

Make perfect square trinomials of the following by supplying the missing terms:

17. $x^2 + ? + y^2$.
18. $a^2 - 2ab + ?$.
19. $? + 2mn + n^2$.
20. $4A^2 + ? + 1$.
21. $16 - 24t + ?$.
22. $25v^2 - ? + 9w^2$.
23. $W^2 + 49 + ?$.
24. $36 + ? - 12x$.
25. $a^2 - ? + 100$.
26. $? + 1 + x^4$.
27. $4P^3 + 1 + ?$.
28. $r_1^2 - 12r_1r_2 + ?$.
29. $? - 32N + ?$.
30. $? + 42b^2 + ?$.

Factor:

31. x^2+6x+9.

32. R^2-4R+4.

33. $t^2+8t+16$.

34. $25-10a+a^2$.

35. $1+12n+36n^2$.

36. $49W^2-28W+4$.

37. v^2+2v+1.

38. $81-18A+A^2$.

39. x^4+4x^2+4.

40. $16-24D^2+9D^4$.

41. $ax^2-6axy+9ay^2$.

(First remove monomial factor a.)

42. $2n^2+20nm+50m^2$.

43. $tS^2-30tSV+225tV^2$.

44. $16H^2+56Hd+49d^2$.

45. $25b^2-60bc+36c^2$.

46. $4w^2v+36wv^2+81v^3$.

47. $M^2-50MN+625N^2$.

48. $9r_1^2+30r_1r_2+25r_2^2$.

49. $x^5-2x^3y^2+xy^4$.

50. $a^4+4a^3b^2+4a^2b^4$.

51. $P^2-6PQ^3+9Q^6$.

52. $81m^4+36m^2n+4n^2$.

53. $25W^4-40W^2g+16g^2$.

54. $x^3y^2+2x^2y+x$.

55. $c^2d^2+2cdmn+m^2n^2$.

56. $R^2r^2-8Rrt+16t^2$.

57. $24v^2+36v^3+4v$.

58. $25z^2+81z^4+90z^3$.

59. $64t_1^2+49t_2^2-112t_1t_2$.

60. $36A^3+144A+144A^2$.

61. $-1-6a-9a^2$.

87. Trinomials of the Form x^2+ax+b. — By the principle in § 76, the product of two binomials having a common term x is always a trinomial in the general form x^2+ax+b.

Thus, $(x+5)(x+7)=x^2+12x+35$.

Here a equals 12 and b equals 35.

It follows that *a trinomial in the form* x^2+ax+b *may be factored into two binomials with a common term* x, *if for the other terms of the factors two numbers can be found whose algebraic sum is the coefficient* a *and whose algebraic product is* b.

EXAMPLE 1. — Factor $x^2-8x+15$.

For the second terms of the factors, we are to find two numbers whose sum is -8 and whose product is $+15$. These are evidently -3 and -5.

Hence, $x^2-8x+15=(x-3)(x-5)$

Check. — When $x = 1$, the trinomial is 8, and the factors -2 and -4, respectively; or mentally multiplying, $(x - 3)(x - 5) = x^2 - 8x + 15$.

EXAMPLE 2. — Factor $m^2 - 4mn - 45n^2$.

We are to find two numbers whose algebraic sum is $-4n$ and product $-45n^2$. These are evidently $-9n$ and $+5n$.

Hence, $m^2 - 4mn - 45n^2 = (m - 9n)(m + 5n)$.

EXERCISES

Factor:

1. $t^2 + 5t + 6$.
2. $a^2 + 5a + 4$.
3. $n^2 + 8n + 12$.
4. $P^2 + 9P + 14$.
5. $x^2 + 9x + 20$.
6. $S^2 + 9S + 8$.
7. $v^2 + 10v + 16$.
8. $a^2 + 10a + 9$.
9. $R^2 + 10\mathrm{R} + 24$.
10. $k^2 + 10k + 21$.
11. $V^2 + 11V + 18$.
12. $d^2 + 11d + 24$.
13. $M^2 + 11M + 30$.
14. $b^2 + 12b + 20$.
15. $y^2 + 12y + 27$.
16. $A^2 + 12A + 32$.
17. $r^2 + 15r + 54$.
18. $m^2 + 17m + 70$.
19 $c^2 + 13c + 30$.
20. $p^2 + 14p + 48$.
21. $B^2 - 5B + 6$.
22. $N^2 - 5N + 4$.
23. $w^2 - 8w + 12$.
24. $Q^2 - 9Q + 14$.
25. $z^2 - 9z + 20$.
26. $t^2 - 10t + 9$.
27. $D^2 - 10D + 16$.
28. $V^2 - 10V + 21$.
29. $a^2 - 10a + 24$.
30. $T^2 - 9T + 8$.
31. $y^2 - 17y + 30$.
32. $H^2 - 14H + 45$.
33. $W^2 - 4W + 3$.
34. $m^2 - 12m + 32$.
35. $c^2 + 35 - 12c$.
36. $B^2 + 55 - 16B$.
37. $t^2 + 7 - 8t$.
38. $F^2 + 2F - 15$.
39. $x^2 + 2x - 3$.
40. $R^2 + 3R - 10$.
41. $K^2 + 3K - 4$.
42. $s^2 - 35 + 2s$.
43. $y^2 + 3y - 18$.
44. $3a - 28 + a^2$.
45. $C^2 - 12 + 4C$.
46. $5m + m^2 - 14$
47. $b^2 + 3b - 40$.
48. $5x - 36 + x^2$.
49. $B^2 - 3B - 18$.
50. $g^2 - 3g - 40$.
51. $m^2 - 14 - 5m$.
52. $T^2 - 4T - 12$.
53. $x^2 + 5xy + 6y^2$.
54. $a^2 + 7ab + 10b^2$.
55. $w^2 - 5wv + 6v^2$.
56. $r^2 - 7rs + 10s^2$.
57. $p^2 - 3pq - 28q^2$

58. $C^2 - 5\,CD - 14\,D^2$.

59. $y^2 + 5\,yz - 14\,z^2$.

60. $a^2b^2 - 3\,ab - 18$.

61. $m^2n^2 - 2\,mn - 35$.

62. $H^2 - 19\,H - 120$.

63. $V^2 - V - 30$.

64. $ab^2 - 3\,ab - 70\,a$.

(First remove monomial factor a.)

65. $xy^2 + xy - 90\,x$.

66. $w^2v^2 + 2\,w^2v - 99\,w^2$.

67. $d^3m^2 + d^3m - 56\,d^3$.

68. $2\,st^2 - 56\,st + 230\,s$.

69. $3\,x^2 - 90\,x + 600$.

70. $6\,b^2c + 126\,bc + 660\,c$.

71. $81\,t^2 + 30\,at^2 + a^2t^2$.

72. $4\,wn^2 - 28\,wn - 240\,w$.

73. $90\,vy - 204\,v + 6\,vy^2$.

74. $6 - 5\,a - a^2$.

88. Trinomials of the Form $ax^2 + bx + c$. — There are different methods of factoring trinomials of the form $ax^2 + bx + c$. Two methods are given here.

First Method. — If any two binomials such as $mx + p$ and $nx + q$ are multiplied, their product equals $mnx^2 + mqx + npx + pq$, which may be written $mnx^2 + (mq + np)x + pq$. This is of the general form $ax^2 + bx + c$.

Thus, $(2\,x + 3)(3\,x + 1) = 6\,x^2 + 11\,x + 3$.

If a trinomial of the general form $ax^2 + bx + c$ has binomial factors of the forms $mx + p$ and $nx + q$, these may be found by trial.

Example 1. — Factor $2\,x^2 + 5\,x + 2$. The first terms of the binomial factors must be factors of $2\,x^2$, and the second terms must be factors of 2. The sum of the products of the first term of each by the second of the other, called the *cross products*, must be $5\,x$. Since all signs of the trinomial are positive, only positive terms for the binomial factors need be sought.

The possible trial sets of factors may be arranged as follows:

$$\begin{array}{r}2\,x + 2\\ \underline{x + 1}\end{array} \qquad\qquad \begin{array}{r}x + 2\\ \underline{2\,x + 1}\end{array}$$

It is seen that the second of these gives the right sum of cross products.

Hence, the factors are $x + 2$ and $2\,x + 1$.

Check. — When $x = 1$, the trinomial is 9, and the factors 3 and 3, respectively; or multiplying the factors, $(x + 2)(2\,x + 1) = 2\,x^2 + 5\,x + 2$.

Example 2. — Factor $3\,w^2 + wv - 10\,v^2$.

Since the third term is negative, the factors of it chosen must have unlike signs.

Four of the many possible trial sets of factors are:

$$\begin{array}{cccc} w-2v & w+2v & w-5v & w+5v \\ \underline{3w+5v} & \underline{3w-5v} & \underline{3w+2v} & \underline{3w-2v} \end{array}$$

It is seen that the second of these gives the right sum of cross products. Hence, $3w^2 + wv - 10v^2 = (w+2v)(3w-5v)$.

Each trial set should be tested as written. If done, it usually will be unnecessary to write down all possible sets. Each set should be written down as above, and the sum of cross products found mentally. In a short time the student should be able to factor most trinomials in very few trials. The following principles systematize the work:

I. *If the trinomial contains no monomial factor, neither factor can contain one.*

Thus, in Example 1, since $2x^2 + 5x + 2$ contains no monomial factor, it was useless to write the first trial set, with the binomial $2x + 2$.

II. *If the last term of the trinomial is +, the last terms of the factors will be both + or both −, according as the middle term of the trinomial is + or −.*

III. *If the last term of the trinomial is −, the last terms of the factors will have unlike signs.*

SECOND METHOD. — In $mnx^2 + mqx + npx + pq$, which is the product of $mx + p$ and $nx + q$, the product of the terms mnx^2 and pq equals the product of the terms mqx and npx. From this we may infer that

If we may separate the second term of a trinomial in the form $ax^2 + bx + c$ *into two terms whose product is equal to that of the first and third terms, we may factor the expression as in* § 85.

EXAMPLE 3. — Factor $3x^2 + 11x + 6$.

$3x^2 \cdot 6 = 18x^2$. Now $18x^2 = 9x \cdot 2x = 18x \cdot x$. The sum of the factors $9x$ and $2x$ is the middle term. Hence we write

$$\begin{aligned} 3x^2 + 11x + 6 &= 3x^2 + 9x + 2x + 6 \\ &= 3x(x+3) + 2(x+3) \\ &= (x+3)(3x+2). \end{aligned}$$

Note that it will be necessary only to use the coefficients in determining how to break up the middle term, as shown in the next example.

Example 4. — Factor $12t^2 - 5tv - 2v^2$.

$12(-2) = -24$. $24 = 3 \times 8 = 6 \times 4 =$ etc. Since the product -24 has a negative sign, the two terms into which $-5tv$ is to be separated must have unlike signs. The required terms are $-8tv$ and $3tv$.

Hence,
$$\begin{aligned} 12t^2 - 5tv - 2v^2 &= 12t^2 - 8tv + 3tv - 2v^2 \\ &= 4t(3t - 2v) + v(3t - 2v) \\ &= (3t - 2v)(4t + v). \end{aligned}$$

EXERCISES

Factor:

1. $10a^2 + 17a + 7$.
2. $3t^2 + 11t + 10$.
3. $3x^2 + 10x + 3$.
4. $5m^2 + 12m + 4$.
5. $2V^2 + 9V + 10$.
6. $3A^2 + 10A + 7$.
7. $8r^2 + 22r + 15$.
8. $3B^2 - 11B + 10$.
9. $10y^2 - 17y + 7$.
10. $3S^2 - 10S + 3$.
11. $5g^2 - 12g + 4$.
12. $15 - 22x + 8x^2$.
13. $2P^2 - 9P + 10$.
14. $5 - 11c + 2c^2$.
15. $4 - 8k + 3k^2$.
16. $4R^2 + 3R - 27$.
17. $6m^2 + 17m - 14$.
18. $3D^2 + 2D - 5$.
19. $3a^2 + a - 4$.
20. $3v^2 + v - 2$.
21. $4H^2 - H - 5$.
22. $4b^2 - 3b - 27$.
23. $6n^2 - 17n - 14$.
24. $3L^2 - 2L - 5$.
25. $3F^2 - F - 4$.
26. $3R^2 - R - 2$.
27. $2a^2x - ax - 28x$.

    (First remove the monomial factor.)
28. $6c^2d - cd - 35d$.
29. $6z^2 - 7z - 20$.
30. $20x^2 - 46x - 84$.
31. $36g^2 - 15g - 6$.
32. $4ay^2 - 10ay - 36a$.
33. $15W^2g - 5Wg - 20g$.
34. $30w^2v^2 - 37wv^2 - 7v^2$.
35. $2Q^2t - 5Qt - 25t$.
36. $4C^2 + 32C + 15$.
37. $4x^2 + 23xy + 15y^2$.
38. $2M^2 + 15MN + 7N^2$.
39. $2d^2 - 3dk + k^2$.
40. $6E^2 - 23EF + 20F^2$

41. $14b^2 + 2c^2 - 11bc$.

42. $7A^2 + 4n^2 - 29An$.

43. $21R^2 + 2Rt - 8t^2$.

44. $2r_1^2 - 5r_1r_2 - 18r_2^2$.

45. $3T^2S - 2TT'S - 5T'^2S$.

46. $12G^2n^2 - 5Gdn^2 - 2d^2n^2$.

47. $4b^2x - 2bdx - 20d^2x$.

48. $12Y^2 - 4YZ - 16Z^2$.

49. $3a^4 - a^2 - 2$.

50. $6w^2t + 15wv^2t - 54v^4t$.

89. Binomials of the Form $a^2 - b^2$. Since the product of $a + b$ and $a - b$ is $a^2 - b^2$, the factors of $a^2 - b^2$ are $a + b$ and $a - b$. Hence,

The factors of the difference between the squares of two numbers are the sum and the difference of the numbers.

Example. — Factor $4t^2 - 25$.

$$4t^2 - 25 = (2t)^2 - (5)^2$$
$$= (2t + 5)(2t - 5).$$

Check. — When $t = 1$, the trinomial is -21, and the factors 7 and -3 respectively. Or, check by finding the product of the factors.

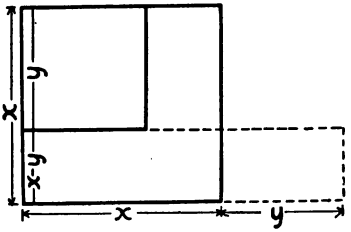
Observe that, by taking the negative square roots of the terms, we might also write $4t^2 - 25 = (-2t)^2 - (-5)^2$

$$= (-2t - 5)(-2t + 5).$$

See § 84 and § 85.

EXERCISES

1. Show by the accompanying diagram that $x^2 - y^2 = (x + y)(x - y)$.



Factor:

2. $N^2 - 4$.
3. $a^2 - 9$.
4. $V^2 - 1$.
5. $t^2 - 16$.
6. $1 - P^2$.
7. $4 - w^2$.
8. $k^2 - 25$.
9. $16 - S^2$.
10. $H^2 - 64$.
11. $49 - m^2$.
12. $100 - v^2$.
13. $x^2 - 81$.
14. $A^2 - 144$.
15. $1 - 16x^2$.
16. $1 - 49t^2$.
17. $9 - 4B^2$.
18. $h^2 - k^2$.
19. $81m^2 - 64n^2$

20. $x^2 - 49y^2$.

21. $16R^2 - 25D^2$.

22. $4c^2 - 25d^2$.

23. $36E^2 - F^2$.

24. $81W^2 - 49t^2$.

25. $C^2 - 100D^2$.

26. $16x^2 - 121y^2$.

27. $50a^2 - 32b^2$. (Remove the monomial factor first.)

28. $x^2y^2 - y^2$.

29. $w^3 - w$.

30. $6at^2 - 24a$.

31. $16m^4 - 36m^2$.

32. $H^2t - 25t^3$.

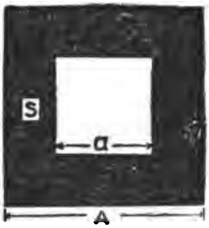
33. $28V^2g^2 - 7g^4$.

34. $x^2y^2 - 1$.

35. $4 - 36a^2t^2$.

36. $200P^2W^2 - 8W^4$.

37. $m^4 - n^4$.



38. $16 - T^4$.

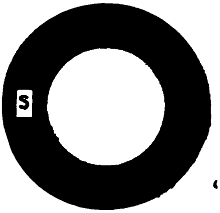
39. $a^4 - 1$.

40. $\pi r_1^2 - \pi r_2^2$.

41. $2\pi R^3 - 8\pi RH^2$.

42. Show that the area of the shaded border in this figure is $S = (A + a)(A - a)$.

43. Show that the area of the ring shown in the figure is $S = \pi(R + r)(R - r)$, where R is the outer radius and r the inner radius of the ring.



44. The cross section of a hollow iron column is 10 inches in outer diameter and 8 inches in inner diameter. Find the area by the formula in Problem 43.

45. The cost of steel construction such as is used in bridges and modern large city buildings is computed at so much per pound of steel used, and hence the weight of steel is always computed.

A hollow steel column is 16 ft. long, 14 in. in outer diameter, and 10 in. in inner diameter. Find its weight, allowing 490 lb. to a cubic foot. Use formula in Problem 43.

46. Find mentally the values of the following:
$28^2 - 22^2$; $37^2 - 33^2$; $49^2 - 41^2$; $76^2 - 74^2$; $97^2 - 93^2$; $116^2 - 114^2$

90. Polynomials which can be Written in the Form $a^2 - b^2$. — Some polynomials, by grouping terms, can be written in the form $a^2 - b^2$, and hence factored by the method of § 89.

Example 1. — Factor $w^2 + 2wv + v^2 - t^2$.

Grouping terms,

$$\begin{aligned} w^2 + 2wv + v^2 - t^2 &= (w^2 + 2wv + v^2) - t^2 \\ &= (w + v)^2 - t^2 \\ &= (w + v + t)(w + v - t). \end{aligned}$$

Check. — When $w = 1$, $v = 1$, $t = 1$, the polynomial is 3, and the factors 3 and 1, respectively.

Example 2. — Factor $A^2 - B^2 + 2BC - C^2$.

$$\begin{aligned} A^2 - B^2 + 2BC - C^2 &= A^2 - (B^2 - 2BC + C^2) \\ &= A^2 - (B - C)^2 \\ &= (A + [B - C])(A - [B - C]) \\ &= (A + B - C)(A - B + C). \end{aligned}$$

Learn to omit all except the first and last steps, and to write out the work as follows:

Example 3. — Factor $x^2 + 6x - m^2 + 4mn - 4n^2 + 9$.

$$\begin{aligned} x^2 + 6x - m^2 + 4mn - 4n^2 + 9 &= (x^2 + 6x + 9) - (m^2 - 4mn + 4n^2) \\ &= (x + 3 + m - 2n)(x + 3 - m + 2n). \end{aligned}$$

EXERCISES

Factor:

1. $(a + b)^2 - 1$.
2. $(x - y)^2 - z^2$.
3. $(S + T)^2 - 25$.
4. $(h - k)^2 - 16t^2$.
5. $(2m + n)^2 - 36$.
6. $1 - (A + B)^2$.
7. $4 - (2w + v)^2$.
8. $25a^2 - (b - c)^2$.
9. $36r_1^2 - (r_2 - 4r_3)^2$.
10. $49V^2 - (3T - S)^2$.
11. $(x + y)^2 - (a + b)^2$.
12. $(M - N)^2 - (W - V)^2$.
13. $(C + D)^2 - (4E - F)^2$.
14. $(1 - x)^2 - (v - t)^2$.
15. $m^2 + 2mn + n^2 - b^2$.
16. $R^2 - 4RA + 4A^2 - B^2$.
17. $1 + 2x + x^2 - y^2$.
18. $r_1^2 + 6r_1r_2 + 9r_2^2 - r_3^2$

19. $x^4+4x^3+4x^2-1.$

20. $W^2+9D^2-4M^2-6WD.$

21. $1-8x+16x^2-z^2.$

22. $1-a^2-2ab-b^2.$

23. $4-M^2+6MV-9V^2.$

24. $1-h^2+14hk-49k^2.$

25. $A^2-9B^2-30BC-25C^2.$

26. $16U^2+36VW-4V^2-81W^2.$

27. $S^2-T^2+4aT-4a^2.$

28. $12pq+16r^2-4p^2-9q^2.$

29. $A^2-4C^2-8AB+16B^2.$

30. $6tm-t^2-9m^2+1.$

31. $a^2+2ab+b^2-c^2-2cd-d^2.$

32. $x^2-4x+4-m^2+6mn-9n^2.$

33. $1+2xy+2p-y^2-x^2+p^2.$

34. $W^2+12at-10Wg-4a^2+25g^2-9t^2.$

35. $M^2-12dM-4kN+36d^2-4N^2-k^2.$

36. $x^4-E^2+10Em^2-16x^2n-25m^4+64n^2.$

91. Binomials of the Form a^3-b^3. — From § 81 it follows that the factors of a^3-b^3 are $a-b$ and a^2+ab+b^2. Test these factors by actual multiplication. Hence,

One factor of the difference between the cubes of two terms is the difference between the terms, and the other factor is the sum of their squares and their product.

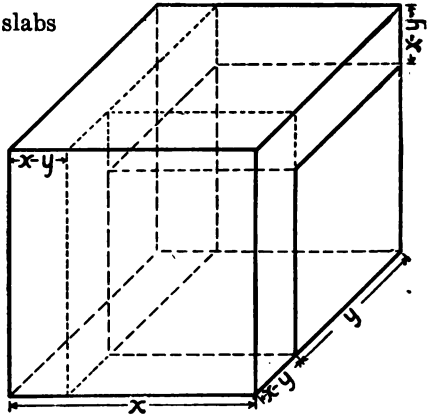
EXAMPLE. — Factor $8m^3-125n^3$.

$$\begin{aligned}8m^3-125n^3&=(2m)^3-(5n)^3\\&=(2m-5n)(4m^2+10mn+25n^2).\end{aligned}$$

Check. — When $m=1$ and $n=1$, the binomial is -117, and the factors -3 and 39, respectively.

EXERCISES

1. Show by the accompanying diagram that $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$. (If y^3 is taken from x^3, it leaves three slabs each $x - y$ thick.)



Factor:

2. $m^3 - n^3$.
3. $x^3 - 8$.
4. $A^3 - 1$.
5. $1 - P^3$.
6. $8 - r^3$.
7. $27 - t^3$.
8. $8w^3 - v^3$.
9. $T^3 - 27U^3$.
10. $64y^3 - 1$.
11. $M^3 - 125N^3$.
12. $8r_1^3 - 27r_2^3$.
13. $64g^3 - 27a^3$.
14. $216V^3 - 125V_1^3$.
15. $a^3b^3 - 1$.
16. $27x^4 - x$.
17. $125a^2 - a^5$.
18. $2Pt^3 - 16P^4$.
19. $81y^4 - 24y$.
20. $m^3 - n^6$.
21. $8A^6 - 27$.
22. $v^6 - 64w^3$.
23. $125 - Z^{12}$.
24. $27R^9 - V^3$.
25. $4\pi r^4 - 32\pi rh^3$.

26. The volume of a sphere whose diameter is D is $\frac{1}{6}\pi D^3$. Show that the volume of a spherical *shell* whose outer diameter is D and inner diameter d, is $V = \frac{1}{6}\pi(D - d)(D^2 + Dd + d^2)$.

27. Compute, by the formula in Problem 26, the volumes of spherical shells whose respective outer and inner diameters are in the following table.

D	4 in.	6 in.	12 in.	15 in.
d	3 in.	4 in.	10 in.	12 in.

92. Binomials of the Form $a^3 + b^3$. — From § 82 it follows that the factors of $a^3 + b^3$ are $a + b$ and $a^2 - ab + b^2$. Test these factors by actual multiplication. Hence,

One factor of the sum of the cubes of two terms is the sum of the terms, and the other factor is the sum of their squares minus their product.

EXAMPLE. — Factor $27\,W^3 + 64\,V^3$.

$$27\,W^3 + 64\,V^3 = (3\,W)^3 + (4\,V)^3$$
$$= (3\,W + 4\,V)(9\,W^2 - 12\,WV + 16\,V^2).$$

Check. — When $W = 1$ and $V = 1$, the binomial is 91 and the factors 7 and 13, respectively.

EXERCISES

Factor:

1. $m^3 + n^3$.
2. $A^3 + 1$.
3. $1 + p^3$.
4. $w^3 + 8$.
5. $H^3 + 64$.
6. $27 + v^3$.
7. $125 + t^3$.
8. $D^3 + 216$.
9. $1 + 8\,a^3$.
10. $1 + 27\,x^3$.
11. $64\,N^3 + 1$.
12. $8\,a^3 + b^3$.
13. $y^3 + 27\,z^3$.
14. $W^3 + 64\,D^3$.
15. $27\,S^3 + 8\,T^3$.
16. $125\,r_1^3 + 27\,r_2^3$.
17. $64\,a^3 + 27\,t^3$.
18. $16\,x^3 + 54\,y^3$.
19. $a^3b^3 + 1$.
20. $27 + m^3n^3$.
21. $27\,P^2 + P^5$.
22. $8\,x^4 + 125\,x$.
23. $81\,D^4 + 24\,D$.
24. $v^3 + t^6$.
25. $8\,N^6 + 1$.
26. $40\,a^6 + 135\,b^3$.
27. $\pi D^3 + \pi d^3$.
28. $54\,R_1^9 + 2\,R_2^3$.
29. $3\pi r^4 + 81\,\pi r h^3$.
30. $w^6 + v^6$.

93. General Suggestions on Factoring. — As shown in the preceding sections, the method of factoring any given expression depends upon the special form of the expression. There are no general rules or principles that can be laid down for factoring all expressions. To factor any expression, wherever the need for it is encountered, requires first that *the expression be identified as having some special form,* such as one of those discussed in this chapter, and second *the application of the special rule for factoring expressions of that form.* The following miscellaneous exercises are given for practice in identifying the special forms of expressions to be factored.

General Suggestions:

I. *In factoring any expression, first see if there is a monomial factor; and if there is one, separate the expression into this monomial and the corresponding polynomial factor.*

II. *Then try to identify the polynomial factor as having one of the special forms discussed in the preceding sections; and if successful, factor it by the appropriate rule.*

III. *Repeat the process with each factor obtained, until the prime factors of the given expression are found. Write the given expression as the indicated product of all of the prime factors.*

MISCELLANEOUS EXERCISES

Factor:

1. $A^5 + 3A^3 + A^2$.
2. $2x^3y + 6x^2y^2 + 4xy^3$.
3. $4w^2 + 12wv + 9v^2$.
4. $m^2n + 6mn + 9n$.
5. $P^2 + 8P + 12$.
6. $t^2 - 8t + 12$.
7. $R^2 + 4R - 12$.
8. $n^2 - 4n - 12$.
9. $a^3 + 8a^2 + 15a$.
10. $M^3 - 8M^2 + 15M$.
11. $d^3 + 2d^2 - 15d$.
12. $v^2 - v^4$.
13. $5x^3y - 10x^2y^2 + 5xy^3$.
14. $a - a^4$.
15. $D^3 - 2D^2 - 15D$.
16. $M^2 + 13M + 42$.
17. $42 - 13y + y^2$.
18. $W^2 - 42 - W$.
19. $a^2 - 42 + a$.
20. $t^3 - 4t^2 + 4t$.
21. $8 - B^3$.
22. $x^3 + 2x^2 - 48x$.
23. $x^3 - 2x^2 - 48x$.
24. $W^2 - 9Wg + 20g^2$.
25. $W^2 + 9Wg + 20g^2$.
26. $H^4 - 16t^4$.
27. $a^3 + a^2 + 8a + 8$.
28. $1 - 4m^2 + 4mv - v^2$.
29. $x^2y + 3xy - 28y$.
30. $72 + N^2 + 17N$.

31. $17\,at + 72\,t + a^2t.$
32. $A^2 + 5\,A - 50.$
33. $A^2 - 5\,A - 50.$
34. $2\,x^2 + 3\,x + 1.$
35. $t^3 - 125.$
36. $a^2z^2 - b^2w^2.$
37. $144\,L^2 - M^4.$
38. $ax - bx - a + b.$
39. $v^2 - 15\,v + 50.$
40. $x^2y^2 - 10\,xy - 56.$
41. $2\,a^2 - 3\,a + 1.$
42. $Q^4t^6 - 9\,V^2.$
43. $xy - x - y + 1.$
44. $m^3 + n^6.$
45. $2\,A^2 + 5\,AB + 2\,B^2.$
46. $v^3 + 8.$
47. $w^2v^2 + 5\,wvt + 6\,t^2.$
48. $D^4 - 7\,D^2 + 10.$
49. $x^4y^4 + 7\,x^2y^2 + 10.$
50. $V^4 + 11\,V^2 - 26.$
51. $6\,a^2 + 11\,a + 3.$
52. $12\,w^2 - 23\,w + 10.$
53. $27 + T^3.$
54. $36\,M^2 - 64\,N^2M^2.$
55. $3\,x^2 + 7\,xy + 2\,y^2.$
56. $81\,B^4 - 16\,C^4.$
57. $ab - 3\,a - 2\,b + 6.$
58. $2\,m^2 - 9\,mn + 10\,n^2.$
59. $V^3 + V^2 - 42\,V.$
60. $x^2 + 2\,xy - 35\,y^2.$
61. $a^4 - (a - 6)^2.$
62. $p^2q^2 - p^2 - q^2 + 1.$
63. $7\,w^2 + 36\,w + 5.$
64. $2\,A^2 + 5\,A - 3.$
65. $2\,A^2 - 5\,A + 3.$
66. $a^2 + 4\,ab - c^2 + 4\,b^2.$
67. $t^3 + 125.$
68. $5\,m^2 - 9\,mn - 2\,n^2.$
69. $81\,y^3 - 3.$
70. $12\,H^2 + 5\,H - 2.$
71. $216 + v^3.$
72. $3\,abx^2 + 2\,abx - ab.$
73. $8\,D^3 + 1.$
74. $6\,bc - 11\,st + 5\,t^2.$
75. $81\,x^4 - 100\,v^4.$
76. $N^2 + aN - 2\,N - 2\,a.$
77. $4\,x^2 - (a + b)^2.$
78. $27\,E^3 + 1.$
79. $4\,A^2 + 32\,AB + 39\,B^2.$
80. $125\,v^3 + t^3.$
81. $4\,(a - b)^2 - (y + z)^2.$
82. $8\,B^2 + 15\,B - 2.$
83. ${r_1}^2 + 6\,r_1r_2 - 1 - 4\,r_3 - 4\,{r_3}^2 + 9\,{r_2}^2.$
84. $40\,M^3 + 135\,V^3.$
85. $H^2 - t^2 + vH - vt.$
86. $3\,x^2 + x - 2.$

87. $A^3 + A^2n - A - n$.
88. $15 + 2z - z^2$.
89. $xy + z^2 - xz - yz$.
90. $9x^2 - 27x + 20$.
91. $D^3 - 216$.
92. $W^6 - V^6$.
93. $mn^2 + 7mn - 30m$.
94. $1 + 4a^2t^2 - a^4 - 4t^4$.
95. $9a^2 + 6bd - d^2 - 9b^2$.
96. $v^3 - 216t^6$.
97. $5Q^2 - 79Qt - 16t^2$.
98. $x + x^4 - ax^3 - a$.
99. $v^3 + v^2w + vw^2 + w^3$.
100. $R^2 + 2R - Rr - 2r$.

HIGHEST COMMON FACTOR AND LOWEST COMMON MULTIPLE

94. Highest Common Factor. — An expression that is a factor of each of two or more expressions is called a **common factor** of those expressions. The expression with the greatest numerical coefficient and the highest powers of its literal factors that is a common factor of two or more expressions is called their **highest common factor** (H.C.F.).

We shall consider here only the highest common factors of expressions whose factors may be obtained by the methods of this chapter.

EXAMPLE 1. — Find the H.C.F. of $8A^3BC^4$ and $12A^4B^2C^2$.

The largest number contained in both 8 and 12 is 4.
The highest power of A contained in both A^3 and A^4 is A^3.
The highest power of B contained in both B and B^2 is B.
The highest power of C contained in both C^4 and C^2 is C^2.
Hence, the H.C.F. is $4A^3BC^2$.

EXAMPLE 2. — Find the H. C. F. of $x^3 - x^2y - xy^2 + y^3$ and $2x^3 - 4x^2y + 2xy^2$.

$$x^3 - x^2y - xy^2 + y^3 = (x - y)^2(x + y).$$
$$2x^3 - 4x^2y + 2xy^2 = 2x(x - y)^2.$$

Hence, $\text{H.C.F.} = (x - y)^2$, or $x^2 - 2xy + y^2$.

To find the highest common factor of two or more monomials, take the product of the greatest common divisor of their numerical coefficients and each letter raised to the lowest power to which it appears in any of the monomials.

In polynomials, factor each expression and proceed as with monomials, treating each factor as you would a single letter.

EXERCISES

Find the highest common factor of:

1. $4a^3b^2$, $6a^2b^4$.
2. $9a^4b^3c^2$, $12a^3b^4c^5$.
3. $15W^7V^6$, $20W^4V^8U^2$.
4. $8mn^4p$, $96m^3np^2$.
5. $16M^5N^5$, $24M^2N^6P^2$.
6. $2x^{10}y^9z^2$, y^8z^8.
7. $49wv^4t^2s$, $21w^4v^3t^3u$.
8. $35C^7DP^5$, $-56CD^2P^3$.
9. $24m^2np^4$, $42mp^3$, $18m^3p^2q$.
10. $12(v+t)^2$, $8(v+t)^3$.
11. $14(E-F)^6$, $21(E-F)^4$.
12. $6(x-1)^2$, $8(x-1)(x-2)$.
13. $10(y+1)^3$, $4(y+1)^2(y-1)$.
14. $39(m-n)^2(m+n)^3$, $26(m-n)(m+n)^4$.
15. A^2-1, A^2+2A-3.
16. $1-t^2$, $1-t^3$, $1-t^4$.
17. $V^2+2V-15$, V^2-4V+3.
18. a^2-b^2, a^3-b^3, $a^2-3ab+2b^2$.
19. $2x^3+4x$, $4x^3+12x^2+8x$.
20. R^2-3R, $9-R^2$, R^2-5R+6.
21. w^2-1, w^3+1, $w+1$.
22. $2a^2+a-6$, $6a^2-7a-3$.
23. $3r_1r_2^3-3r_1^3r_2$, $4r_1^2r_2^3+2r_1^3r_2^2-6r_1^4r_2$.
24. A^5-AB^4, $A^3+A^2B+AB+B^2$.
25. $6m^3n+14m^2n^2+4mn^3$, $4m^4+16m^3n+16m^2n^2$.

95. Lowest Common Multiple. — A **multiple** of a given expression is any expression of which the given expression is a factor. A **common multiple** of two or more expressions is an expression that is a multiple of each. The **lowest common multiple** (L. C. M.) of two or more given expressions is the expression with the least numerical coefficient and the lowest powers of its literal factors that is a common multiple of the given expressions.

Example 1. — Find the L. C. M. of $24\,A^2B^3C^4$ and $80\,A^4B^2C^2$.
The least number that will contain both 24 and 30 is 120.
The lowest power of A of which A^2 and A^4 are factors is A^4.
The lowest power of B of which B^3 and B^2 are factors is B^3.
The lowest power of C of which C^4 and C^2 are factors is C^4.
Hence, the L. C. M. is $120\,A^4B^3C^4$.

Example 2. — Find the L. C. M. of $2\,v^2 - 4\,v + 2$ and $8\,v^2 + 8\,v - 6$.

$$\begin{aligned} 2\,v^2 - 4\,v + 2 &= 2(v-1)^2. \\ 3\,v^2 + 3\,v - 6 &= 3(v+2)(v-1). \\ \text{Hence, L. C. M.} &= 6(v-1)^2(v+2). \end{aligned}$$

To find the lowest common multiple of two or more monomials, take the product of the least common multiple of the numerical coefficients, and each letter raised to the highest power to which it appears in any one of the monomials.

In polynomials, factor each expression, and proceed as with monomials, treating each factor as you would a single letter.

EXERCISES

Find the lowest common multiple of:

1. $2\,a^2b,\ 3\,ab^3$.
2. $6\,x^3y^2,\ 10\,xy^4$.
3. $8\,w^6vu^3,\ 12\,wv^2$.
4. $81\,M^2N^5,\ 18\,MN^6$.
5. $7\,pq^3r,\ 3p^2q,\ 6\,p^3r^2$.
6. $21\,A^5BC^3, 4\,A^3BC^2,\ 6\,AB^3$.
7. $25\,m^2n^3,\ 4\,amn^2,\ 10\,a^2mn^2$.
8. $15\,R^5r^5,\ 6\,R^2r^3,\ 4\,R^3r^2$.
9. $(x+y)^3,\ 2(x+y)^2$.
10. $(A-B)^3, 3(A-B)(A+B)^2$.
11. $n(n-1)(n+1),\ 2\,n^2(n+1)$.
12. $t^3(1+t)^2(1-t),\ (1+t)(1-t)$.
13. $S^2-1,\ S^2-2\,S+1$.
14. $D^2-1,\ D^2-D-2$.
15. $1-y^2,\ y+y^2-2\,y^3$.
16. $P^2-Q^2,\ 2\,P^2+3\,PQ+Q^2$.
17. $t^2+5t+6,\ t^2+7\,t+12$.
18. $v^2+v-30,\ v^2+5\,v-6$.
19. $2\,A^2+3\,A-2,\ 3\,A^2+7\,A+2$.
20. $2\,x^2-x-10,\ 2\,x^2+x-3$.
21. $2\,N^2+2\,N,\ N^2+5\,N+6$.
22. $k^3-1,\ k^3-k,\ 2\,k^2$.

23. $P^4 - 1,\ P^2 + 3P + 2,\ P^2 + P - 2.$

24. $1 - d^2,\ 1 - d^3,\ 1 - d.$

25. $z^2 - z^4,\ 2z^2 - 4z,\ z^2 + 1.$

26. $B^2 + 5B - 14,\ 4B^3 - 16B^2 + 16B,\ 3B^2.$

27. $a^2 - b^2,\ a^3 - b^3,\ a^4 - b^4.$

28. $a^2 + ab + ac + bc,\ a^2 + 2ab + b^2.$

29. $x^2 - y^2,\ x^2 - xy + xv - yv,\ x^2 + xy + yv + xv.$

30. $A^4 - B^4,\ A^4 + 2A^2B^2 + B^4,\ A^4 - 2A^2B^2 + B^4.$

EQUATIONS SOLVED BY FACTORING

96. Equations solved by Factoring. — One important use of factoring is in the solution of equations of the second or higher degrees. In solving an equation by factoring, use is made of the principle that:

If one of the factors of an expression is zero, the whole expression must be zero.

Example. — Solve $n^2 + n = 6$.

Transposing, $n^2 + n - 6 = 0$.

Factoring, $(n - 2)(n + 3) = 0$.

Now this equation is satisfied by any value of n that will make either of the factors $n - 2$ or $n + 3$ zero, since then their product must be zero. Such values of n are obtained by setting each factor equal to zero, and solving the resulting linear equations.

Setting each factor equal to zero,

$$n - 2 = 0;\ \text{whence } n = 2.$$
$$n + 3 = 0;\ \text{whence } n = -3.$$

Check. — When $n = 2$, the equation becomes $4 + 2 = 6$.

When $n = -3$, the equation becomes $9 - 3 = 6$.

To solve an equation by factoring:

(1) *Transpose all terms to the first member, making the second member* 0.

(2) *Factor the resulting first member.*

(3) *Set each of the factors equal to* 0, *and solve the resulting equations.*

It will be found that an equation of the second degree has two roots, an equation of the third degree three roots, etc.

Note.—Care must be exercised not to divide both members of an equation by an expression containing the unknown number, because in so doing roots are lost.

EXERCISES

Solve:

1. $n^2 - 6n + 8 = 0$.
2. $b^2 + 4b = 12$.
3. $x^2 - 9 = 0$.
4. $P^2 = 16$.
5. $4a^2 - 25 = 0$.
6. $9t^2 - 49 = 0$.
7. $S^2 = 64$.
8. $4 - r^2 = 0$.
9. $5B^2 + 6 = 4B^2 + 7$.
10. $7d^2 - 28 = 0$.
11. $A(A + 4) = 4A + 49$.
12. $c^2 = 8c + 9$.
13. $t^2 = 22 + 9t$.
14. $V^2 - 15V = 54$.
15. $D^2 + 12D + 20 = 0$.
16. $p^2 = 6 - p$.
17. $W^2 + 14W + 40 = 0$.
18. $h^2 - 11h + 30 = 0$.
19. $N^2 + 5N = 14$.
20. $y^2 + 12 = 7y$.
21. $10 = 3A + A^2$.
22. $6k = 9 + k^2$.
23. $Q^2 = 3Q + 54$.
24. $2x^2 + 2 = 5x$.
25. $2p^2 + 5p = 3$.
26. $5B^2 = 1 + 4B$.
27. $21w^2 = 10 + 29w$.
28. $x^3 - x = 0$.
29. $m^3 + 2m^2 = 3m$.
30. $V^3 = 3V^2 + 10V$.
31. $4t^3 + 10t^2 + 4t = 0$.
32. $a^3 + a^2 = 9a + 9$.
33. $R^3 + 4 = R^2 + 4R$.
34. $3p^3 - p^2 = 27p - 9$.
35. $32 + x^3 = 16x + 2x^2$.
36. $N^4 - 5N^2 + 4 = 0$.
37. $v^4 = 9v^2$.
38. $A^4 + 12A^2 = 7A^3$.
39. $t^4 - 18t^2 + 81 = 0$.
40. $2D^4 + 18 = 20D^2$.

41. Explain the fallacy in the following reasoning, by which it is shown that $2 = 1.$

Let $n = 1.$
Multiplying by n, $n^2 = n.$
Hence, $n^2 - 1 = n - 1.$
Dividing by $n - 1$, $n + 1 = 1.$
Substituting 1 for n, $2 = 1.$

42. If to the square of a certain number five times the number is added, the sum is 104. Find the number.

43. Find two consecutive whole numbers whose product is 182.

44. The sum of the squares of two consecutive odd numbers is 202. Find the numbers.

45. The sum of the squares of three consecutive numbers is 194. Find the numbers.

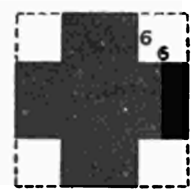
46. The area of a triangle is 144 sq. in., and the altitude is twice as long as the base. Find the base and altitude.

47. The base of a triangle is 7 ft. longer than the altitude, and the area is 60 sq. ft. Find the base and altitude.

48. The base of a right triangle is 3 ft. greater than the altitude, and the hypotenuse is 15 ft. Find the length of the base and altitude.

49. One base of a trapezoid is 2 in. greater than the altitude, and the other base is 4 in. greater than the altitude. The area is 54 sq. in. Find the bases and the altitude.

50. The altitude of a trapezoid equals one base and exceeds the other base by 4 in. The area is 120 sq. in. Find the bases and altitude.



51. A square box without a lid and 6 in. deep is to be made from a square piece of tin by cutting out a square from each corner, bending up the sides and soldering along the edges. The box is to hold 384 cu. in Find how large to cut the piece of tin from which to make the box.

52. An open box, to be 4 in. deep, twice as long as wide, and to hold 512 cu. in., is to be made from a rectangular piece of cardboard by cutting out a square from each corner, bending up the sides and pasting along the edges. Find how large to cut the cardboard from which to make it.

53. A farmer has a field 60 rd. wide and 80 rd. long, which he is plowing to sow with wheat. How wide a strip must he plow around the field in order to have it half done?

54. Two boys have a lawn to mow that is 60 ft. long and 32 ft. wide. The first boy is to mow half of it by cutting a strip of uniform width around it, and the other is to finish it. How wide a strip must the first boy cut?

55. By measuring the side of a room 15 ft. square, a man found that he got an area 3500 sq. in. too small. Find the amount of his error in measuring the side of the room.

SUGGESTION. — If e inches is the error, show that $e^2 - 360e + 3500 = 0$.

56. A room is 10 ft. square. What error (in inches) in measuring the side would make the computed area 476 sq. in. too small?

57. In the room in Problem 56, what error in measuring the side would make the computed area 729 sq. in. too large?

58. The horse power of a gasoline engine is computed by the formula $H = \frac{D^2N}{2.5}$, where H = horse power, D = diameter of cylinders in inches, and N = number of cylinders. In an engine of one cylinder, find the diameter of the cylinder required to yield 40 horse power.

59. In a gasoline engine of 4 cylinders, find the diameter of the cylinders required to yield 160 horse power.

60. From a group of n children two leaders for a game may be selected in $n(n-1)$ ways. If the number of ways in which the two leaders may be chosen is 240, how many children are there in the group?

SUPPLEMENTARY EXERCISES

Factor:

1. $4A^2B^3 - 8B^5$.
2. $ax^n + bx^n$.
3. $v^n - wv^n$.
4. $a^mt + a^{m-1}t^2$.
5. $P^{2x} - 5P^xQ^x$.
6. $r^{n-1}s^{n+1} + r^{n+1}s^{n-1}$.
7. $5v^ay^{a-1} - 10v^{a-1}y^a$.
8. $8M^{2n+1} + 12M^{2n}N$.
9. $a^2b^2 - (a-b)^2$.
10. $m^2 - n^2 + mx - nx$.

Suggestion. — Group and factor the first two terms, then group and factor the last two terms.

11. $V^2 - W^2 + tV - tW$.
12. $x - a + (x-a)^2$.
13. $x^2 - x - y^2 + y$.
14. $m^4n^4 - m^4 - n^4 + 1$.
15. $r_1^4 - r_2^4 - (r_1^2 - r_2^2)^2$.
16. $At^2 + At + Bt^2 + Bt + A + B$.

Suggestion. — Group the first, second, and fifth terms; also group the third, fourth, and sixth terms. One factor is a trinomial.

17. $2a - 2b + ad - bd + 2c + cd$.
18. $ny^2 + 3ny + 5y^2 + 15y + 2n + 10$.
19. $p^3 + 4p^2 + 4p - p^2q - 4pq - 4q$.
20. $x^2 - y^2 + z^2 - w^2 - 2(xz - yw)$.
21. $W^4 + W^2 + V^2 - V^4$.
22. $A^3 + A^2C + ABC + B^2C - B^3$.

Suggestion. — Group the first and last terms together, and the other three terms together.

23. $4m^2n^2 - (m^2 + n^2 - p^2)^2$.
24. $a(a+c) - b(b+c)$.
25. $w^2 - v^2 + u(u - 2w)$.
26. $x^{2n} - 2x^ny^n + y^{2n}$.
27. $a^n - 6a^nt^n + 9a^nt^{2n}$.
28. $6P - 9P^2 - 1$.

Suggestion. — First remove the monomial factor -1.

29. $-t^4 + 8t^2 - 16$.
30. $-x^2 + 2xy^2 - y^4$.
31. $-4M^2 - 12MN - 9N^2$.
32. $(a+b)^2 + 7(a+b) + 10$.
33. $(x-y)^2 - 6(x-y) - 40$.
34. $ax^2 + (a+b)x + b$.
35. $ax^2 + (b-a)x - b$.

36. $2y^2+(4a+b)y+2ab$.

37. $2W^2-(2m+n)W+mn$.

38. $av^2+(ab+1)v+b$.

39. $x^{2n}-y^{2n}$.

40. $4A^{4n}-9B^{6n}$.

41. $w^{3n}+v^{3n}$.

42. $1-Q^{3n}$.

43. x^6-y^6.

Suggestion. — First factor as the difference of two squares, then factor the factors.

$x^6-y^6=(x^3-y^3)(x^3+y^3)=$ etc.

44. $1-n^6$.

45. D^6-1.

46. a^6-64b^6.

47. $R^{12}-r^{12}$.

Solve the following equations for x:

48. $x^2-3ax+2a^2=0$.

49. $n^2x^2-n^4=0$.

50. $4vx=x^2+4v^2$.

51. $2x^2-3t^2=5xt$.

52. $x^2-w^2=2wv+v^2$.

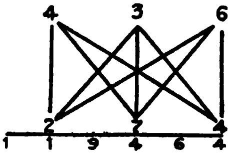
53. $x^2+9+4ab=6x+a^2+4b^2$.

54. $x^2=2rx+63r^2$.

55. $x^3+nx^2=b^2x+b^2n$.

56. Any two arithmetical numbers with three figures each may be written in the polynomial forms $100C+10B+A$ and $100c+10b+a$, respectively. Show that their product is
$10000\,Cc+1000(Cb+cB)+100(Ca+Bb+Ac)+10(Ab+aB)+Aa$.

From the form of this expression may be deduced a process, called "cross-multiplication," for finding the product of two arithmetical numbers in an abbreviated form. The process is applicable to numbers with any number of figures. By this process practically all of the work in multiplication is done without the aid of pencil, but it requires great mental concentration.



Thus, to find 274 × 436 the work is done mentally as follows, and only the answer written: To get the ones, take 4 × 6 = 24 and write the 4; to get the tens, take 2 + 4 × 3 + 6 × 7 = 56 and write the 6; to get the hundreds, take 5 + 4 × 4 + 6 × 2 + 7 × 3 = 54 and write the 4; then take 5 + 7 × 4 + 3 × 2 = 39, and write the 9; then take 3 + 2 × 4 = 11.

57. Find, by "cross-multiplication," the products of: 34 × 52; 61 × 36; 47 × 54; 67 × 62; 52 × 83; 245 × 326; 318 × 241; 234 × 571; 384 × 615; 724 × 318; 27 × 356 (same as 027 × 356)

CHAPTER IX

FRACTIONS

97. Algebraic Fractions. — The indicated quotient of two numbers or algebraic expressions, as $\frac{m}{n}$ or $\frac{a^2 - b^2}{a^2 + 2ab + b^2}$, is called a **fraction.** In algebra, as in arithmetic, the dividend is called the **numerator,** and the divisor the **denominator,** of the fraction. The numerator and denominator are called the **terms** of the fraction.

The fraction $\frac{2x - 1}{x^2 + x + 1}$ is read either "$2x - 1$ divided by $x^2 + x + 1$" or, more briefly, "$2x - 1$ over $x^2 + x + 1$."

98. Signs of a Fraction. — In an algebraic fraction there are three signs involved: (1) the *sign before the fraction,* which is the plus or minus sign written (or understood) before the line separating the numerator from the denominator; (2) the *sign of the numerator;* (3) the *sign of the denominator.*

Since a fraction is an indicated quotient, it follows from the laws of signs in division that $\frac{+a}{+b}$ and $\frac{-a}{-b}$ are both positive and are equal, and that $\frac{-a}{+b}$ and $\frac{+a}{-b}$ are both negative and are equal.

Hence,
$$\frac{+a}{+b} = \frac{-a}{-b} = -\frac{-a}{+b} = -\frac{+a}{-b}.$$

It follows that to preserve the algebraic value of a fraction

(1) *If the signs of both terms of the fraction are changed, the sign before the fraction must be left unchanged.*

(2) *If the sign of only one term of the fraction is changed, the sign before the fraction must be changed.*

Thus, $\frac{+12}{+4} = \frac{-12}{-4} = -\frac{-12}{+4} = -\frac{+12}{-4} = 3.$

If the numerator or denominator be a polynomial, its sign is changed by changing the sign of each of its terms, because this is equivalent to multiplying the polynomial by -1.

Thus, $$\frac{x^2-4x+2}{-x^2+3x-1} = \frac{-x^2+4x-2}{x^2-3x+1} \text{ or } -\frac{x^2-4x+2}{x^2-3x+1}.$$

EXERCISES

Write with positive numerators and denominators:

1. $\frac{-3}{8}$.
2. $\frac{-2}{5}$.
3. $\frac{4}{-9}$.
4. $\frac{10}{-12}$.
5. $\frac{-6}{-7}$.
6. $\frac{-3x}{-5y}$.
7. $\frac{-2a^2}{10b^3}$.
8. $\frac{7n}{-9t^2}$.
9. $\frac{-1}{-V^2}$.
10. $\frac{d}{-M^2}$.
11. $\frac{-100}{P^4}$.
12. $\frac{-at^2}{-vg^2}$.

Write with denominators $a-b$:

13. $\frac{1}{b-a}$.
14. $\frac{-x}{b-a}$.
15. $\frac{m-n}{b-a}$.

In each of the following expressions write all of the fractions with the same denominator:

16. $\frac{6}{x-y} + \frac{2}{y-x}$.
17. $\frac{a}{a-b} + \frac{b}{b-a} - \frac{a+b}{b-a}$.
18. $\frac{2x+1}{x^2-1} - \frac{3x}{1-x^2} + \frac{1-x}{1-x^2}$.
19. $\frac{5at}{a^2-t^2} - \frac{a^2}{t^2-a^2} + \frac{a^2+t^2}{a^2-t^2}$.
20. $\frac{1+r}{1-r} - \frac{r}{r-1} + \frac{1}{r-1}$.

Write for each of the following an equal fraction with positive denominator and the sign + before the fraction:

21. $-\frac{x}{-y}$. 23. $-\frac{2x}{-3y}$. 25. $-\frac{-1}{-A}$. 27. $\frac{\pi r^2}{-\pi r^3}$.

22. $\frac{w}{-v}$. 24. $-\frac{-R}{-N}$. 26. $\frac{-5}{-n}$. 28. $-\frac{-vg^2}{-100}$.

99. Reduction of Fractions to Lowest Terms. — As in arithmetic, *to change a fraction to lowest terms* is to change it to a fraction with equal value, but whose numerator and denominator have no common factor.

Thus, $\frac{6}{8}=\frac{2\times 3}{2\times 4}=\frac{3}{4}.\quad \frac{9}{18}=?\quad \frac{16}{24}=?\quad \frac{36}{100}=?$

The principle involved in the process is that:

Dividing both terms of a fraction by the same expression does not change the value of the fraction.

Example 1. — Reduce $\frac{12a^2b^6}{18a^3b^4}$ to lowest terms.

Dividing both terms by $6a^2b^4$, their highest common factor,

$$\frac{12a^2b^6}{18a^3b^4}=\frac{2b^2}{3a}.$$

Example 2. — Reduce $\frac{x^2+2x-3}{x^2+5x+6}$ to lowest terms.

Factoring both terms, and dividing by $x+3$, their H. C. F.,

$$\frac{x^2+2x-3}{x^2+5x+6}=\frac{(x+3)(x-1)}{(x+3)(x+2)}=\frac{x-1}{x+2}.$$

It is evident that:

To reduce an algebraic fraction to its lowest terms, factor the numerator and denominator, and divide both terms by their highest common factor.

The division may be indicated by *cancellation* of the common factors of the terms.

Thus, $\frac{am^2 - an^2}{am^3 - an^3} = \frac{\cancel{a}\cancel{(m-n)}(m+n)}{\cancel{a}\cancel{(m-n)}(m^2+mn+n^2)} = \frac{m+n}{m^2+mn+n^2}.$

NOTE.—It must be remembered in using cancellation that only *factors* common to the numerator and denominator may be cancelled. *Terms* in polynomial numerators and denominators cannot be cancelled—a common mistake made by students. Thus, in $\frac{3+5}{4+5}$ the 5's cannot be cancelled, because the fraction equals $\frac{8}{9}$, and if the 5's were cancelled, it would become $\frac{3}{4}$. Similarly, in $\frac{x^2+1}{x^2+4}$ no cancellation is possible.

EXERCISES

Reduce to lowest terms:

1. $\frac{10}{12}$.
2. $\frac{9}{12}$.
3. $\frac{15}{18}$.
4. $\frac{21}{28}$.
5. $\frac{12}{20}$.
6. $\frac{36}{96}$.
7. $\frac{42}{54}$.
8. $\frac{72}{100}$.
9. $\frac{240}{800}$.
10. $\frac{60}{720}$.
11. $\frac{ab}{ac}$.
12. $\frac{3x}{6x^2}$.
13. $\frac{4m^2n}{6mn^2}$.
14. $\frac{12p^4q^2}{16p^3q^5}$.
15. $\frac{4abc}{6a^2bd}$.
16. $\frac{5xyz^2}{10yz}$.
17. $\frac{3w^2v^3}{12wv^4}$.
18. $\frac{7rst^2}{21r^2st}$.
19. $\frac{8Mv^2}{36v^5}$.
20. $\frac{4\pi R^2}{\pi R^3}$.
21. $\frac{4\pi R(R+H)}{8\pi R^2(R+H)}$.
22. $\frac{a^2-b^2}{a^2+2ab+b^2}$.
23. $\frac{xn-xm}{n^2-m^2}$.
24. $\frac{v^2t-t^3}{v^4-t^4}$.
25. $\frac{A^3-AB^2}{A^2+2AB+B^2}$.
26. $\frac{R^2-1}{2Ry+2y}$.
27. $\frac{t^2+2t-15}{t^2-t-6}$.
28. $\frac{1-p^2}{1-2p+p^2}$.
29. $\frac{4n^2+4n+1}{4n^2-1}$.
30. $\frac{a^2-b^2}{a^3-b^3}$.
31. $\frac{1-x}{1-x^3}$.
32. $\frac{v^2-w^2}{(v+w)^2}$.

33. $\dfrac{V^2+8V+12}{V^3+6V^2}$.

34. $\dfrac{ab+b^2}{ac^2+bc^2}$.

35. $\dfrac{M^5-M^3N^2}{M^4-N^4}$.

36. $\dfrac{r^2-1}{r^2+3r+2}$.

37. $\dfrac{r_1^3r_2-r_1r_2^3}{r_1^3r_2^2+r_1^2r_2^3}$.

38. $\dfrac{m^4-n^4}{m^4+2m^2n^2+n^4}$.

39. $\dfrac{D^2-D-2}{4-D^2}$.

40. $\dfrac{ac-bc-ad+bd}{ac+ad-bc-bd}$.

41. $\dfrac{R^2-Rr-R+r}{R^2-Rr-2R+2r}$.

42. $\dfrac{6x^2+7x-5}{3x^2+17x+20}$.

43. $\dfrac{10a^2-33at-7t^2}{12a^2-52at+35t^2}$.

44. $\dfrac{1+P^3}{P^2+8P+7}$.

100. Reduction of Fractions to Mixed Expressions. — An algebraic expression containing no fractions with literal denominators is an **integral expression**; an expression consisting of one or more such fractions is a **fractional expression**; and an expression consisting of an integral part and a fractional part is a **mixed expression.**

Thus, $x^2-\dfrac{x-1}{x^2+1}$ and $\dfrac{A}{A-1}-\dfrac{A}{A+1}+A^2-1$ are mixed expressions.

NOTE. — In arithmetic the mixed number $4\frac{3}{5}$ means $4+\frac{3}{5}$, the sign + being omitted. In algebra the sign between the integral and fractional parts of a mixed expression must never be omitted. Thus, $x+\dfrac{y}{z}$ may not be written $x\dfrac{y}{z}$, because the latter implies multiplication.

An algebraic fraction whose numerator is of as high degree as the denominator, or of higher degree, may often be reduced to a mixed expression. The process is similar to that in arithmetic of reducing an improper fraction to a mixed number.

EXAMPLE 1. — Reduce $\frac{731}{46}$ to a mixed number.

Dividing 731 by 46 gives a quotient 15 and remainder 41.

Hence, $\frac{731}{46}=15\frac{41}{46}$.

EXAMPLE 2. — Reduce $\dfrac{x^2+5x-9}{x+3}$ to a mixed expression.

Dividing x^2+5x-9 by $x+3$ gives a quotient $x+2$ and remainder -15.

Hence, $\dfrac{x^2+5x-9}{x+3}=x+2+\dfrac{-15}{x+3}$.

This answer may be written $x+2-\dfrac{15}{x+3}$.

To reduce a fraction to a mixed expression, divide the numerator by the denominator. The mixed expression equals the quotient plus the fraction whose numerator is the remainder and denominator the denominator of the given fraction.

EXERCISES

Reduce to mixed numbers:

1. $\frac{7}{3}$. 3. $\frac{88}{9}$. 5. $\frac{126}{84}$. 7. $\frac{192}{81}$. 9. $\frac{1264}{150}$.

2. $\frac{24}{5}$. 4. $\frac{98}{17}$. 6. $\frac{847}{86}$. 8. $\frac{640}{230}$. 10. $\frac{2487}{652}$.

Reduce to mixed expressions:

11. $\frac{x^2+1}{x-1}$. 16. $\frac{P^2}{P-3}$. 21. $\frac{9A^2+3AB+4B^2}{3A-2B}$.

12. $\frac{2A^2-3A-7}{2A-4}$. 17. $\frac{M^2+N^2}{M+N}$. 22. $\frac{v^3-t^3}{v+t}$.

13. $\frac{n^2-5n+1}{n-2}$. 18. $\frac{r^3+3r^2+5r+2}{r+1}$. 23. $\frac{n^3-3n^2+3n-1}{n+1}$.

14. $\frac{1-x^3}{1+x-x^2}$. 19. $\frac{D^3+16}{D+2}$. 24. $\frac{S^5-16}{S-2}$.

15. $\frac{w^3+8}{w^2-w+2}$. 20. $\frac{32+a^4}{a-2}$. 25. $\frac{E^3+4E^2+3}{E^2-3E}$.

101. Reduction of Fractions to Higher Terms.—It sometimes is necessary to change a fraction to an equivalent fraction with denominator of higher degree. The principle involved is that:

If both terms of a fraction are multiplied by the same expression, the value of the fraction is not changed.

The process is the same as that in arithmetic of reducing a fraction to higher terms.

Example 1.—Reduce $\frac{7}{8}$ to 48ths.

Dividing 48 by 8 gives 6. Hence, multiplying both terms by 6,

$$\frac{7}{8}=\frac{42}{48}.$$

Example 2. — Change $\frac{a+b}{2a+b}$ to a fraction with denominator $6a^2+ab-b^2$.

Dividing $6a^2+ab-b^2$ by $2a+b$ gives $3a-b$. Hence, multiplying both terms by $3a-b$ gives

$$\frac{a+b}{2a+b}=\frac{3a^2+2ab-b^2}{6a^2+ab-b^2}.$$

Divide the required denominator by the given denominator, and multiply both terms of the fraction by the quotient.

Note. — The division may be quickly performed by factoring the required denominator and cancelling from it the factors of the given denominator. Thus, in Example 2, the required denominator $6a^2+ab-b^2=(2a+b)(3a-b)$. Cancelling $2a+b$, the quotient is $3a-b$.

EXERCISES

1. Change $\frac{3}{5}$ to 20ths.
2. Change $\frac{6}{7}$ to 56ths.
3. Change $\frac{9}{16}$ to 64ths.
4. Change $\frac{3}{32}$ to 64ths.
5. Change $\frac{5}{18}$ to 126ths.
6. Change each of the following to 72ds:

$$\tfrac{1}{2},\quad \tfrac{3}{4},\quad \tfrac{5}{6},\quad \tfrac{7}{8},\quad \tfrac{11}{12},\quad \tfrac{17}{18}.$$

7. Change $\frac{a}{b^2c}$ to a fraction with denominator $a^3b^3c^3$.
8. Change $\frac{1}{x-1}$ to a fraction with denominator x^2-1.
9. Change $\frac{M-N}{M+N}$ to a fraction with denominator M^2-N^2.
10. Change $\frac{R}{R+1}$ to a fraction with denominator R^2+3R+2.
11. Change $\frac{v+t}{v-t}$ to a fraction with denominator v^3-t^3.
12. Change $\frac{x}{2x+5y}$ to a fraction with denominator $6x^2+7xy-20y^2$.

13. Change each of the following to a fraction with denominator $a^3b - ab^3$:

$$\frac{1}{a+b},\ \frac{1}{a-b},\ \frac{1}{a},\ \frac{1}{b}.$$

14. Change each of the following to a fraction with denominator $(x-y)^2(x+y)^2$:

$$\frac{x}{x-y},\ \frac{y}{x+y},\ \frac{x+y}{(x-y)^2},\ \frac{x-y}{(x+y)^2}.$$

102. Reduction of Fractions to a Common Denominator. — It is clear that by the process of § 101 two or more fractions may be changed to equivalent fractions having for the denominator of each the lowest common multiple of all of the denominators of the given fractions. This lowest common multiple of the denominators of the given fractions is called the **lowest common denominator** (**L. C. D.**) of the fractions.

Example. — Reduce $\frac{x+3}{x^2-1}$ and $\frac{1-x}{x^2-4x-5}$ to equivalent fractions having the lowest common denominator.

$$x^2 - 1 = (x+1)(x-1).$$
$$x^2 - 4x - 5 = (x+1)(x-5).$$

Hence, $\text{L. C. D.} = (x+1)(x-1)(x-5).$

Cancelling the factors of the first denominator that are found in the L. C. D., it is seen that the terms of the first fraction must be multiplied by $x - 5$. Similarly, the terms of the second fraction must be multiplied by $x - 1$.

Hence, $$\frac{x+3}{x^2-1} = \frac{(x-5)(x+3)}{(x-5)(x^2-1)} \text{ or } \frac{x^2-2x-15}{x^3-5x^2-x+5};$$

and $$\frac{1-x}{x^2-4x-5} = \frac{(x-1)(1-x)}{(x-1)(x^2-4x-5)} \text{ or } \frac{-x^2+2x-1}{x^3-5x^2-x+5}.$$

EXERCISES

Reduce to equivalent fractions having the least common denominator:

1. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$. 3 $\frac{3}{4}, \frac{5}{6}, \frac{2}{5}$. 5. $\frac{8}{9}, \frac{3}{7}, \frac{5}{21}$. 7. $\frac{1}{2}, \frac{3}{5}, \frac{7}{16}, \frac{5}{12}$.

2. $\frac{1}{3}, \frac{5}{6}, \frac{7}{8}$. 4. $\frac{7}{8}, \frac{5}{9}, \frac{11}{12}$. 6. $\frac{2}{15}, \frac{3}{20}, \frac{3}{4}$. 8. $\frac{3}{16}, \frac{9}{40}, \frac{4}{5}, \frac{2}{3}$.

Reduce to equivalent fractions having the lowest common denominator:

9. $\frac{2}{3x}, \frac{1}{2x^2}, \frac{5}{6x}.$

10. $\frac{3}{8t^3}, \frac{5}{6t^3}, \frac{1}{4t^4}.$

11. $\frac{2a^2}{3b^2c}, \frac{3b^2}{2a^2c}, \frac{c^2}{4a^2b^2}.$

12. $\frac{z}{xy}, \frac{x}{yz}, \frac{y}{xz}.$

13. $\frac{A}{A+1}, \frac{A}{A-1}, \frac{1}{A^2-1}.$

14. $\frac{3}{n^2+3n+2}, \frac{5}{2n^2+5n+2}.$

15. $\frac{V-1}{V^2+5V+4}, \frac{V+3}{V^2+V-12}.$

16. $\frac{2R+7}{6R^2+13R-5}, \frac{3R-4}{12R^2-13R+3}.$

17. $\frac{1}{a-b}, \frac{2}{a+b}, \frac{1}{a^2+b^2}.$

18. $\frac{2}{c+1}, \frac{c}{(c+1)^2}, \frac{2c^2}{(c+1)^3}.$

19. $\frac{w}{v}, \frac{w^2}{wv-v^2}, \frac{wv}{w^2-v^2}.$

20. $\frac{x}{x-y}, \frac{y}{2x-2y}, \frac{x^2}{6(x^2-y^2)}.$

21. $\frac{p-5}{p^2+3p+2}, \frac{p+1}{p^2-3p-10}, \frac{p+2}{p^2-4p-5}.$

22. $\frac{1}{6S^2+S-2}, \frac{1}{15S^2+10S}, \frac{1}{12S^2-6S}.$

103. Addition and Subtraction of Fractions. — Algebraic fractions are added or subtracted by the same rules as fractions are added or subtracted in arithmetic.

Example 1. — Add $\frac{2}{3}$, $\frac{3}{8}$, and $\frac{5}{6}$.

Reducing all of the fractions to the least common denominator,

$$\tfrac{2}{3} + \tfrac{3}{8} + \tfrac{5}{6} = \tfrac{16}{24} + \tfrac{9}{24} + \tfrac{20}{24}.$$

Adding the numerators and placing the sum over the common denominator,

$$\tfrac{16}{24} + \tfrac{9}{24} + \tfrac{20}{24} = \tfrac{45}{24} \text{ or } 1\tfrac{7}{8}.$$

Example 2. — Add $\frac{2x+1}{x-1}$, $\frac{x-2}{x+1}$, and $\frac{3}{x}$.

Reducing the fractions to their lowest common denominator,

$$\frac{2x+1}{x-1} + \frac{x-2}{x+1} + \frac{3}{x} = \frac{2x^3+3x^2+x}{x^3-x} + \frac{x^3-3x^2+2x}{x^3-x} + \frac{3x^2-3}{x^3-x}.$$

Adding the numerators, this becomes $\frac{3x^3+3x^2+3x-3}{x^3-x}$.

Example 3. — Perform the indicated addition and subtraction in

$$\frac{5}{a^2+5a+4} + \frac{1}{a^2-a-2} - \frac{3}{a^2+2a-8}.$$

$$\frac{5}{a^2+5a+4} + \frac{1}{a^2-a-2} - \frac{3}{a^2+2a-8}$$

$$= \frac{5}{(a+1)(a+4)} + \frac{1}{(a-2)(a+1)} - \frac{3}{(a+4)(a-2)}$$

$$= \frac{5a-10}{(a+1)(a+4)(a-2)} + \frac{a+4}{(a+1)(a+4)(a-2)} - \frac{3a+3}{(a+1)(a+4)(a-2)}$$

$$= \frac{3a-9}{(a+1)(a+4)(a-2)} \text{ or } \frac{3a-9}{a^3+3a^2-6a-8}.$$

After reducing the fractions to a common denominator, the first two numerators are added, and the third one subtracted from their sum.

To add or subtract algebraic fractions, reduce them to their lowest common denominator, then find the sum or difference of their numerators, and place the result over the common denominator.

EXERCISES

Simplify by adding or subtracting, as indicated:

1. $\frac{3}{4}+\frac{5}{6}$. 3. $\frac{9}{20}+\frac{7}{10}-\frac{3}{4}$. 5. $\frac{5}{7}-\frac{1}{3}+\frac{3}{4}$. 7. $\frac{7}{9}-\frac{1}{15}-\frac{1}{6}$.

2. $\frac{7}{8}-\frac{5}{12}$. 4. $\frac{5}{18}+\frac{1}{6}-\frac{1}{12}$. 6. $\frac{4}{21}-\frac{1}{14}+\frac{9}{20}$. 8. $\frac{11}{12}+\frac{8}{9}-\frac{15}{16}$.

9. $\dfrac{3a}{4}+\dfrac{5a}{6}+\dfrac{a}{3}$. 11. $\dfrac{3v}{20}+\dfrac{v}{5}$. 13. $\dfrac{a}{b}-\dfrac{c}{d}$.

10. $\dfrac{P}{2}+\dfrac{P}{3}+\dfrac{P}{4}$. 12. $\dfrac{4R}{7}+\dfrac{R-2}{5}$. 14. $\dfrac{1}{A}+\dfrac{A-1}{A^2}+\dfrac{3}{2A}$.

15. $\dfrac{x}{yz}+\dfrac{y}{xz}+\dfrac{z}{xy}$.

16. $\dfrac{n+1}{n^2}+\dfrac{1}{7n}+\dfrac{2n-1}{2n^2}$.

17. $\dfrac{1}{r_1r_2}+\dfrac{1}{r_1r_3}+\dfrac{1}{r_2r_3}$.

18. $\dfrac{3}{4\pi R}+\dfrac{2\pi R+H}{2\pi R^2}+\dfrac{H+R}{\pi R^2}$.

19. $\dfrac{x}{x+y}+\dfrac{y}{x-y}$. 23. $\dfrac{3}{s^2-s}-\dfrac{2}{s^2+s}+\dfrac{4}{s}$.

20. $\dfrac{1}{1+a}-\dfrac{1}{1-a}$. 24. $\dfrac{2x+1}{x-2}-\dfrac{x+2}{2x-1}$.

21. $\dfrac{m+n}{m-n}+\dfrac{m-n}{m+n}$. 25. $\dfrac{3v}{w+v}-\dfrac{v}{w-v}+\dfrac{2wv}{w^2-v^2}$.

22. $\dfrac{v-1}{v+1}-\dfrac{v+1}{v-1}$. 26. $\dfrac{1}{a^2+6a+5}-\dfrac{1}{a^2+2a-15}$.

27. $\dfrac{M-2N}{M^2+5MN+4N^2}+\dfrac{M+4N}{M^2-MN-2N^2}$.

28. $\dfrac{1}{2b^2+5bc-3c^2}-\dfrac{1}{4b^2-13bc+3c^2}$.

29. $\dfrac{2\pi R}{\pi R+\pi H}+\dfrac{2\pi R}{\pi R-\pi H}.$

30. $\dfrac{3}{y^2+2yz+z^2}-\dfrac{2}{2y^2+yz-10z^2}.$

31. $\dfrac{a+2}{a+1}+\dfrac{a-2}{a-1}+\dfrac{4}{1-a^2}.$

Suggestion. — In problems such as this it is best to arrange the terms of all denominators according to either the descending or ascending powers of some letter. In order to have the first term of the denominator of each fraction positive when rearranged, it may be necessary to change the signs of the terms of the fraction, as in § 98. Thus, the last term in this problem may be changed to $+\dfrac{-4}{a^2-1}.$

32. $\dfrac{P}{1-P^2}-\dfrac{1}{P-1}.$

33. $\dfrac{3v}{u+v}-\dfrac{v}{u-v}+\dfrac{2uv}{v^2-u^2}.$

34. $\dfrac{5}{1+x}+\dfrac{x}{1-x^2}+\dfrac{2x}{x^2+1}.$

35. $\dfrac{M}{M^2-N^2}-\dfrac{1}{M+N}-\dfrac{1}{N-M}.$

36. $\dfrac{3+2y}{2-y}+\dfrac{16y-y^2}{y^2-4}-\dfrac{2-3y}{2+y}.$

37. $\dfrac{R}{1+R}-\dfrac{R}{1-R}+\dfrac{R^2}{R^2-1}.$

38. $\dfrac{b^3}{1-b^3}-\dfrac{2b}{b^2+b+1}+\dfrac{1}{b-1}.$

39. $\dfrac{x+2a}{2b-x}+\dfrac{x-2a}{2b+x}+\dfrac{4ab}{x^2-4b^2}.$

40. $\dfrac{1}{a-b}-\dfrac{1}{a+b}+\dfrac{1}{a-2b}-\dfrac{1}{a+2b}.$

Suggestion. — In certain cases, such as this, it is best to combine only part of the fractions at a time. It avoids long multiplications. Here, combine the first and second fractions, then the third and fourth, then the two results obtained.

41. $\dfrac{1}{n+1}+\dfrac{1}{n+3}-\dfrac{1}{n-1}-\dfrac{1}{n-3}$.

42. $\dfrac{1}{R-r}-\dfrac{1}{R+r}+\dfrac{1}{S-s}-\dfrac{1}{S+s}$.

43. $\dfrac{2}{y-5}+\dfrac{3}{y+1}-\dfrac{2}{y+5}-\dfrac{3}{y-1}$.

44. $\dfrac{1}{t^2-1}-\dfrac{1}{t^2+1}+\dfrac{1}{t^2-a^2}-\dfrac{1}{t^2+a^2}$.

45. $\dfrac{2A+1}{A^4-1}-\dfrac{A-3}{A^4+A^2}+\dfrac{3A-2}{A^4-A^2}$.

104. Reduction of Mixed Expressions to Fractions. — A mixed expression may be changed to a fraction by the process of § 103. Compare the process as illustrated in the following example to that of reducing an arithmetical mixed number to an improper fraction.

EXAMPLE. — Reduce $m + n + \dfrac{n^2}{m-n}$ to a fraction.

Writing the integral part as a fraction with the denominator 1,

$$\begin{aligned} m+n+\frac{n^2}{m-n} &= \frac{m+n}{1}+\frac{n^2}{m-n} \\ &= \frac{m^2-n^2}{m-n}+\frac{n^2}{m-n} \\ &= \frac{m^2}{m-n}. \end{aligned}$$

To reduce a mixed expression to a fraction, write the integral part as a fraction with the denominator 1, *and proceed as in addition and subtraction of fractions.*

EXERCISES

Reduce to a fraction:

1. $1+\frac{1}{x}$.

2. $B+\frac{B}{1+B}$.

3. $r+1-\frac{r^2}{r-1}$.

4. $\frac{T}{T+2}+2$.

5. $\frac{m^2}{m-n}-m$.

6. $w+5+\frac{1-2w}{w-3}$.

7. $\pi+\frac{\pi R^2}{H^2}$.

8. $r_1-\frac{r_1 r_2}{r_1+r_2}$.

9. $\frac{2ab}{a-b}+a-b$.

10. $\frac{B^2}{4A}-AC$.

11. $1-y+\frac{2y^2}{1+y}-\frac{1}{1-y}$.

12. $5+\frac{t}{1-t^2}+\frac{2t}{t^2-1}$.

13. $a^2+b^2-\frac{a^3-b^3}{a+b}-ab$.

14. $V^2-\frac{V^2-3V^4}{1+V+V^2}-V+1$.

15. $4+\frac{u^2-2v^2}{u^2-uv+v^2}-\frac{u^2v+uv^2}{u^3+v^3}-3$.

16. $\frac{p^2}{(p+q)^2}+2+\frac{p}{p+q}$.

17. $1+\frac{b^2}{B^2}+\frac{b}{B}$.

105. Multiplication of Fractions.—The product of two or more algebraic fractions is obtained by the same process as the product of two arithmetical fractions.

Thus, $\frac{2}{3}\times\frac{7}{9}=\frac{2\times 7}{3\times 9}=\frac{14}{27}$; $\frac{3}{5}\times\frac{2}{7}\times\frac{9}{11}=\frac{3\times 2\times 9}{5\times 7\times 11}=\frac{54}{385}$.

Similarly, $\frac{a}{b}\times\frac{m}{n}=\frac{am}{bn}$; $\frac{x+2}{x+5}\times\frac{3x-1}{x-4}=\frac{(x+2)(3x-1)}{(x+5)(x-4)}$.

The product of two or more fractions is the fraction whose numerator is the product of their numerators and denominator the product of their denominators.

As in arithmetical fractions, the product often may be reduced to lower terms. In such a case, work may be saved by factoring the terms of the fractions, and cancelling any factor of any numerator by a like factor of any denominator, before performing the multiplications.

Example. — Find the product of $\frac{2a^2+ab-b^2}{2a^2-ab-b^2}$ and $\frac{3a^2-ab-2b^2}{2a^2+5ab-3b^2}$.

$$\frac{2a^2+ab-b^2}{2a^2-ab-b^2}\times\frac{3a^2-ab-2b^2}{2a^2+5ab-3b^2}=\frac{(a+b)\cancel{(2a-b)}}{(2a+b)\cancel{(a-b)}}\times\frac{\cancel{(a-b)}(3a+2b)}{\cancel{(2a-b)}(a+3b)}$$

$$=\frac{(a+b)(3a+2b)}{(2a+b)(a+3b)}$$

$$=\frac{3a^2+5ab+2b^2}{2a^2+7ab+3b^2}.$$

EXERCISES

Find the product of:

1. $\frac{5}{6}\times\frac{8}{7}$.
2. $\frac{4}{9}\times\frac{3}{8}$.
3. $\frac{1}{6}\times\frac{12}{17}$.
4. $\frac{1}{3}\times\frac{3}{4}\times\frac{6}{7}$.
5. $\frac{5}{8}\times\frac{16}{21}\times\frac{3}{5}$.
6. $\frac{24}{35}\times\frac{1}{12}\times\frac{5}{6}$.
7. $\frac{1}{4}\times\frac{3}{8}\times\frac{32}{39}$.
8. $\frac{5}{6}\times\frac{7}{8}\times\frac{12}{15}$.
9. $\frac{1}{2}\times\frac{6}{15}\times\frac{20}{21}$.
10. $\frac{A}{B}\times\frac{C}{D}$.
11. $\frac{w^2}{v^3}\times\frac{v}{w}$.
12. $\frac{2x}{3y}\times\frac{9y^3}{10x^2}$.
13. $\frac{n^3}{4}\times\frac{n^5}{5}$.
14. $\frac{3pq}{8r}\times\frac{12r^3}{q}$.
15. $\frac{\pi R^2}{4}\times\frac{2\pi R}{3}$.
16. $\frac{2}{c}\times\frac{c}{2}$.
17. $\frac{5v}{t}\times\frac{n}{v}$.
18. $\frac{7}{15x^3}\times\frac{20x}{21}$.
19. $\frac{a}{3d}\times\frac{6d}{ay}$.
20. $\frac{x}{z^2}\times\frac{y^2}{xz}$.
21. $\frac{a^2}{b^2}\times\frac{bc}{cd}\times\frac{d^2}{a^2}$.
22. $\frac{1}{W}\times\frac{Wg^2}{4}\times\frac{2t^2}{g}$.
23. $\frac{1}{a^2}\times\frac{ab}{a+b}$.
24. $\frac{M}{N}\times\frac{M-N}{5MN}$.
25. $\frac{x+1}{x+2}\times\frac{x+2}{x^2-1}$.
26. $\frac{a^2-b^2}{a^2+b^2}\times\frac{a+b}{a-b}\times\frac{1}{a+b}$.
27. $\frac{m+n}{m^2-n^2}\times\frac{m-n}{m+2n}$

28. $\frac{2t-1}{s+3} \times \frac{2s+3}{t-1} \times \frac{s+3}{2t-1}$.

30. $\frac{x-y}{x+2y} \times \frac{x^2-4y^2}{x^3-y^3}$.

29. $\frac{A^2-1}{A^2+A+1} \times \frac{A^3-1}{A-1}$.

31. $\frac{v^2+4v}{v+3} \times \frac{v^3+3v^2}{v+4}$.

32. $\frac{x^2+2x-3}{x^2+3x-10} \times \frac{x^2-9x+14}{x^2+7x+12}$.

33. $\frac{r^2-5r}{r^2+8r+12} \times \frac{r+2}{r^3+r^2+r}$.

34. $\frac{\pi R^2-\pi RH}{R+H} \times \frac{1}{\pi R^2-\pi H^2}$.

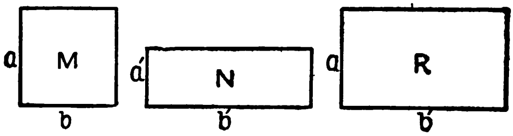
35. $\frac{6m^2-5mn-4n^2}{4m^2-17mn-15n^2} \times \frac{4m^2-mn-3n^2}{6m^2+7mn+2n^2}$.

36. $\frac{V^2+6V+9}{16-8V+V^2} \times \frac{V-4}{9-V^2}$.

37. $\frac{b+c}{a+b} \times \frac{a^2+ab+ac+bc}{b^2+bc+bd+cd}$.

38. $\frac{1-y^3}{y+1} \times \frac{y^2+3y+2}{y^2+y+1} \times \frac{1}{y-1}$.

39. M, N, and R are the areas of the rectangles with the dimensions shown in the figures. It is known from Geometry that $\frac{M}{R}=\frac{b}{b'}$ and that $\frac{R}{N}=\frac{a}{a'}$.



Show by multiplication that $\frac{M}{N}=\frac{ab}{a'b'}$.

106. Multiplication of Mixed and Fractional Expressions. — Mixed and fractional expressions to be multiplied should first be reduced to fractions.

Example. — Multiply $\frac{9n^2}{v^2} - 1$ by $\frac{v}{3n - v} + 1$.

$$\begin{aligned}\left(\frac{9n^2}{v^2} - 1\right)\left(\frac{v}{3n - v} + 1\right) &= \frac{9n^2 - v^2}{v^2} \times \frac{3n}{3n - v}\\ &= \frac{(3n - v)(3n + v)}{v^2} \times \frac{3n}{3n - v}\\ &= \frac{9n^2 + 3nv}{v^2}.\end{aligned}$$

EXERCISES

Find the product of:

1. $\left(1 + \frac{a}{b}\right)\left(\frac{b^2}{b^2 - a^2}\right)$.
2. $\left(1 - \frac{R^2}{D^2}\right)\left(\frac{1}{R - D}\right)$.
3. $\left(\frac{uv}{u^2 + v^2}\right)\left(1 - \frac{u^4}{v^4}\right)$.
4. $\left(r_1 - \frac{r_2^2}{r_1}\right)\left(\frac{r_1}{r_2} + \frac{r_2}{r_1}\right)$.
5. $\left(\frac{9h^2}{b^2} - 1\right)\left(\frac{b}{3h - b} + 1\right)$.
6. $\left(M + \frac{1}{N}\right)\left(M - \frac{1}{N}\right)$.
7. $\left(\frac{1}{x} - 1\right)\left(\frac{1}{x} - 1\right)$.
8. $\left(\frac{y^2 + 2yz + z^2}{yz} - 4\right)\left(1 + \frac{4yz}{y^2 - 2yz + z^2}\right)$.
9. $\left(\frac{36}{2k + 3} + 2k - 9\right)\left(2k + \frac{6k - 9}{2k - 3}\right)$.
10. $\left(1 + \frac{1}{E} + \frac{1}{E^2}\right)\left(\frac{8E^3}{E^3 - 1}\right)$.

11 $\left(A + \frac{AB}{A + B}\right)\left(A - \frac{AB}{A + B}\right)\left(\frac{A^2 - B^2}{A^2 + B^2}\right)$.

12. $\left(\frac{1}{t^2}-1\right)\left(\frac{t}{1+t}\right)\left(\frac{1-t^2}{1-2t+t^2}\right)$.

13. $\left(\frac{P}{1+P}+\frac{1-P}{P}\right)\left(\frac{P}{1+P}-\frac{1-P}{P}\right)$.

14. $\frac{s}{2}\left(\frac{s^2-t^2}{s^2t+st^2}\right)\left(\frac{1}{s-t}-\frac{1}{s+t}\right)$.

15. $\left(\frac{1}{p-q}-\frac{1}{p+q}+\frac{2}{p}\right)\left(\frac{p^2-q^2}{4pq}\right)$.

107. Multiplication of Integral Expressions and Fractions. — In the multiplication of a fraction by an integral expression, the latter may first be written as a fraction with the denominator 1, and then the product found as in the multiplication of fractions.

Thus, $12 \times \frac{3}{56} = \frac{12}{1} \times \frac{3}{56} = \frac{9}{14}$.

And $(n^2-4)\left(\frac{n^2+4n-5}{n^2-n-6}\right) = \frac{n^2-4}{1} \times \frac{n^2+4n-5}{n^2-n-6} =$ etc.

It frequently is found necessary, as in solving equations containing fractions, to multiply a fractional expression by the L. C. D. of the fractions involved.

EXAMPLE. — Multiply $\frac{2a^2}{a^2-1}+\frac{a}{a-1}-\frac{a}{a+1}$ by the lowest common denominator of all three fractions.

The L. C. D. is a^2-1. Multiplying by this gives

$$\frac{a^2-1}{1}\times\frac{2a^2}{a^2-1}+\frac{a^2-1}{1}\times\frac{a}{a-1}-\frac{a^2-1}{1}\times\frac{a}{a+1}$$
$$=2a^2+a^2+a-a^2+a$$
$$=2a^2+2a.$$

EXERCISES

Multiply:

1. $3\times\frac{5}{6}$.
2. $14\times\frac{16}{21}$.
3. $9\times\frac{11}{12}$.
4. $24\times\frac{5}{16}$.
5. $15\times\frac{32}{36}$.
6. $A^2\times\frac{B}{A}$.

7. $z^3 \times \frac{3xy}{z^2}$.

8. $5w \times \frac{3v}{20w}$.

9. $\pi R^2 \times \frac{R+H}{2\pi R}$.

10. $(n^2-9)\left(\frac{n^2+4n-12}{n^2-4n+3}\right)$.

11. $(\pi R^2-\pi H^2)\left(\frac{R+H}{R-H}\right)$.

12. $(t^3-1)\left(\frac{2t}{t^2-7t+6}\right)$.

13. $(2v^2-8)\left(\frac{v}{v+2}\right)$.

14. $(2x^2+xy-6y^2)\left(\frac{x^2}{4x^2+9xy+2y^2}\right)$.

15. $(r_1+r_2)\left(\frac{r_1r_2}{r_1^2-2r_1r_2-3r_2^2}\right)$.

Multiply by the lowest common denominator:

16. $\frac{1}{a+2}+\frac{1}{a-2}$.

17. $\frac{2y}{y-3}+\frac{3y}{y+3}-\frac{y^2}{y^2-9}$.

18. $\frac{1}{m+1}-\frac{1}{m-1}+\frac{1}{m+2}$.

19. $\frac{8}{P}+\frac{P-3}{P+3}-\frac{P+1}{P-1}$.

20. $\frac{g}{3}-\frac{g^2-5g}{3g-7}-\frac{2}{3}$.

21. $\frac{2D+4}{3D-5}-\frac{5}{2}$.

22. $\frac{3R+5}{4R-3}-\frac{3R^2+5R-4}{4R^2-3R+2}$.

23. $\frac{2k+3}{3k-4}-\frac{4k+5}{6k-1}$.

24. $\frac{x+y}{x^2+xy+y^2}+\frac{1}{x-y}-\frac{xy+2y^2}{x^3-y^3}$.

25. $\frac{2M}{2M+3}-\frac{1-M}{2-M}+\frac{7}{2M^2-M-6}$.

108. Division of Fractions.—The quotient of two algebraic fractions is obtained by the same process as the quotient of two arithmetical fractions; *i.e.*,

To find the quotient of two algebraic fractions, multiply the dividend by the divisor inverted.

This follows from the principle that the product of the quotient and divisor must equal the dividend. If $\frac{a}{b}$ and $\frac{c}{d}$ represent any two fractions, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$, because $\frac{a}{b} \times \frac{d}{c}$ multiplied by $\frac{c}{d}$ gives $\frac{a}{b}$ by cancellation.

Example.—Divide $\frac{A^2 - 4\,AB + 3\,B^2}{A^2 + 2\,AB + B^2}$ by $\frac{A - 3\,B}{A + B}$.

$$\frac{A^2 - 4\,AB + 3\,B^2}{A^2 + 2\,AB + B^2} \div \frac{A - 3\,B}{A + B} = \frac{(A - 3\,B)(A - B)}{(A + B)^2} \times \frac{A + B}{A - 3\,B}$$
$$= \frac{A - B}{A + B}.$$

Note.—If a dividend or divisor is a mixed or fractional expression, it may be changed to a fraction, and the division performed by the above process.

EXERCISES

Perform the indicated divisions:

1. $\frac{2}{3} \div \frac{5}{6}$.
2. $\frac{7}{8} \div \frac{3}{4}$.
3. $\frac{2}{9} \div \frac{4}{5}$.
4. $\frac{11}{12} \div \frac{1}{6}$.
5. $\frac{15}{16} \div \frac{3}{8}$.
6. $\frac{7}{24} \div \frac{5}{6}$.
7. $\frac{3}{17} \div \frac{2}{5}$.
8. $\frac{81}{100} \div \frac{3}{25}$.
9. $\frac{a}{2} \div \frac{3\,a}{4\,m}$.
10. $\frac{3\,\pi R}{4\,D^2} \div \frac{6\,R^2}{2\,D^2}$.
11. $\frac{4\,v^2}{9\,t} \div \frac{2\,v}{3}$.
12. $\frac{8\,W}{9\,P} \div \frac{W}{3}$.
13. $\frac{3\,x^2y}{5\,z^2} \div \frac{2\,xy^2}{10\,zw}$.
14. $\frac{20\,H^2}{21\,L^2} \div \frac{4\,HL}{3\,H^2}$.
15. $\frac{36\,r_1^4r_2^2}{35\,R_1^3R_2^2} \div \frac{12\,r_1^2r_2}{7\,R_1^2R_2}$.
16. $\frac{a+1}{a} \div \frac{a^2-1}{a^2}$.
17. $\frac{6\,M^2 - 2\,M}{4 - M^2} \div \frac{M^2}{2 + M}$.
18. $\frac{W^2 - V^2}{W + 2\,V} \div \frac{W - V}{3\,W + 6\,V}$.
19. $\frac{x^2 - 18\,x + 80}{x^2 - 5\,x - 50} \div \frac{x^2 - 15\,x + 56}{x^2 - 6\,x - 7}$.
20. $\frac{R^2 - 1}{R^2 - 3\,R - 10} \div \frac{R^2 - 12\,R + 35}{R^2 + 3\,R + 2}$

21. $\dfrac{2t^2}{a^3+t^3} \div \dfrac{t}{a+t}$.

22. $\dfrac{b^2+c^2}{b^2-c^2} \div \dfrac{3b^2+3c^2}{b+c}$.

23. $\dfrac{2mn-n^2}{mn+m^2} \div \dfrac{n^2-m^2}{n+m}$.

24. $\dfrac{A^2-B^2}{A} \div (A-B)$.

25. $\left(1-\dfrac{a^2}{b^2}\right) \div \dfrac{a+b}{ab}$.

26. $\dfrac{n-1}{3n^2+8n+5} \div \dfrac{n^2-1}{n^2+4n+3}$.

27. $\dfrac{x^2-9y^2}{x^2+4y^2} \div \dfrac{x^2+xy-6y^2}{x^2-xy-6y^2}$.

28. $\dfrac{p^3-q^3}{p^2-pq+q^2} \div \dfrac{p^2+pq+q^2}{p^3+q^3}$.

29. $(t-3v) \div \dfrac{2t^2-5tv-3v^2}{t^2}$.

30. $\left(\dfrac{V^2}{t^2}+\dfrac{2V}{t}+1\right) \div \left(1+\dfrac{V^3}{t^3}\right)$.

109. Complex Fractions. — Fractions whose numerators, denominators, or both, contain fractions are called **complex fractions.**

Thus, $\dfrac{\frac{3}{5}}{2\frac{1}{3}}$, $\dfrac{\frac{a}{b}}{\frac{c}{d}}$, $\dfrac{W}{\frac{W+V}{WV}}$, and $\dfrac{1+\frac{1}{t}}{1-\frac{1}{t}}$ are complex fractions.

A complex fraction is said to be *simplified* when it is reduced to an equivalent fraction whose terms are integral expressions, or to an integral expression. Since a complex fraction may be considered an indicated quotient, it may be simplified by dividing the numerator by the denominator.

EXAMPLE. — Simplify $\dfrac{A-\frac{1}{A}}{1+\frac{1}{A}}$.

$$\frac{A-\frac{1}{A}}{1+\frac{1}{A}}=\frac{\frac{A^2-1}{A}}{\frac{A+1}{A}}=\frac{A^2-1}{A}\times\frac{A}{A+1}=A-1.$$

EXERCISES

Simplify:

1. $\dfrac{5}{\frac{2}{3}}$.
2. $\dfrac{\frac{4}{7}}{\frac{2}{5}}$.
3. $\dfrac{2\frac{3}{4}}{3\frac{2}{3}}$.
4. $\dfrac{8\frac{1}{3}}{\frac{5}{6}}$.
5. $\dfrac{\frac{A}{B}}{\frac{A^2}{B^2}}$.
6. $\dfrac{x^2}{\frac{1}{x}}$.
7. $\dfrac{w+1}{1+\frac{1}{w}}$.
8. $\dfrac{\frac{m+n}{x}}{\frac{m^2-n^2}{x^2}}$.
9. $\dfrac{y}{\frac{ay+b}{a}}$.
10. $\dfrac{\frac{1}{R-H}}{\frac{1}{R+H}}$.
11. $\dfrac{\frac{u}{u-v}}{\frac{u^2}{u^2-v^2}}$.
12. $\dfrac{1+\frac{M}{N}}{\frac{M}{N}-1}$.
13. $\dfrac{c+2}{\frac{c^2-4}{2}}$.
14. $\dfrac{1-\frac{r_1}{r_2}}{r_2^2-r_1^2}$.
15. $\dfrac{H+R}{\pi-\frac{\pi R^2}{H^2}}$.
16. $\dfrac{\frac{x}{1+x}+1}{\frac{1}{1+x}+x}$.
17. $\dfrac{A-1+\frac{6}{A-6}}{A-2+\frac{3}{A-6}}$.
18. $\dfrac{y+\frac{y}{z}}{1+\frac{1}{z}}$.
19. $\dfrac{\frac{x-y}{x}-\frac{x+y}{y}}{\frac{x-y}{y}+\frac{x+y}{x}}$.
20. $\dfrac{\frac{R}{1+R}+\frac{1-R}{R}}{\frac{R}{1+R}-\frac{1-R}{R}}$.
21. $\dfrac{\frac{r_1^2}{r_2^2}-\frac{r_2^2}{r_1^2}}{r_1+\frac{r_2^2}{r_1}}$.

SUPPLEMENTARY EXERCISES

Note. — In some problems in the addition or subtraction of fractions, such as the following, it is advisable to change the signs of factors of one or more denominators before finding the lowest common denominator. In this change of signs use is made of the principle that, *if the sign of one factor of a product is changed, the sign of the product is changed; and if the signs of two factors of a product are changed, the sign of the product is not changed.*

Thus, $(a-2)(a-4)=a^2-6a+8$. If the sign of $a-2$ is changed, by changing the signs of its terms, we get $(2-a)(a-4)=-a^2+6a-8$. If the signs of both factors are changed, we get $(2-a)(4-a)=a^2-6a+8$.

Perform the indicated additions and subtractions:

1. $\dfrac{1}{(A-B)(A-C)}+\dfrac{1}{(B-C)(B-A)}+\dfrac{1}{(C-A)(C-B)}.$

Solution. — Writing all factors of denominators as $A-B$, $B-C$, or $C-A$,

$$\frac{1}{(A-B)(A-C)}+\frac{1}{(B-C)(B-A)}+\frac{1}{(C-A)(C-B)}$$

$$=\frac{-1}{(A-B)(C-A)}+\frac{-1}{(B-C)(A-B)}+\frac{-1}{(C-A)(B-C)}$$

$$=\frac{-1(B-C)}{(A-B)(B-C)(C-A)}+\frac{-1(C-A)}{(A-B)(B-C)(C-A)}$$

$$+\frac{-1(A-B)}{(A-B)(B-C)(C-A)}$$

$$=\frac{-B+C-C+A-A+B}{(A-B)(B-C)(C-A)}=0.$$

2. $\dfrac{1}{(x-y)(x-z)}-\dfrac{1}{(y-x)(y-z)}+\dfrac{1}{(x-z)(z-y)}.$

3. $\dfrac{mn}{(p-m)(p-n)}+\dfrac{np}{(m-n)(m-p)}+\dfrac{pm}{(n-p)(n-m)}.$

4. $\dfrac{u}{(v-w)(w-u)}+\dfrac{v}{(w-u)(u-v)}+\dfrac{w}{(u-v)(v-w)}.$

5. $\dfrac{r_1+r_2}{(r_2-r_3)(r_3-r_1)}+\dfrac{r_2+r_3}{(r_3-r_1)(r_2-r_1)}+\dfrac{r_3+r_1}{(r_2-r_1)(r_3-r_2)}.$

6. $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-c)(b-a)}+\dfrac{c^2}{(c-a)(c-b)}.$

7. $\dfrac{u(v+w)}{(u-v)(w-u)}+\dfrac{v(w+u)}{(v-w)(u-v)}+\dfrac{w(u+v)}{(w-u)(v-w)}.$

8. $\dfrac{1}{(x-1)(x-2)}+\dfrac{1}{(2-x)(3-x)}+\dfrac{1}{(x-1)(3-x)}.$

9. $\dfrac{6}{a^2-4a+3}+\dfrac{3}{7a-6-a^2}+\dfrac{1}{9a-a^2-18}.$

NOTE.—In the following problems invert divisors, and find the products of all factors at one step by cancellation.

Simplify:

10. $\dfrac{x^2}{y^2}\times\dfrac{3y}{5x}\div\dfrac{9x}{10y}.$

11. $\dfrac{a}{b}\div\dfrac{a^2}{b^2}\div\dfrac{a^3}{b^3}.$

12. $\dfrac{n-1}{n-2}\times\dfrac{n^2+3n-10}{n^2+n-2}\div\dfrac{n^2+5n}{n^2-n-6}$

13. $(3u-2v)\left(2-\dfrac{u-2v}{2u+v}\right)\div\dfrac{9u^2+6uv-8v^2}{2u+v}.$

14. $\dfrac{t^2-16}{t^2-5t+6}\times\dfrac{t^2-3t-10}{t^2-11t+30}\div\dfrac{t^2-9t-52}{t^2+5t-66}.$

15. $\dfrac{6P^2-11P-10}{3P^2+2P-5}\div\dfrac{7P^2+17P-12}{5P^2+9P-2}\div\dfrac{10P^2-27P+5}{21P^2+23P-20}.$

16. $\left(1+\dfrac{3}{a}+\dfrac{2}{a^2}\right)\div\left(1+\dfrac{1}{a^3}\right)\div(a+2)^2.$

17. $\left(4-\dfrac{16N}{M}+\dfrac{15N^2}{M^2}\right)\div\left(\dfrac{2M-5N}{2M+5N}\right)\div\left(\dfrac{4N}{M}+4-\dfrac{15N^2}{M^2}\right).$

NOTE.—A good method of simplifying a complex fraction (see § 109) consists of multiplying both terms of the fraction by the lowest common denominator of all simple fractions in the terms, as suggested in the following problem.

18. $$\frac{\frac{a+b}{a}+\frac{a-b}{b}}{\frac{a-b}{a}-\frac{a+b}{b}}.$$

SUGGESTION. — Multiplying both terms by the L. C. D. gives

$$\frac{b(a+b)+a(a-b)}{b(a-b)-a(a+b)} = \text{etc.}$$

19. $$\frac{\frac{1}{n+2}+\frac{2}{n+3}}{\frac{3}{n+2}}.$$

20. $$\frac{\frac{n}{y^3}}{1+\frac{n^3}{y^3}}.$$

21. $$\frac{\frac{3x}{2y}+2+\frac{2y}{3x}}{\frac{1}{2y}+\frac{1}{3x}}.$$

22. $$\frac{\frac{2A^2-AB-3B^2}{3A^2-AB-2B^2}}{\frac{A^2+3AB+2B^2}{4A^2-5AB+B^2}}.$$

CHAPTER X

FRACTIONAL EQUATIONS. PROBLEMS. FORMULÆ

110. Fractional Equations. — Some problems are expressed and solved by means of equations in which the unknown number appears in one or more denominators of fractions. An equation of which the unknown number appears in one or more denominators is called a **fractional equation.**

Thus, $\frac{n+1}{n-2} = \frac{n}{n+2} + 5$ is a fractional equation.

111. Clearing of Fractions. — To solve a fractional equation it is necessary first to change it so that it is free of fractions. This process is called **clearing the equation of fractions.**

It was found in § 107 that if a mixed or fractional expression is multiplied by the lowest common denominator of all of the fractions involved, the product is integral. We know also, as an axiom, that if equal quantities, such as the members of an equation, are multiplied by the same quantity, the products are equal. These principles are used in clearing an equation of fractions.

EXAMPLE. — Solve $\frac{2t^2}{t^2-1} + \frac{t}{t-1} = \frac{t}{t+1} + 2.$

The L. C. D. of all fractions in the equation is $t^2 - 1$. Multiplying both members of the equation by $t^2 - 1$, by multiplying every term of each member, gives

$$(t^2-1)\left(\frac{2t^2}{t^2-1}\right) + (t^2-1)\left(\frac{t}{t-1}\right) = (t^2-1)\left(\frac{t}{t+1}\right) + 2(t^2-1),$$

or $$2t^2 + t^2 + t = t^2 - t + 2t^2 - 2.$$

Solving, $$2t = -2,$$

$$t = -1.$$

To clear an equation of fractions, multiply each member, by multiplying every term of each member, by the lowest common denominator of all fractions in the equation.*

EXERCISES

Solve:

1. $2 = \frac{6}{n}.$

2. $\frac{3}{2y} = \frac{1}{4}.$

3. $\frac{4}{A} = \frac{5}{A} - 1.$

4. $\frac{1}{r} + \frac{2}{r} + \frac{3}{r} = 1.$

5. $\frac{x+1}{x^2} + \frac{2}{3x} = \frac{1}{x}.$

6. $\frac{4}{5c} - \frac{7}{10c} = \frac{1}{10}.$

7. $\frac{42}{T} - \frac{1}{T} + \frac{1}{3T} = 20.$

8. $\frac{m+12}{2m} = \frac{2m+12}{3m}.$

9. $\frac{2}{1-z} = 1.$

10. $\frac{2}{v} = \frac{1}{v-1}.$

11. $\frac{S+3}{S-3} = 3.$

12. $\frac{3}{k+1} = \frac{2}{k-1}.$

13. $\frac{6}{2t+3} = \frac{2}{t-4}.$

14. $\frac{n-3}{n+9} = \frac{n-5}{n+5}.$

15. $\frac{D-1}{2D-5} = \frac{D-5}{2D-2}.$

16. $\frac{9}{5r+2} - \frac{7}{3r+4} = 0.$

17. $\frac{B}{B-2} + \frac{6}{B+2} = 1.$

18. $1 = \frac{y}{y-2} + \frac{4}{y+1}.$

19. $\frac{2P+3}{3P-4} - \frac{4P+5}{6P-1} = 0.$

20. $\frac{w}{w+1} = \frac{3w}{w+2} - 2.$

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* It is important that the teacher should see that no pupil gets a wrong idea of the process of clearing of fractions through careless thinking. Thus, a pupil sometimes thinks that clearing an equation such as $\frac{x-1}{2x+4} = \frac{3-x}{1+x}$ of fractions consists of multiplying the first member by $1+x$ and the second by $2x+4$, rather than each member by $(1+x)(2x+4)$.

21. $\frac{4}{A-5}=\frac{5}{A}.$

22. $\frac{t+1}{t^2}+\frac{2}{3t}=\frac{1}{t}.$

23. $\frac{15y^2-5y-8}{3y^2+6y+4}=5.$

24. $\frac{3}{N+1}-\frac{2}{N-1}=0.$

25. $\frac{c+3}{c}=\frac{c+9}{c+4}.$

26. $\frac{2n+1}{4}-\frac{1}{n-1}=\frac{n}{2}.$

27. $\frac{1}{2i+3}+\frac{1}{2i-3}=\frac{1}{4i^2-9}.$

28. $\frac{5}{z+1}=\frac{10-7z}{z-1}+7.$

29. $\frac{3}{m+1}-\frac{2}{m-2}=\frac{1}{m+2}.$

30. $\frac{6}{V+2}+\frac{2}{V-2}=\frac{16}{V^2-4}.$

31. $\frac{8}{p}+\frac{p-3}{p+3}=\frac{p+1}{p-1}.$

32. $\frac{3a+5}{4a-3}=\frac{3a^2+5a-4}{4a^2-3a+2}.$

33. $\frac{2x+1}{2x-1}-\frac{8}{4x^2-1}=\frac{2x-1}{1+2x}.$

34. $\frac{v^2-3}{1-v^2}=\frac{4}{1+v}+\frac{v+1}{1-v}.$

35. $\frac{37}{R^2+5R+6}-\frac{4}{R+2}=\frac{7}{R+3}.$

36. $\frac{1}{T-3}=\frac{T-6}{T+3}-\frac{T}{T+3}.$

37. $\frac{7}{4L-12}+\frac{47}{220}+\frac{3}{5L-15}=0.$

38. $$\frac{x-5}{x^2-10x+21}-\frac{7-x}{x^2-8x+15}=\frac{2x-15}{x^2-12x+35}.$$

39. $$\frac{9A+17}{A^2-2A-48}+\frac{2A-1}{2A+12}=\frac{2A+1}{2A-16}.$$

40. $$\frac{t^2-2t+5}{t^2-2t-3}+\frac{t^2+3t-7}{t^2+3t+2}=2.$$

41. $$\frac{N-4}{N-5}+\frac{N-15}{N+4}=\frac{2N^2-10N-1}{N^2-N-20}.$$

42. $$\frac{7a^2+11a+4}{6a^2+13a+5}+\frac{a+3}{2a+1}=\frac{5a+11}{3a+5}.$$

112. Problems Leading to Fractional Equations. — Types of problems such as those in the following list lead to fractional equations, which are solved by the method of § 111.

EXERCISES

1. The sum of two numbers is 265, and if the larger be divided by the smaller, the quotient is 14 and remainder 10. Find the numbers.

SUGGESTION. — Let $n =$ the smaller number.
Then $265 - n =$ the larger number.
Hence, $$\frac{265-n}{n}=14+\frac{10}{n}.$$

2. The sum of two numbers is 160, and if the larger be divided by the smaller, the quotient is 4. Find the numbers.

3. The difference of two numbers is 324, and if the larger be divided by the smaller, the quotient is 3 and remainder 80. Find the numbers.

4. Separate 84 into two parts whose quotient is $\frac{3}{4}$.

5. Separate 148 into two parts such that if the larger be divided by the smaller, the quotient is 5 and remainder 16.

6. Find where to divide a beam 18 ft. long into two parts such that their quotient is $3\frac{1}{2}$.

7. A board is 42 in. long. A hole is to be bored in the board so that its distance from one end divided by its distance from the other end shall be $\frac{3}{4}$. How far from one end must the hole be bored?

8. A surveyor has two stakes set in the ground 150 ft. apart. He wishes to drive a third stake between them such that its distance to the nearer of the two divided by its distance to the other shall be $\frac{2}{3}$. Where must he set the third stake?

9. What number must be subtracted from each term of the fraction $\frac{10}{13}$ in order that the result shall be equal to $\frac{1}{2}$?

10. What number must be subtracted from each term of the fraction $\frac{17}{37}$ in order that the result shall be equal to $\frac{3}{8}$?

11. What number must be subtracted from each term of the fraction $\frac{21}{26}$ in order that the result shall be equal to $\frac{3}{4}$?

12. What number must be added to each term of the fraction $\frac{13}{25}$ so that the result shall be equal to $\frac{5}{9}$?

13. What number must be added to each term of the fraction $\frac{19}{136}$ so that the result shall be equal to $\frac{3}{16}$?

14. What number must be subtracted from each of the numbers, 14, 20, 32, and 50, so that the quotient of the first two remainders shall be equal to the quotient of the last two remainders?

15. In a certain factory a large machine can turn out a number of articles in 4 days, and a smaller sized machine can turn them out in 6 days. If both machines are operated at once, how long will it take them to turn out the articles?

SUGGESTION. — Let n = number of days required for the machines together to turn out the articles.

Show that $$\frac{1}{n}=\frac{1}{4}+\frac{1}{6}.$$

16. In a factory which manufactures a certain kind of goods there are 3 large machines and 5 small ones. To manufacture the goods required to fill an order would take either one of the

large machines alone 60 days, and either of the small machines alone 90 days. If all 8 machines are operated, how long does it take to fill the order?

17. In a city water system, two pumps pump the water into a reservoir. One pump would fill the reservoir in 16 hr., and the other would fill it in 12 hr. If both pumps were driven at once, how long would it require to fill the reservoir?

18. A tank is fitted with two pipes. One pipe alone could fill the tank in 5 hr., and the other pipe alone in 8 hr. If both pipes were opened at once, how long would it require to fill the tank?

19. A tank can be filled by one pipe in 4 hr. and emptied by another in 5 hr. If both pipes are open, how long will it take to fill the tank?

20. A railroad water tank can be pumped full in $4\frac{1}{2}$ hr. But locomotives draw out and use, on the average, a tankful every 8 hr. At this rate, how long would it take to get the tank pumped full?

21. Of two furnaces in a school building one will burn a given quantity of coal in 8 days and the other will burn it in 10 days. If both furnaces are fired, how long will the quantity of coal last?

22. A farmer has a quantity of corn which he feeds to his horses and hogs. It would last the horses alone 160 days, and the hogs alone 60 days. How long will it last if he feeds it to both horses and hogs?

23. Assuming that they work at constant speeds, A can do a piece of work in 7 days that it would take B 9 days to do. If they work together, how long should it require them both to do the work?

24. A can do a piece of work in 10 days; but after he has worked 2 days, B comes to help him, and together they finish it in 3 more days. In how many days could B alone have done the work?

25. John could remove the snow from the sidewalk in 30 minutes. His larger brother James could do it in 20 minutes

John began the work, but later James took his place, and the snow was all removed in 25 minutes from the beginning. How long did John work?

26. A steamboat makes 8 miles an hour against the wind on a journey of 40 miles. After it has gone 15 miles, the wind ceases. The entire time consumed on the journey is $4\frac{1}{2}$ hr. Find how many miles per hour the wind retards the boat.

27. A train runs into a city from a suburban town, a distance of 40 mi., in 1 hr. 40 min. Of this distance 8 mi. are within the city limits. The train makes 50 mi. an hour outside of the city limits. What is its speed within the city limits?

28. Two trains, approaching each other, leave stations 225 miles apart at the same moment. One train runs 5 miles an hour faster than the other, and they meet at a point 120 miles from the station from which the faster train starts. Find the speeds of the trains.

29. A motorcyclist was overtaken 75 miles from his starting point by an automobile which started from the same place 2 hours later, and traveled 10 miles an hour faster. Find the speed of each.

30. A messenger was started on a journey with a message that later was found to be wrong. One hour and 20 minutes after he left, a second messenger started to overtake him, traveling 2 miles an hour faster. The first messenger was overtaken after he had gone a distance of 16 miles. Find the speed of each.

31. A man started on a journey of 72 miles. After going 42 miles, he stopped and rested 2 hours. Including the 2 hours that he rested, to finish the journey at the same speed took him as long as to go the first 42 miles. Find how fast he traveled.

32. An aviator flew to a point 60 miles away, and back, in 5 hr. 24 min. His rate going was 25 miles an hour. What was his rate returning?

33. An aviator started on a trip to a point 80 miles away. After going 30 miles he increased his speed 20 miles an hour, and made the remaining distance in the same time that it took him to fly the first 30 miles. What was his speed the first 30 miles?

34. An aviator made 25 miles an hour against the wind on a journey of 45 miles. After he had gone 20 miles, the wind ceased. The entire time required for the trip was 1 hr. $25\frac{1}{2}$ min. What was the speed of the wind?

35. In the National League of baseball teams, at one point of the season, Chicago and New York had records as follows:

	Won	Lost
Chicago	64	30
New York	58	32

If these two teams were to play a final series of 6 games together, how many would Chicago have to win in order that the quotient of the number of games won and the number lost should be greater for Chicago than for New York?

Suggestion. — Let x be the number Chicago must win in order that the quotients of the numbers of games won and the numbers lost should be the same for both teams.

Show that $$\frac{64+x}{30+6-x}=\frac{58+6-x}{32+x}.$$

Show that $$x=1\tfrac{15}{49}.$$

Hence, Chicago must win two more games in order to finish ahead of New York.

36. The Pittsburg and Philadelphia baseball teams of the National League had records as follows:

	Won	Lost
Pittsburg	54	34
Philadelphia	50	36

If these two teams were to play a final series of 5 games together, how many would Pittsburg have to win in order that the quotient of the number of games won and the number lost should be greater for Pittsburg than for Philadelphia?

113. Literal Equations: Formulæ. — An equation that contains two or more letters, such as the practical formulæ encountered in the earlier part of the book, is called a **literal equation.**

In a literal equation the number represented by any one of the letters involved may be considered as the unknown number, and its value solved for. It is clear that in such an equation the value found for the unknown number will, in general, be an expression containing the other letters involved in the equation. If a literal equation is linear with respect to the unknown number, it may be solved by the method of this chapter.

EXAMPLE. — Solve $x = \frac{na+b}{n}$ for n.

Clearing of fractions, $nx = na + b.$

Transposing, $nx - na = b.$

Uniting terms, $(x-a)n = b.$

Dividing by $x - a$, $n = \frac{b}{x-a}.$

NOTE. — The process of uniting the similar terms in a literal equation, as in step 3 of the above example, is equivalent to factoring the terms.

EXERCISES

1. Solve $x = \frac{na+b}{n}$ for a.

2. Solve $2a - \frac{b}{x} = \frac{cy}{x}$ for x.

3. Solve $a = \frac{x-a}{x-1}$ for a.

4. Solve $2 + \frac{3}{2t-1} = \frac{v}{2t-1}$ for t.

5. Solve $\frac{R}{R-2}+\frac{V}{R+1}=1$ for V.

6. Solve $\frac{y}{m}+\frac{y}{n}=m+n$ for y.

7. Solve $\frac{A}{B}=\frac{t-B^2}{t-A^2}$ for t.

8. Solve $\frac{m+r}{n+r}-\frac{2}{r^2-n^2}=\frac{m-r}{n-r}$ for r.

9. Solve $\frac{1}{P}+\frac{1}{Q}+\frac{1}{R}=1$ for P.

10. Solve $\frac{1}{N}+1-\frac{N}{d}+\frac{1}{dN}=0$ for d.

11. Solve $\frac{9+3x}{2xy}+\frac{y}{y(y-t)}=\frac{3}{x(t-y)}$ for t.

12. Solve $\frac{3A+4C}{3A}-\frac{4A+3C}{4A}=\frac{1}{10}$ for A.

13. Solve $\frac{2P+2W}{P^2-PW}-\frac{P+W}{P^2-2PW+W^2}=\frac{P+W}{P^2-W^2}$ for P.

14. Solve $\frac{2c^2+d^2}{c^2-d^2}=\frac{c}{c+d}+1$ for c.

Note. — The following formulæ express important principles in science, geometry, etc. It often is necessary, in using such formulæ, to express one of the quantities involved in terms of the others. The pupil will find it of decided advantage to learn that process here.

15. If C and c are the circumferences, and R and r the radii, respectively, of any two circles, then

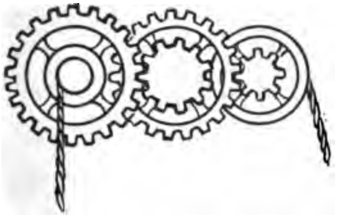
$$\frac{C}{c}=\frac{R}{r}.$$

Solve for r.

16. The figure bounded by two radii of a circle and the arc subtended by them is a sector of the circle. If s and a are the area and arc, respectively, of a sector of a circle whose area is S and circumference C, then $$\frac{s}{S}=\frac{a}{C}.$$

Solve for s; for S; for a; for C.

17. If one gear or cogwheel drives another, and S and T are the speed and number of teeth of the driving wheel, and s and t the speed and number of teeth of the driven wheel, respectively,



then $$\frac{S}{s}=\frac{t}{T}.$$

Solve for S; for s; for t; for T.

If the number of teeth of the driving wheel is 96, of the driven wheel 12, and the speed of the driving wheel 40 revolutions a minute, find the speed of the driven wheel.

18. The formula for computing the area L of a lune, the portion of the surface of a sphere bounded by two semicircles such as the meridians on the earth's surface, is

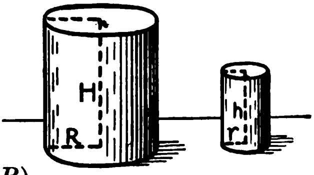
$$\frac{L}{8T}=\frac{A}{4}.$$

Solve for L; for A.

19. In two similar right cylinders whose volumes, altitudes, and radii are V, H, R, and v, h, r, respectively,

$$\frac{V}{v}=\frac{\pi R^2H}{\pi r^2h}.$$

Solve for V; for v; for H; for h.



20. If T and t are the areas of the entire surfaces of the above cylinders,

$$\frac{T}{t}=\frac{R(H+R)}{r(h+r)}.$$

Solve for T; for t; for H; for h.

21. If the radius of a right cylinder is denoted by R, the area of the curved surface by S, the area of the total surface by T, the altitude by H, and the volume by V,

$$(1)\ R=\frac{V}{S};\quad (2)\ RH=\frac{S}{2\pi};\quad (3)\ R^2H=\frac{V}{\pi};$$

$$(4)\ R^2+HR=\frac{T}{2\pi};\quad (5)\ R^2+HR=\frac{T+S}{2\pi}.$$

Solve (1) for V; for S. Solve (2) for R; for H; for S. Solve (3) for H; for V. Solve (4) for H; for T. Solve (5) for T; for S; for H.

22. In the above cylinder it is known also that $\frac{H}{R}=\frac{T-S}{S}$. Solve for H; for R; for T; for S.

23. The formula $F=\frac{Mv^2}{r}$ is used in computing centrifugal force, or the force with which an object moving in a circle tends to fly off from the center. It is the force which causes grindstones and flywheels to burst when run at too high a speed, and carriages to overturn when turning a corner at too high a speed.

Solve for M; for r.

24. Solve $f=\frac{wv^2}{gr}$ for w; for g; for r.

25. The densities of some substances are computed by the formula,

$$D=\frac{w}{w-w'}.$$

Solve for w; for w'.

26. The formula $C=\frac{E}{R}$ is of great importance in electricity. It gives the relation between the quantity of current C of electricity flowing through a wire or other conductor, the resistance offered to the current by the conductor, and the electrical pressure (electromotive force) required to overcome that resistance.

Solve for E; for R.

27. If the resistance within an electric battery cell is R, the resistance of the external circuit through which the current flows r, the strength of current C, and the electric pressure E, then

$$C=\frac{E}{R+r}.$$

Solve for E; for R; for r.

28. If n equal electric cells are connected by wires in one way (series), so as to form a battery, the current strength is found by the formula

$$C=\frac{nE}{R+nr}.$$

Solve for n; for E; for R; for r.

29. If the cells in Problem 28 are connected in another way (parallel), the current strength of the battery is found by the formula

$$C=\frac{E}{R+\frac{r}{n}}.$$

Solve for n; for E; for R; for r.

30. If an electric circuit between two points is divided into two branches, so that part of the current flows through one branch and the rest through the other, the total resistance of the circuit is computed by the formula

$$\frac{1}{R}=\frac{1}{r_1}+\frac{1}{r_2},$$

where R is the total resistance, r_1 the resistance of one branch, and r_2 the resistance of the other branch.

Solve for R.

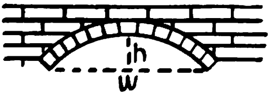
31. If the circuit in Problem 30 is divided into three branches whose resistances are r_1, r_2, and r_3, the formula is

$$\frac{1}{R}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}.$$

Solve for R.

32. Solve $S = \frac{a - rl}{1 - r}$ for r; for a; for l.

33. The radius r of the arch of a bridge or doorway whose span is w and rise h is found from the formula



$$\frac{2r - h}{\frac{1}{2}w} = \frac{\frac{1}{2}w}{h}.$$

Solve for r.

34. Find the radius at which the stones must be cut for an arch whose span is 16 ft. and rise 4 ft.

SUPPLEMENTARY EXERCISES

Solve:

1. $\frac{4}{2n^2 - n - 1} + \frac{1}{2n^2 + 3n + 1} = \frac{1}{n^2 - 1}.$

2. $\frac{1}{A - 2} + \frac{1}{A + 2} = \frac{2}{A - 3}.$

3. $\frac{7w + 2}{6w^2 + 13w - 5} - \frac{w + 2}{2w^2 - w - 15} = \frac{2}{3w - 1}.$

4. $\frac{6r^2 - 1}{r^2 - 9r + 14} = \frac{3r - 5}{r - 7} + 3.$

5. $\frac{18t - 27}{14} + \frac{11t - 1}{3t + 1} = \frac{9t + 11}{7}.$

Suggestion.—In equations like this, in which some denominators are monomials and others polynomials, it often is best to clear of monomial denominators only at first, and then simplify before clearing of the remaining denominators. Thus, multiplying both members by 14,

$$18t - 27 + \frac{154t - 14}{3t + 1} = 18t + 22.$$

This becomes

$$\frac{154t - 14}{3t + 1} = 49.$$

6. $\frac{4P+3}{9}+\frac{7P-29}{5P-12}=\frac{8P+19}{18}.$

7. $\frac{5k+5}{k-5}=\frac{11k-15}{10}-\frac{33k+15}{30}.$

8. $4+\frac{N+7}{2(3N-4)}+\frac{9N+3}{15}=\frac{3N+6}{5}.$

9. $\frac{x-7}{x-9}-\frac{x-9}{x-11}=\frac{x-13}{x-15}-\frac{x-15}{x-17}.$

SUGGESTION. — In equations of this kind much multiplication may be avoided by combining some of the fractions before clearing the equation of fractions. Thus, performing the subtractions indicated in each member,

$$\frac{-4}{x^2-20x+99}=\frac{-4}{x^2-32x+255}.$$

10. $\frac{y-5}{y-7}+\frac{y-7}{y-9}=\frac{y-4}{y-6}+\frac{y-8}{y-10}.$

SUGGESTION. — First transpose in order to form differences such that the fractions when combined will give simple numerators. Thus,

$$\frac{y-5}{y-7}-\frac{y-4}{y-6}=\frac{y-8}{y-10}-\frac{y-7}{y-9}.$$

11. $\frac{B}{B-2}-\frac{B+1}{B-1}=\frac{B-8}{B-6}-\frac{B-9}{B-7}.$

12. $\frac{1}{v-3}-\frac{1}{v-1}=\frac{1}{v-4}-\frac{1}{v-2}.$

13. $\frac{n-3}{n-4}-\frac{n-4}{n-5}=\frac{n-1}{n-2}+\frac{n-2}{3-n}.$

14. $\frac{a+1}{a+2}+\frac{a+6}{a+7}=\frac{a+5}{a+6}+\frac{a+2}{a+3}.$

15. $\frac{P-1}{P-2}+\frac{P-7}{P-8}-\frac{P-5}{P-6}=\frac{P-3}{P-4}.$

16. $\frac{k-9}{k+5}+\frac{k-5}{k+9}=\frac{k-10}{k+4}+\frac{k-4}{k+10}.$

17. $\frac{2a+5}{2a+6}+\frac{2a+2}{2a+3}=\frac{2a+1}{2a+2}+\frac{2a+6}{2a+7}.$

18. $\frac{r-1}{r-2}=\frac{1}{r^2-r-2}+\frac{r-2}{r+1}.$

19. $\frac{4y+1}{4y-1}-\frac{4y-1}{4y+1}=\frac{8}{16y^2-1}.$

20. $\frac{z}{3-z}-\frac{3-2z}{3+z}=\frac{3(2z-7)}{6(3+z)}.$

21. $\frac{s-1}{s-2}=\frac{s}{s+2}+\frac{3}{s+1}.$

22. Solve $\frac{ax+b}{bx+c}-\frac{ax-c}{bx+2c}=\frac{b^2+c(a+b)}{b(bx+3c)}$ for x.

23. Solve $\frac{t^2-at}{t^2+ct-at-ac}+\frac{a-t}{t-c}=0$ for t.

24. Solve $\frac{1}{a(b-n)}+\frac{1}{b(c-n)}=\frac{1}{a(c-n)}$ for n.

25. Solve $\frac{xn}{2}-xy\left(\frac{x}{2}-\frac{y}{5}\right)=\frac{3y}{5}\left(n-\frac{2xy}{3}\right)$ for n.

26. Solve $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}$ for R.

27. Solve $E = \frac{mcT + m'c't}{mc + m'c'}$ for t.

28. Solve $\dfrac{\dfrac{V}{m-n} - m}{\dfrac{V}{m-n} + m} + 1 = \dfrac{2}{1 + mn + m^2}$ for V.

29. In a guessing game, the leader says: "If you will add 20 years to your age, divide the sum by your age, add 3 to the quotient, and tell me the result, I will tell you your age." How did he find it?

Suggestion.—Let $x =$ the age and $a =$ the result given. Write an equation and solve it for x.

30. Make up a guessing game similar to that in Problem 29, and show how the age is found.

31. If a body whose weight is W pounds, moving with a velocity of V feet per second, strike a second body whose weight is w pounds and which is at rest, so that the two bodies move on together, the velocity v with which they continue together is found from the formula

$$v = \frac{W \times V}{W + w}.$$

Solve for W; for V; for w.

32. If a freight car, weighing 40,000 lb. and moving 8 ft. per second, strike a second car that weighs 32,000 lb. and is standing still, and the two are then coupled together, with what speed will they continue?

CHAPTER XI

PROPORTION. VARIABLES

114. Ratio. — The quotient of one number divided by another of the same kind is sometimes called their **ratio**. It usually is written in the form of a fraction, and is subject to all of the rules that apply to fractions.

Thus, the ratio of \$6 to \$100 is written $\frac{\$6}{\$100}$. The ratio of 2 lb. to 5 lb. is written $\frac{2 \text{ lb.}}{5 \text{ lb.}}$. And, in general, the ratio of a to b is written $\frac{a}{b}$.

NOTE. — An old form of writing the ratio of a to b is $a : b$. This notation is less convenient for computation than the fractional form, and is used less now than formerly.

The dividend or numerator of a ratio is sometimes called the **antecedent**, and the divisor or denominator is called the **consequent**.

EXERCISES

1. Express the ratio of 3 qt. to 6 qt. as a fraction in its lowest terms.

2. If the rate of interest on a sum of money is 5 %, that is, the ratio of \$5 to \$100, express the rate as a fraction in its lowest terms.

3. The death rate in Boston in a recent year was 18 to 1000 population. Express this ratio as a fraction.

4. An alloy consists of copper and tin in the ratio of 2 to 3. What part of it is each?

5. A solution consists of alcohol and water in the ratio of 3 to 5. What part of it is water?

6. In a city of 84,000 population the number of births in a year is 560. What is the birth rate? How many per 1000?

7. When a man 5 ft. 10 in. tall casts a shadow 35 ft. long, find the ratio of his height to the length of his shadow.

8. The *specific gravity* of a solid or liquid is the ratio of the weight of any volume of it to the weight of an equal volume of water. A cubic foot of water weighs 62.5 lb. A cubic foot of steel weighs 490 lb. Find the specific gravity of steel. Express the answer first as a common fraction, then as a decimal computed to tenths.

9. Find the specific gravity, computed decimally to tenths, of each of the following substances:

SUBSTANCE	WEIGHT IN POUNDS OF 1 CU. FT.	SUBSTANCE	WEIGHT IN POUNDS OF 1 CU. FT.
Cast Iron	450.0	Oak	65.0
Brass	523.8	Ash	52.8
Gold	1200.9	Cork	15.0

10. The diameter of a circular plate is 6 in., and the circumference is measured with a tape line and found to be 18.85 in. Find the ratio of the circumference to the diameter.

11. A boy measures the wheel of his bicycle, and finds that its diameter is 28 in. and circumference 88 in. What is the ratio of the circumference to the diameter?

12. Simplify the ratio of $x + y$ to $x^2 - y^2$ by writing it as a fraction reduced to its lowest terms.

13. Simplify the ratio of $\frac{a}{b}$ to $\frac{b'}{a'}$ by writing it as a simple fraction.

14. Which ratio is the greater, $\frac{7}{12}$ or $\frac{26}{45}$?

SUGGESTION. — Reduce them to a common denominator, and compare the numerators.

15. Write in descending order of magnitude $\frac{5}{6}$, $\frac{13}{15}$, $\frac{21}{25}$.

16. Two partners in business, A and B, divide a profit of \$ 2135 between them so that A's part and B's part are in the ratio of 3 to 4. How many dollars does each receive?

SUGGESTION. — Let d be the amount that A receives.
Then, $2135 - d$ is the amount that B receives.

Hence, $$\frac{d}{2135 - d} = \frac{3}{4}.$$

17. If two partners, A and B, divide a profit of \$1200 in the ratio of 1 to 2, how many dollars does each receive?

18. In a certain city with a population of 247,520, the ratio of the Germans to all other nationalities together is 2 to 15. What is the German population?

19. Separate 40 into two parts which are in the ratio of 3 to 7.

20. Separate 85 into two parts which are in the ratio of 5 to 12.

21. The sides of a triangle are 5 in., 9 in., and 7 in. Divide the side 7 in. long into two parts whose ratio equals the ratio of the other two sides.

115. Proportion. — An equation whose members consist of two ratios is called a **proportion**.

Thus, $\frac{3}{4} = \frac{6}{8}$, $\frac{5a}{6b} = \frac{15a}{18b}$, and $\frac{a}{b} = \frac{c}{d}$ are proportions.

The proportion $\frac{a}{b} = \frac{c}{d}$, is read either "$a$ over b equals c over d," or "a is to b as c is to d."

The numbers forming one of the ratios are said to be "proportional to" the numbers forming the other ratio; or the four numbers forming the ratios are said to be "in proportion."

In any proportion $\frac{a}{b} = \frac{c}{d}$, a, b, c, and d are called the **terms**. The first and fourth terms, a and d, are called the **extremes**; and the second and third terms, b and c, are called the **means**.

Since a proportion is an equation, all of the operations which may be performed upon an equation, such as clearing of fractions, etc., may be performed upon a proportion.

NOTE. — Since the ratio of any two concrete numbers is the same as that of the corresponding abstract numbers, in performing any operation upon a proportion, confusion may be avoided by using only abstract values for the terms.

EXERCISES

Find the value of the unknown term in each of the following proportions:

1. $\frac{n}{2}=\frac{15}{10}$.
2. $\frac{x}{12}=\frac{1}{8}$.
3. $\frac{4}{y}=\frac{16}{21}$.
4. $\frac{7}{9}=\frac{42}{t}$.
5. $\frac{3}{10}=\frac{A}{40}$.
6. $\frac{24}{35}=\frac{r}{70}$.
7. $\frac{36}{v}=\frac{1}{4}$.
8. $\frac{9}{16}=\frac{12}{P}$.
9. $\frac{20}{36}=\frac{8}{L}$.

Find the values of n in the proportions:

10. $\frac{a}{b}=\frac{c}{n}$:
11. $\frac{x}{n}=\frac{y}{z}$.
12. $\frac{t}{v}=\frac{n}{w}$.
13. $\frac{n-a}{a}=\frac{n+b}{b}$.
14. $\frac{1}{r}=\frac{r}{n}$.
15. $\frac{an}{b}=\frac{b}{c}$.

16. A family of three members and a family of four members camp out together. The total cost of provisions is $112, which they wish to divide in proportion to the sizes of the families. How much must each family pay?

116. Definitions. — In any proportion $\frac{a}{b}=\frac{c}{d}$, the fourth term d is called a **fourth proportional** to the other three terms, a, b, and c.

A proportion such as $\frac{a}{x}=\frac{x}{b}$, in which the means are equal, is called a **mean proportion.** The mean term x is called the **mean proportional** between the extremes a and b.

In a mean proportion such as $\frac{a}{x}=\frac{x}{b}$, b is called the **third proportional** to a and x.

EXERCISES

Find the fourth proportional to:

1. 2, 3, and 10.

Suggestion. — Let x be the fourth proportional. Then $\frac{2}{3}=\frac{10}{x}$.

2. 4, 5, and 12.
3. 7, 10, and 28.
4. 16, 12, and 4.
5. 1, a, and a^2.
6. m, n, and m^2n.
7. 1, $1-t$, and $1+t$.

Find the third proportional to:

8. 3 and 6.

Suggestion. — Let x be the third proportional. Then $\frac{3}{6}=\frac{6}{x}$.

9. 8 and 12.
10. x and $3x^3$.
11. P and Pt.
12. a and b.
13. $1+n$ and $1-n^2$.
14. 10 and 1.

Find the mean proportional between:

15. 2 and 8.

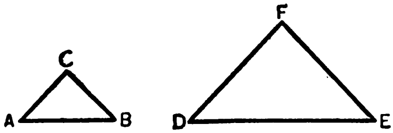
Solution. — Let x be the mean proportional. Then $\frac{2}{x}=\frac{x}{8}$.

Clearing of fractions, $x^2=16$.

Taking square root, $x=\pm 4$.

16. 3 and 12.
17. 7 and 28.
18. 9 and 1.
19. n and n^5.
20. $2A^3$ and $50A$.
21. R and $R+2R^2+R^3$.

117. Distances found by Proportion. — Two triangles that are drawn with the angles of one equal to the corresponding angles of the other have the same shape, and are called **similar triangles**, as ABC and DEF. Let the student draw two such triangles, making the



side DE of one three times as long as the corresponding side AB of the other. Measure the other sides, and compare them. It will be found that:

In two similar triangles, the corresponding sides are proportional.

Thus, in the above figure, $\frac{AB}{DE} = \frac{AC}{DF}$, etc.

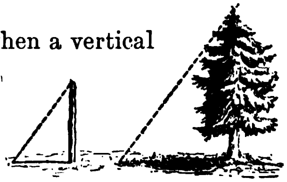
This principle is much used in computing unknown distances.

EXERCISES

1. In two similar triangles, the sides of the larger are 12 in., 14 in., and 20 in., and the shortest side of the smaller is 3 in. Find the other sides of the smaller triangle.

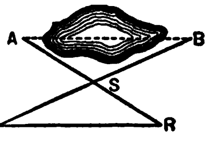
2. The sides of a triangular field are 16 rd., 24 rd., and 30 rd. The smallest side of a similar field is 40 rd. Find the other two sides.

3. A tree casts a shadow 48 ft. long when a vertical rod 6 ft. high casts a shadow 4 ft. long. How high is the tree?



4. A church spire casts a shadow 23 ft. long when a man 5 ft. 10 in. tall who is passing by casts a shadow 2 ft. long. Find the height of the spire.

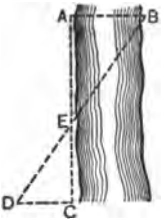
5. The distance from A to B on opposite sides of a lake may be found as follows:



The distances from A to R and from B to T are measured off, making the triangles ASB and TSR similar. If AS is taken 400 yd., and SR 300 yd., and TR is measured and found to be 580 yd., how far is it from A to B?

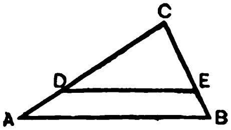
6. If A and B are two points on opposite sides of a hill, and out in the plain at the foot of the hill distances are measured as in Problem 5, so that AS is taken 4050 ft. and SR 1350 ft., and TR is found to be 1600 ft., how far is it between A and B?

7. To find the distance AB across a stream, measure off a distance AC several yards long, along the bank. Then at C measure off a distance CD at right angles to AC. By sighting across from D to B, locate a stake at a point E of AC in line with D and B. Get the lengths of AE and EC. Since the triangles are similar, write the proportion by means of which AB may be computed.



If AE is 80 yd., EC 20 yd., and DC 15 yd., what is the width of the stream?

8. It is known in geometry that a line parallel to one side of a triangle divides the other two sides into four proportional parts. That is, if DE is parallel to AB, $\frac{CD}{DA}=\frac{CE}{EB}$.



(1) If $AD = 6$ in., $DC = 3$ in., and $BE = 4$ in., find EC.

(2) If $AD = 10$ in., $BE = 12$ in., and $EC = 20$ in., find DC.

(3) If $DC = 9$ in., $BE = 7$ in., and $EC = 5$ in., find AD.

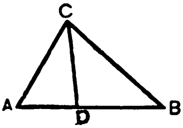
9. In the figure of Problem 8, if $AD = 8$ in., $DC = 5$ in., and $BC = 26$ in., find BE and EC.

SUGGESTION. — Let $BE = x$. Then $EC = 26 - x$.

Hence, $$\frac{8}{5}=\frac{x}{26-x}.$$

10. In the figure of Problem 8, if $AC = 40$ ft., $BE = 12$ ft., and $EC = 15$ ft., find AD and DC.

11. It is known in geometry that in any triangle ABC, if the line CD divides the angle at C into two equal parts, it divides the opposite side into parts proportional to the other two sides; that is, $\frac{AD}{DB}=\frac{AC}{BC}$.

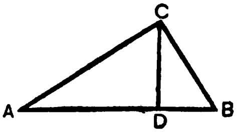


If $AC = 15$ in., $BC = 10$ in., and $AB = 12$ in., find AD and DB.

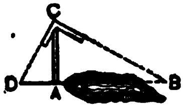
12. In the figure of Problem 11, if $AC = 40$ yd., $BC = 32$ yd., and $AB = 60$ yd., find AD and DB.

13. In the triangle ABC, the angle at C is a right angle, and CD is perpendicular to AB. It is known in geometry that CD is a mean proportional between AD and DB.

If $AD = 20$ in., and $DB = 5$ in., find CD.

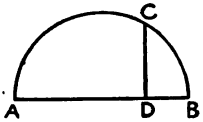


14. A method used several hundred years ago of finding the distance from A to the inaccessible point B was to erect a vertical staff AC, place upon it an instrument resembling a carpenter's square, pointing one blade towards B, and note the place D on the ground to which the other blade pointed. AC and DA were measured. If $AC = 52$ in., and $DA = 6$ in., find AB.



15. In the semicircle with the diameter AB, CD, a perpendicular to AB, is a mean proportional between AD and DB.

If $AD = 12$ in., and $DB = 3$ in., find CD.
If $AD = 8$ in., and $CD = 6$ in., find DB.
If $AB = 20$ in., and $CD = 8$ in., find AD and DB.



118. Proportion in Simple Machines. — Simple machines, such as the lever, the wheel and axle, etc., are instruments by means of which a small **effort** exerted may be made to overcome a **resistance** of much greater size. Proportion is involved in the use of simple machines.

EXERCISES

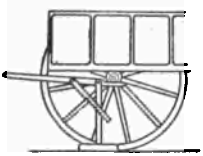
1. A lever is a stiff bar of wood or metal that is movable about a fixed point or pivot called the fulcrum. The resistance W to be overcome is applied at one end of the bar, and the effort w, exerted to overcome it, at the other end. If D is the distance of W from the fulcrum, and d the distance of w from the fulcrum, then

$$\frac{w}{W} = \frac{D}{d}.$$

What effort is required to lift a weight of 600 lb. by a lever, if the weight is 8 in. from the fulcrum and the effort applied 40 in. from the fulcrum?

2. What effort is required to lift a stone weighing 840 lb. by means of a crowbar 60 in. long, if the fulcrum is placed 4 in. from the stone?

3. A blacksmith weighing 180 lb. lifts wagons, etc., by means of a wagon jack. The weight lifted is applied at a point 6 in. from the pivot or fulcrum, and he grasps the handle 32 in. from the pivot. By throwing his whole weight upon the jack, how much can he lift?



4. The entire length of a lever is 48 in. Where must the fulcrum be placed in order that a resistance of 360 lb. may be overcome by an effort of 120 lb.?

5. The wheel and axle consists of a wheel and a cylindrical axle passing through its center, the two being fastened rigidly together, so that the axle is turned by revolving the wheel. The weight or resistance to be overcome is applied at the circumference of the axle through a cord wrapped around the axle, and the effort required to overcome it is applied at the circumference of the wheel. If the resistance is W, the effort required to overcome it w, the radius of the axle R, and the radius of the wheel r, then $\frac{w}{W} = \frac{R}{r}$.

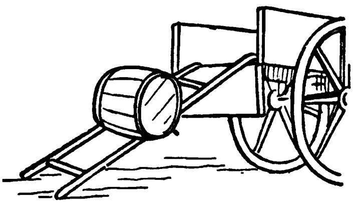
If the radius of the wheel is 20 in., the radius of the axle 6 in., and the resistance to be overcome 500 lb., what is the effort required?

6. The axle of a windlass shown in the above figure, which is used for drawing water from a well, has a radius of $4\frac{1}{2}$ in., and the radius of the wheel 15 in. What effort is required to lift a bucket of water weighing 54 lb.?

7. A capstan, a form of wheel and axle used in lifting anchors on ships, has an axle 12 in. in diameter and a lever arm or spoke 45 in. long. How much effort is required to lift by it an anchor weighing a ton?

8. The inclined plane is a smooth sloping surface that is used in raising a heavy object through the application of a comparatively small effort, the object being raised by sliding or rolling it along the surface of the plane. If the effort is applied parallel to the surface of the plane, the effort applied is to the weight lifted as the height of the plane is to its length; that is, $\frac{w}{W}=\frac{H}{L}$, where w = the effort applied, W = the weight, H = the height of the plane, or distance the weight is lifted, and L = the length of the plane.

If the length of an inclined plane is 12 ft., the height 3 ft., and the weight of the object lifted 400 lb., what effort is required to move it up the plane?



9. A barrel of material weighing 180 lb. is loaded into a wagon bed 32 in. from the ground by rolling it up a board 10 ft. long with one end resting on the ground and the other on the wagon bed. What effort is required?

10. Two men place a 580 lb. building stone upon a wagon by sliding it up a board 8 ft. long, one end of the board being on the ground and the other resting on the bed of the wagon. The wagon bed is 20 in. from the ground. What effort do they use in loading the stone in addition to that required to overcome the friction?

11. Ice is stored in an ice house by dragging it up an inclined plane. The incline is 200 ft. long and 40 ft. high. What effort is required to pull up a block of ice weighing 250 lb.?

12. A locomotive pulls a train weighing 720 tons up a grade with a rise of $2\frac{1}{2}$ ft. to 100 ft. of grade. How much more pull must the locomotive exert than if the train were on a level road bed?

119. Important Principles in Proportion. — The following important principles in proportion are often used in geometry, and elsewhere, where proportion is applied. Let the pupil establish each of them.

(1) *If four numbers are in proportion, the product of the extremes equals the product of the means.* That is,

$$\text{if } \frac{a}{b} = \frac{c}{d}, \text{ then } ad = bc.$$

Suggestion. — Clear $\frac{a}{b} = \frac{c}{d}$ of fractions.

(2) *If the product of two numbers equals the product of two other numbers, the four numbers are in proportion.* That is,

$$\text{if } ad = bc, \text{ then } \frac{a}{b} = \frac{c}{d}.$$

Suggestion. — Divide both members of $ad = bc$ by bd.

(3) *If four numbers are in proportion, they are in proportion by inversion.* That is,

$$\text{if } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{b}{a} = \frac{d}{c}.$$

Suggestion. — Divide $1 = 1$ by the members of $\frac{a}{b} = \frac{c}{d}$.

(4) *In any proportion, the means may be interchanged, or the extremes interchanged, without destroying the proportion.* That is,

$$\text{if } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a}{c} = \frac{b}{d} \text{ and } \frac{d}{b} = \frac{c}{a}.$$

Suggestion. — Multiply both terms of $\frac{a}{b} = \frac{c}{d}$ by $\frac{b}{c}$; by $\frac{d}{a}$.

(5) *The terms of any proportion are in proportion by addition* That is,

$$\text{if } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a+b}{b} = \frac{c+d}{d}.$$

Suggestion. — Add 1 to each member of $\frac{a}{b} = \frac{c}{d}$.

(6) *The terms of any proportion are in proportion by subtraction.* That is,

$$\text{if } \frac{a}{b}=\frac{c}{d}, \text{ then } \frac{a-b}{b}=\frac{c-d}{d}.$$

SUGGESTION. — Subtract 1 from each member of $\frac{a}{b}=\frac{c}{d}$.

(7) *Like powers of the terms of a proportion are in proportion.* That is,

$$\text{if } \frac{a}{b}=\frac{c}{d}, \text{ then } \frac{a^2}{b^2}=\frac{c^2}{d^2}, \ \frac{a^3}{b^3}=\frac{c^3}{d^3}, \text{ etc.}$$

SUGGESTION. — Raise both members of $\frac{a}{b}=\frac{c}{d}$ to the same power.

(8) *If two or more ratios are equal, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.* That is,

$$\text{if } \frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\text{ etc., then } \frac{a+c+e+\text{ etc.}}{b+d+f+\text{ etc.}}=\frac{a}{b}=\text{ etc.}$$

For, since $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=$ etc., let each ratio equal r.

Then $a=rb,\ c=rd,\ e=rf,$ etc.

Hence, $a+c+e+\text{etc.}=rb+rd+rf+\text{etc.}$

$=r(b+d+f+\text{etc.}).$

Hence, $\frac{a+c+e+\text{etc.}}{b+d+f+\text{etc.}}=r=\frac{a}{b}=\frac{c}{d}=\text{etc.}$

EXERCISES

1. If $ad=bc$, show that $\frac{a}{c}=\frac{b}{d}$.

2. If $ad=bc$, show that $\frac{d}{b}=\frac{c}{a}$.

3. If $ad=bc$, show that $\frac{d}{c}=\frac{b}{a}$.

4. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

SUGGESTION. — Divide the members of $\frac{a+b}{b}=\frac{c+d}{d}$ (5) by the members of $\frac{a-b}{b}=\frac{c-d}{d}$ (6).

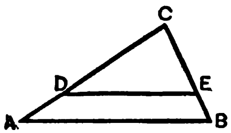
5. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a+b}{c+d}=\frac{b}{d}$.

6. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a-b}{c-d}=\frac{b}{d}$.

7. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{3a+b}{b}=\frac{3c+d}{d}$.

8. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a-2b}{b}=\frac{c-2d}{d}$.

9. It is known that, in triangle ABC, if DE is parallel to AB, $\frac{AD}{DC}=\frac{BE}{EC}$. Show from this that $\frac{AC}{DC}=\frac{BC}{EC}$.



10. In the similar triangles in § 117, it is known that $\frac{AB}{DE}=\frac{BC}{EF}=\frac{CA}{FD}$. Show from this that the ratio of the perimeters of the triangles equals the ratio of any two corresponding sides, such as AB and DE. See § 119, (8).

120. Variable Quantities. — Any quantity such as one's age, the temperature, the population of the country, etc., that is always changing is called a **variable quantity**, or a **variable**. Any quantity that does not change is called a **constant**.

Many of the quantities encountered in business, in science, etc., are variables.

Thus, the prices at which many foodstuffs are retailed usually vary from day to day, or from week to week. Butter may sell at 24 ¢ one week, 27 ¢ the next week, 29 ¢ the next week, etc.

As an illustration of variable quantities encountered in science, suppose that a small heavy object such as a lead ball be dropped from a great height toward the earth. The farther it falls, the faster it falls. At the end of one second its speed will be about 32 ft. per second, at the end of the second

second its speed will be 64 ft. per second, at the end of the third second its speed will be 96 ft. per second, etc. That is, the speed is a variable constantly increasing.

121. Related Variables. — Some variable quantities are so related to each other that the values of one depend upon the values of the other.

Thus, if a train runs 40 miles an hour, the distance that it has traveled depends upon the time elapsed since it started. As the time increases the distance increases.

The relation between two such variables may be expressed by an equation.

Thus, in the above example, if the distance in miles is denoted by d, and the time in hours by t, then the relation between them is expressed by $d = 40t$.

One of the simplest cases of related variables is that in which the *ratio of two variables is constant.* If x and y are the variables, then $\frac{x}{y} = k$, or $x = ky$, where k is a constant. In this case it is said that "x varies directly as y."

It is evident that *when one quantity varies directly as another, the ratio of any two corresponding values of the variables equals the ratio of any other two corresponding values,* since each ratio equals the same constant.

Thus, if x varies directly as y, and x_1 and y_1, and x_2 and y_2 are any two pairs of corresponding values of x and y, then

$$\frac{x_1}{y_1} = \frac{x_2}{y_2}.$$

Hence, the fact that x varies directly as y is sometimes expressed by saying that "x is proportional to y."

EXAMPLE. — m varies directly as n, and when m is 3, n is 5. Write the equation expressing the relation between m and n. Find m when n is 15.

$$\frac{m}{n} = \frac{3}{5}.$$

If n is 15, $$\frac{m}{15} = \frac{3}{5}.$$

Solving, $$m = 9.$$

EXERCISES

1. If a train runs 35 miles an hour, write the equation expressing the relation between the variable time t and the distance d.

2. If the price of coffee is 30 ¢ a pound, the cost of any quantity varies as the number of pounds bought. If $c =$ cost and $w =$ weight in pounds, express the relation between them by an equation.

3. If the price of sugar is 5 ¢ a pound, express by an equation the relation between the number of pounds bought and the cost.

4. If the rate of interest is 6 %; the amount of interest on any sum of money varies as the time. Express by an equation the relation between the interest and the time.

5. The velocity of a body let fall toward the ground varies as the time during which it has fallen from rest, and the velocity at the end of 3 sec. is 96 ft. per second. As the time increases, the velocity increases at the same rate. Write the equation between the velocity and time.

6. The force with which a moving body of any given velocity strikes a stationary body varies as the mass or weight of the moving body. If I strike a nail with a force of 15 lb. by using a hammer weighing $\frac{1}{2}$ lb., with what force would a 2 lb. hammer strike it when swinging with the same speed?

7. The resistance offered by a wire of given size to a current of electricity varies directly as the length of the wire. If a wire 50 ft. long gives a resistance of 6 ohms, what will be the resistance of 75 ft.?

8. If R is the resistance in Problem 7 and L the length of wire, write the equation between them.

9. If x varies directly as y, and $x = 2$ when $y = 9$, find x when $y = 36$. Write the equation between x and y.

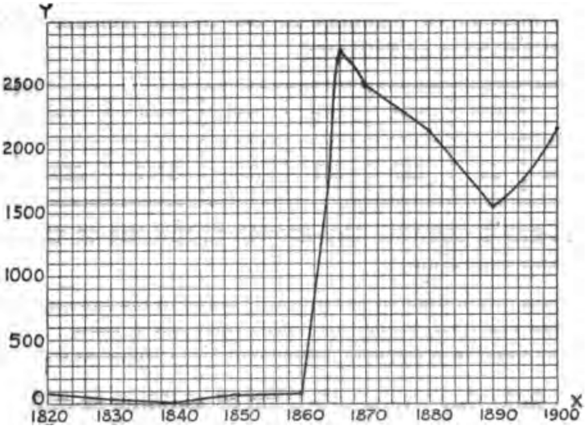
10. If P varies directly as V, and $P = 7$ when $V = 5$, write the equation between P and V. Find V when $P = 35$.

122. Variation shown by Graphs.—In dealing with variable quantities, their variation often is depicted to the eye by means of a **chart** or **graph**. See § 29.

For example, the public debt of the United States, in millions of dollars, from 1820 to 1900 is given in the following table:

YEAR	1820	1830	1840	1850	1860	1866	1870	1880	1890	1900
Debt	91	49	4	63	65	2773	2480	2120	1552	2136

On squared paper draw a horizontal line OX and a line OY at right angles to it. These are called the **axes.** The point O is called the **origin.**



On the line OX let a distance of one space represent 2 years of time. Then the dates in the above table will be represented on OX as in the chart. On OY let a distance of one space represent 100 million dollars, On the 1820 vertical line measure off a distance representing 91 million dollars, and mark a point. On the 1830 vertical line measure off a distance representing 49 million dollars, and mark a point. Similarly, locate a point on each of the other date lines corresponding to the amount of the debt of that date. Draw a line connecting all of the points obtained.

This curved line, called the *graph* of the national debt, depicts clearly to the eye the variation of the national debt throughout the period of time considered.

Note. — Various kinds of apparatus are made for constructing graphs automatically. The *thermograph* or recording thermometer is a good illustration. A pen connected to a thermometer moves up and down as the temperature rises and falls. At the same time a clock mechanism runs a strip of ruled paper under the pen, so that the pen traces a continuous curve or graph on the paper.

Among other instruments for making graphs is the *seismograph*, used for recording earthquakes.

The pupil should be provided with several sheets of squared paper for use in the exercises of this and subsequent chapters.

EXERCISES

1. The number of teachers in the public schools of Chicago since 1850 has been as follows:

Year	1850	1860	1870	1880	1890	1900	1909
Teachers	21	123	557	898	2711	5806	6296

Draw a graph showing the growth in the size of the teaching force during that time.

Suggestion. — On squared paper draw axes as in the example above. On one represent the time, letting one space represent 2 years. On the other represent the number of teachers, letting one space represent 200 teachers.

2. The following average heights of children at different ages have been determined from measurements of thousands of individuals:

Age	½ Yr.	1 Yr.	2 Yr.	3 Yr.	4 Yr.	5 Yr.	6 Yr.	7 Yr.	8 Yr.
Height	24 in.	29 in.	36 in.	40 in.	42 in.	44 in.	46 in.	48 in.	50 in.

Draw a graph showing the rate of growth of the average individual.

3. The population, to the nearest million, of the United States since 1800 is given by decades as follows:

1800,	5	1840,	17	1880,	50
1810,	7	1850,	23	1890,	63
1820,	10	1860,	31	1900,	76
1830,	13	1870,	39	1910,	93

Draw a graph showing the growth of the population.

4. The population, to the nearest thousand, of the city of Chicago since 1840 is recorded as follows:

1840,	4	1860,	109	1880,	503	1900,	1699
1850,	28	1870,	299	1890,	1100	1910,	2185

Draw a graph showing the growth of the city.

5. The following table gives, in billions of dollars, the total value of the farm property and products of the United States by decades from 1850 to 1900:

YEAR	1850	1860	1870	1880	1890	1900
Value	4	8	11	12	16	21

Show by a graph the growth of agriculture in the United States.

6. The amount of the annual exports of the United States, in millions of dollars, has been as follows:

1790,	20	1820,	70	1850,	152	1880,	853
1800,	71	1830,	74	1860,	400	1890,	910
1810,	67	1840,	132	1870,	451	1900,	1499

Show the growth of our exports by a graph.

7. The temperature at a place was recorded as follows: 6 A.M., 8°; 7 A.M., 8°; 8 A.M., 10°; 9 A.M., 11°; 10 A.M., 14°; 11 A.M., 18°; 12 M., 20°; 1 P.M., 20°; 2 P.M., 20°; 3 P.M., 18°; 4 P.M., 16°; 5 P.M., 15°; 6 P.M., 13°. Draw a graph to show the variation of temperature during the 12 hours.

123. Price Curves, etc. — Graphs may be constructed and used for determining costs of different quantities of goods, interest on money for different periods of time, etc., without computation. This is shown by an example.

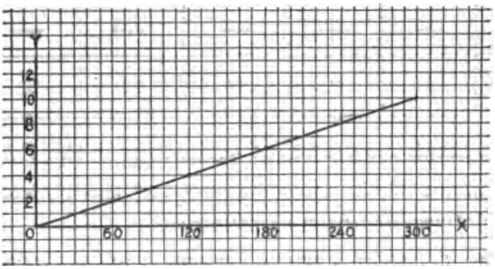
If eggs sell at 30¢ a dozen, the relation between the number of dozens and the cost may be expressed by the equation

$$c = 30d,$$

where d is the number of dozens and c is their cost. If values are given to d, corresponding values may be found for c, as given in the table:

d	0	2	4	6	8	10	12
c	0	60	120	180	240	300	360

On squared paper draw two axes, OX and OY, at right angles, as in § 122.



On OY let a space represent 1 doz., and on OX let a space represent 10¢. Then, on the 60¢ line mark a point representing 2 doz. On the 120¢ line mark a point representing 4 doz., etc. Draw a line through the points thus marked. It is seen that this line or graph is a straight line. It is called the "price curve." Most price curves are straight lines.

By looking at this price curve we can get the cost of any number of dozens, even of a fractional number. For example, to find the cost of 7 doz., observe the point where the horizontal line 7 spaces up meets the price curve; observe the point directly beneath this on the axis OX; this is 21 spaces from O, and hence represents $2.10. Similarly, the cost of $9\frac{1}{2}$ doz. is seen to be $2.85.

EXERCISES

1. Since the price curve in the example in § 123 is a straight line, how many of the points would have to be located through which to draw it? Should these be taken close together or far apart, in order to get the position of the price curve most exact?

2. From the price curve in § 123, give the costs of the following: 3 doz.; 5 doz.; $10\frac{1}{2}$ doz.; $6\frac{3}{4}$ doz.; $8\frac{1}{2}$ doz.

3. If eggs sell at 25 ¢ a dozen, draw the price curve.

4. On this price curve, find the cost of 11 doz.; of $7\frac{1}{2}$ doz.; of $9\frac{3}{4}$ doz.

5. If coffee sells at 28 ¢ a pound, draw the price curve.

6. On this price curve find the cost of $8\frac{1}{2}$ lb.; 17 lb.; 24 lb.

7. If sugar sells at $5\frac{3}{4}$ ¢ a pound, draw the price curve.

8. On this price curve find the cost of 16 lb.; of 42 lb.; of $8\frac{1}{2}$ lb.

9. If money is loaned at 4 % interest, draw a graph showing the interest on $1 for different periods of time.

SUGGESTION. — Let distances on one axis represent time, a space denoting a month; and let distances on the other axis represent interest on $1, a space denoting 1 ¢.

10. On this graph find the interest on $1 for 18 mo.; for 9 mo.; for 22 mo.; for 3 yr. 5 mo.; for 5 yr. 2 mo.

11. On the graph in Problem 9 find the interest on $450 at 4 % for 15 mo.

SUGGESTION. — Find the interest on $1, and multiply this by 450.

12. On the graph in Problem 9 find the interest on $275 at 4 % for 1 yr. 7 mo.; on $1260 at 4 % for 3 yr. 10 mo.

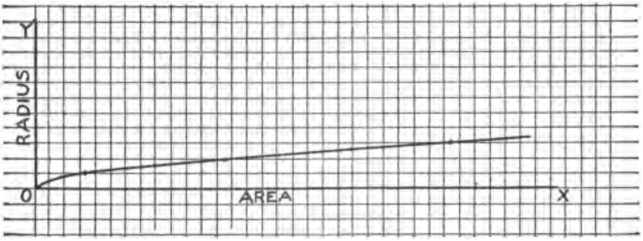
13. If money is loaned at 6 %, draw a graph showing the interest on $1 for different periods of time. Use this to find the interest on different sums of money at 6 % for different lengths of time.

14. The relation between the circumference of a circle and the diameter is expressed by $c = \pi d$, where c denotes the circumference and d the diameter. Using $\pi = 3\frac{1}{7}$, draw the graph expressing this relation. On this graph give at sight the approximate values of the circumferences of circles whose diameters are 4 in.; $5\frac{1}{4}$ in.; $8\frac{3}{4}$ in.

15. The area of a circle is found from $A = \pi R^2$, where A denotes the area and R the radius. Using $\pi = 3\frac{1}{7}$, a few values of A for corresponding values of R are given in the table:

R	0	1	2	3	4
A	0	$3\frac{1}{7}$	$12\frac{4}{7}$	$28\frac{2}{7}$	$50\frac{2}{7}$

Representing values of the radius R on the axis OY, one space denoting 1 in., and values of the area A on the axis OX, one space denoting 1 sq. in., the graph showing the relation between the radius and the area is found to be the *curve* in the figure. The approximate value of the area of a circle with any given radius or of the radius of a circle with any given area, may be determined at sight by this graph.



Give at sight the area of a circle whose radius is $1\frac{1}{2}$ in.; $2\frac{1}{2}$ in.

Give at sight the radius of a circle whose area is 10 sq. in.

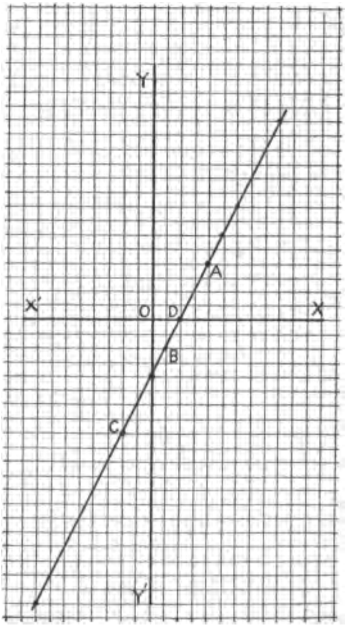
Note. — Observe that the graph in Problem 16 is not a straight line. Many problems lead to graphs that are not straight lines. Some of these will be encountered in more advanced work.

124. Graphs of Linear Equations. — As shown in the preceding sections, the relation between two quantities may be expressed by an *equation*, also the relation may be shown by a *graph*. The graph is often spoken of as **the graph of the equation.**

The two literal quantities in an equation may, in some cases, represent *negative* as well as *positive* values. The process of drawing the graph of an equation between two quantities which have both positive and negative values will be given in this section. The construction of such a graph is given below.

x	2	4	5	7	10	1	0	−2	−5	−7
y	0	4	6	10	16	−2	−4	−8	−14	−18

Let the equation be $2x = y + 4$. By assigning values to y, compute the corresponding values of x and tabulate the results as shown in the table. Thus, when $y = 0$, the equation becomes $2x = 4$, from which $x = 2$. When $y = 6$, $2x = 10$, from which $x = 5$. When $y = -8$, $2x = -4$, from which $x = -2$, etc.



Two axes, XX' and YY', are drawn at right angles and meeting at O. Corresponding to each set of values of x and y, a point is located, as in § 122 and § 123, the values of x being measured along or parallel to XX', and the values of y along or parallel to YY'. Positive values of x are measured to the right of YY' and negative values to the left, as in § 28. Positive values of y are measured above XX' and negative values below XX'. For example, the point A corresponding to $x = 4$ and $y = 4$ is obtained by measuring 4 spaces to the right and 4 spaces upward. The point B corresponding to $x = 1$ and $y = -2$ is obtained by measuring 1 space to the right and 2 spaces downward. The point C corresponding to $x = -2$ and $y = -8$ is obtained by measuring 2 spaces to the left and 3 spaces downward. The point D corresponding to $x = 2$ and $y = 0$ is obtained by measuring 2 spaces to the right on XX'; etc. By drawing a line through all of the points located the graph of the equation is obtained.

It is seen that the graph of the above equation is a straight line. It is shown in more advanced mathematics, and may be assumed here, that

The graph of every linear equation containing two variables is a straight line.

NOTE. — It is seen that in locating a point corresponding to a set of values of the two variables in an equation, the signs of the numbers serve to tell the *directions* of the point from the two axes. This is analogous to the method of locating a point on the earth's surface by knowing its longitude and latitude. If the latitude of a place is minus, the place is south of the equator, etc.

EXERCISES

Draw two axes and locate the following points:

1. $x = 4, y = 6.$
2. $x = 10, y = 7.$
3. $x = 15, y = 2.$
4. $x = 0, y = 3.$
5. $x = 12, y = 0.$
6. $x = 0, y = 0.$
7. $x = -6, y = 5.$
8. $x = -10, y = 12.$
9. $x = 9, y = -4.$
10. $x = 15, y = -5.$
11. $x = -6, y = -6.$
12. $x = -10, y = -12.$

13. By assigning eight different values to y in the equation $x - 2y = 7$, including at least two negative values, compute the eight corresponding values of x. Draw axes and locate the eight points corresponding to these sets of values of x and y. Are all of these points in one straight line? Draw a line through all of the points.

By assigning eight values to y in each of the following equations, including at least three negative values, and computing the corresponding values of x, locate eight points of the graph of the equation, and draw the graph.

14. $x = y$. 16. $x + 4y = 10$. 18. $5x - 4y = 20$.
15. $x + y = 0$. 17. $3x - y = 12$. 19. $2x + 3y = 12$.

20. Since the graph of a linear equation in two variables is a straight line, how many points of it must be located in order to draw it accurately?

By locating only two points of each, draw the graphs of all of the following equations on one pair of axes:

21. $3x - y = 12$. 23. $2x - 4y = 5$. 25. $5x - y = 4$.
22. $y = 2x$. 24. $2x + 4y = 7$. 26. $4x = 3y$.

27. If the cost of setting the type for printing a circular is 75¢, and the cost of paper and press work in printing it is $\frac{1}{4}$¢ a copy, then

$$c = \tfrac{1}{4}n + 75,$$

where c denotes the cost in cents of printing any number of copies and n denotes the number of copies printed.

Draw the graph of this equation.

Give at sight from the graph the cost of 12 copies; 20 copies; 100 copies.

SUPPLEMENTARY EXERCISES

1. Express in simplest form the ratio of $m - \frac{n^2}{m}$ to $1 - \frac{n}{m}$.

2. Separate 72 into three parts which are in the ratio of 2 : 3 : 4.

3. If n is a positive number, which is the greater ratio, $\frac{3+3n}{3+4n}$ or $\frac{3+4n}{3+5n}$?

SUGGESTION. — Reduce to a common denominator, then compare the numerators.

4. If x and y are positive numbers, which is the greater ratio, $\frac{x+5y}{x+6y}$ or $\frac{x+7y}{x+8y}$?

5. Find the mean proportional between $4a^3$ and $9a(a-b)^2$.

6. Find the fourth proportional to v^2-w^2, $\frac{1}{v+w}$, and $v-w$.

7. Write in three ways the proportion between 3, 4, 20, and 15.

8. Write as a proportion $x^2-y^2=5\times 6$.

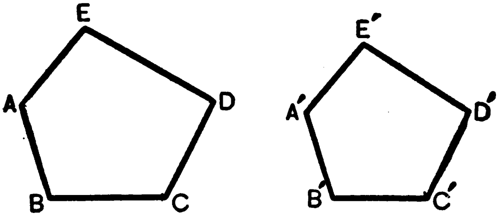
9. If $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$, show that $\frac{a+3b+5c}{a'+3b'+5c'}=\frac{a}{a'}$.

10. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a^2-b^2}{b^2}=\frac{c^2-d^2}{d^2}$.

11. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a}{a+c}=\frac{a+b}{a+b+c+d}$.

12. In the similar polygons $ABCDE$ and $A'B'C'D'E'$ it is known that

$$\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}=\frac{DE}{D'E'}=\frac{EA}{E'A'}.$$



Show that the ratio of the perimeters of the polygons equals the ratio of any two corresponding sides.

13. x varies as y, and when $x=2$, $y=5$. Find y when $x=15$.

14. The distance that a body falls from rest varies as the square of the time. In 2 seconds it falls 64 ft. How far will it fall in 3 sec.? In 4 sec.? In 10 sec.?

15. One quantity is said to *vary inversely* as another when, during all of their changes, their product remains constant. As one increases, the other decreases. If x varies inversely as y, then $xy=k$, where k is a constant.

If x varies inversely as y, and $y=2$ when $x=4$, find y when $x=32$.

16. The volume of any gas varies inversely as the pressure upon it. When the pressure is 8 lb. the volume is 8 cu. in. What is the volume when the pressure is 4 lb.?

17. The number of vibrations made by the pendulum of a clock in a given time varies inversely as the square root of its length. A pendulum 39.1 inches long makes one vibration in a second. How long must a pendulum be to make 4 vibrations in a second?

CHAPTER XII

SYSTEMS OF LINEAR EQUATIONS

125. Systems of Equations. — In Chapter VI it was shown that some problems may be expressed by means of two linear equations containing two unknown numbers. These equations were called *simultaneous,* and were said to form a *system.* It was found that the equations in each system in that chapter had one set of values of the unknown numbers that satisfied both equations, called a *solution of the system.* This solution was discovered through a process called *elimination.*

There are three principal methods of elimination in common use that will now be discussed, and applied in the solution of problems.

126. Elimination by Addition or Subtraction. — The method of elimination shown in § 68 is known as the **method of elimination by addition or subtraction.** The student should now study again the rule on page 110. The following exercises are given for review of that process.

The student should check every answer by seeing if the values of the unknown numbers found satisfy both of the given equations.

EXERCISES

Eliminate by addition or subtraction, solve, and check:

1. $\begin{cases} 3a - b = 21, \\ 2a + b = 4. \end{cases}$

2. $\begin{cases} m + 4n = 4, \\ m - 2n = 16. \end{cases}$

3. $\begin{cases} 2x - y = 5, \\ 5x - 2y = 14. \end{cases}$

4. $\begin{cases} 7s + t = 42, \\ 3s - t = 8. \end{cases}$

5. $\begin{cases} 4w + v = 25, \\ 5w - 2v = 35. \end{cases}$

6. $\begin{cases} 4A - 3B = 1, \\ 3A - 4B = 6. \end{cases}$

7. $\begin{cases} 7x = 2y - 15, \\ 2x = y - 6. \end{cases}$

8. $\begin{cases} 3p + 2q = 12, \\ 4p = 3q - 1. \end{cases}$

9. $\begin{cases} 5R - 2r = 1, \\ 8R = 5r - 11. \end{cases}$

10. $\begin{cases} 3M - 2N + 4 = 0, \\ 2M - N + 1 = 0. \end{cases}$

11. $\begin{cases} 2h + k = 35, \\ 5h - 3k = 27. \end{cases}$

12. $\begin{cases} 5y - 5x = 15; \\ 3x + 5y = 71. \end{cases}$

13. $\begin{cases} 3r_1 + 2r_2 = 4, \\ 4r_2 - 3r_1 + 1 = 0. \end{cases}$

14. $\begin{cases} -4B + 3C = 45, \\ 2C + 6B = 4. \end{cases}$

15. $\begin{cases} m = 4 - 2n, \\ 2n + 12 = m. \end{cases}$

16. $\begin{cases} 8W + w = 7, \\ 11W + 2w = 28. \end{cases}$

17. $\begin{cases} 10f + 4t = 22, \\ t + 5f = 11. \end{cases}$

18. $\begin{cases} 2x_1 - x_2 = 9, \\ 5x_1 - 3x_2 = 14. \end{cases}$

19. $\begin{cases} \frac{a}{2} + \frac{b}{7} = 9, \\ \frac{a}{7} - \frac{b}{2} + 5 = 0. \end{cases}$

20 $\begin{cases} \frac{w}{6} + \frac{v}{4} = 6, \\ \frac{w}{4} + \frac{v}{6} = 5\frac{2}{3}. \end{cases}$

127. Elimination by Comparison. — The method of **elimination by comparison** is illustrated in the following example.

EXAMPLE. — Solve $\begin{cases} 4x + y = 34, & (1) \\ 4y + x = 16. & (2) \end{cases}$

Eliminate x.

Solving (1) for x, $x = \frac{34 - y}{4}$. (3)

Solving (2) for x, $x = 16 - 4y$. (4)

Comparing the two values of x given in (3) and (4),

$$\frac{34 - y}{4} = 16 - 4y. \qquad (5)$$

Solving (5) for y, $y = 2$.

Replacing y in (2) by its value 2,

$$8 + x = 16.$$

Solving, $x = 8$.

Hence the solution is $x = 8$, $y = 2$.

This example illustrates the rule:

Solve each of the equations for the value of either one of the unknown numbers, expressed in terms of the other, and write one of the *values equal to the other. Solve the resulting equation. Substitute the value found for the one unknown number in either of the original equations, and solve for the other unknown number.*

EXERCISES

Eliminate by comparison, solve, and check:

1. $\begin{cases} m+2n=8, \\ m-n=-1. \end{cases}$

2. $\begin{cases} x-5a=7, \\ 2x-15a=9. \end{cases}$

3. $\begin{cases} r+2s=4, \\ 3r-s=5. \end{cases}$

4. $\begin{cases} 2r_1-r_2=9, \\ 5r_1-3r_2=14. \end{cases}$

5. $\begin{cases} 3A+2B=2, \\ A-B=9. \end{cases}$

6. $\begin{cases} 3p-q=1, \\ 2p+5q=41. \end{cases}$

7. $\begin{cases} v+2t=4, \\ 2t-v+12=0. \end{cases}$

8. $\begin{cases} 9M-5N=13, \\ 5M+N=11. \end{cases}$

9. $\begin{cases} 3e-5f=5, \\ 7e+f=265. \end{cases}$

10. $\begin{cases} 2R-T=5, \\ R+2T=25. \end{cases}$

11. $\begin{cases} 10V+3n=174, \\ 3V+10n=125. \end{cases}$

12. $\begin{cases} 5x+2x'=1, \\ 13x+8x'=11. \end{cases}$

13. $\begin{cases} 2h+k=9, \\ 5h+3k=25. \end{cases}$

14. $\begin{cases} B+2Q+2=0, \\ 4B-7Q=37. \end{cases}$

15. $\begin{cases} 6y-5z=1, \\ 9y+10z=12. \end{cases}$

16. $\begin{cases} 7L+3Q+9=0, \\ 6Q-9L=28. \end{cases}$

17. $\begin{cases} 3W+2D=17, \\ 4W+D=16. \end{cases}$

18. $\begin{cases} 8a+5t=5, \\ 3a-2t=29. \end{cases}$

19. $\begin{cases} 5x+9y=8, \\ 6y-9x+7=0. \end{cases}$

20. $\begin{cases} 5V+4P=22, \\ 3V+P=9. \end{cases}$

21. $\begin{cases} \frac{T}{6}+\frac{5}{3}=t, \\ \frac{2T}{5}+3=\frac{7t}{5}. \end{cases}$

22. $\begin{cases} \frac{R}{3}-\frac{r}{2}=\frac{19}{15}, \\ \frac{7R}{4}+r=\frac{1}{2}. \end{cases}$

23. $\begin{cases} W-\frac{2}{3}+\frac{7w}{3}=0. \\ \frac{7W}{2}+1+4w=0 \end{cases}$

128. Elimination by Substitution. — The method of **elimination by substitution** is illustrated in the following example.

EXAMPLE. — Solve
$$\begin{cases} 4w - 5v = 26, & (1)\\ 3w - 6v = 15. & (2)\end{cases}$$

Eliminate w.

Solving (1) for w,
$$w = \frac{26 + 5v}{4}. \qquad (3)$$

Substituting $\frac{26+5v}{4}$ in place of w in (2),
$$3\left(\frac{26 + 5v}{4}\right) - 6v = 15. \qquad (4)$$

Solving (4), $v = 2.$

Replacing v by its value 2 in (1),
$$4w - 10 = 26.$$

Solving, $w = 9.$

Hence, the solution is $w = 9$, $v = 2$.

This example illustrates the rule:

Solve one of the equations for the value of either of the unknown numbers, expressed in terms of the other, and substitute this value in place of that number in the other equation. Solve the resulting equation. Substitute the value found for the one unknown number in either of the original equations, and solve for the other unknown number.

EXERCISES

Eliminate by substitution, solve, and check:

1. $\begin{cases} a + b = 5,\\ a + 3b = 9.\end{cases}$

2. $\begin{cases} x - y = 4,\\ x + 2y = 16.\end{cases}$

3. $\begin{cases} m + 4n = 7,\\ m + 6n = 9.\end{cases}$

4. $\begin{cases} x_1 + x_2 = 30,\\ 3x_1 - 2x_2 = 25.\end{cases}$

5. $\begin{cases} R - S = 4,\\ 2R + S = 14.\end{cases}$

6. $\begin{cases} 2V + t = 7,\\ V + 3t = 16.\end{cases}$

7. $\begin{cases} 3a - 7z = 40,\\ 4a - 3z = 9.\end{cases}$

8. $\begin{cases} 4p - 5q = 26,\\ 3p - 6q = 15.\end{cases}$

9. $\begin{cases} 3D + 7Q = 16,\\ 2D + 5Q = 13.\end{cases}$

10. $\begin{cases} 3g + 2k = 26,\\ 5g - 2k = 38.\end{cases}$

11. $\begin{cases} 7A-9B=13, \\ 5A+2B=10. \end{cases}$

12. $\begin{cases} 7r_1+4r_2=1, \\ 9r_1+4r_2=3. \end{cases}$

13. $\begin{cases} T+t=10, \\ T-t=4. \end{cases}$

14. $\begin{cases} 7v+s=42, \\ 3v-s=8. \end{cases}$

15. $\begin{cases} 2M-b=5, \\ 5M-2b=14. \end{cases}$

16. $\begin{cases} A-8p=45, \\ 3A-p=20. \end{cases}$

17. $\begin{cases} 3x+2y=26, \\ 5x=38+2y. \end{cases}$

18. $\begin{cases} 7m=26+3a, \\ 6a=6m-20. \end{cases}$

19. $\begin{cases} 8y+d=60, \\ 7y=9-20d. \end{cases}$

20. $\begin{cases} 16-7t=3r, \\ 2r-13+5t=0. \end{cases}$

21. $\begin{cases} a+k=30, \\ \dfrac{a}{2}=\dfrac{25}{3}+\dfrac{2k}{3}. \end{cases}$

22. $\begin{cases} \dfrac{2}{a-5}+\dfrac{5}{b-4}=0, \\ 3a=2b+7. \end{cases}$

23. $\begin{cases} \dfrac{2w}{3}-v=5, \\ \dfrac{4w+v}{6w+v}=\dfrac{2}{5}. \end{cases}$

MISCELLANEOUS EXERCISES

Eliminate by the method that seems best suited, and solve:

1. $\begin{cases} x+y=10, \\ x-y=4. \end{cases}$

2. $\begin{cases} 8v+w=60, \\ 7v-10w=9. \end{cases}$

3. $\begin{cases} 21R+8t+66=0, \\ 49R+53=15t. \end{cases}$

4. $\begin{cases} 5M+63=2b, \\ 2M+b=18. \end{cases}$

5. $\begin{cases} 2a-2b=6\frac{2}{3}, \\ 7a=26+3b. \end{cases}$

6. $\begin{cases} \frac{1}{3}p-\frac{1}{4}q=2, \\ \frac{1}{4}p-\frac{1}{2}q=7. \end{cases}$

7. $\begin{cases} 3n+15z=7, \\ 12-n+5z=0. \end{cases}$

8. $\begin{cases} \dfrac{5L}{6}-7+\dfrac{d}{4}=0, \\ 3+\dfrac{d}{8}=\dfrac{2L}{3}. \end{cases}$

9. $\begin{cases} \dfrac{k}{3}=9m-5\frac{3}{4}, \\ 2\frac{1}{4}-\dfrac{7m}{8}=\dfrac{3k}{4}. \end{cases}$

10. $\begin{cases} \dfrac{1}{x}+\dfrac{3}{y}=\dfrac{1}{xy}, \\ 2x-3y=1. \end{cases}$

11. $\begin{cases} 2-5P=4V, \\ \dfrac{3}{P-1}=\dfrac{4}{V+2}. \end{cases}$

129. Problems solved by Systems of Equations. — For the steps in the process of expressing a problem by means of a system of equations with two unknown numbers see § 69. In the following problems use that method of elimination which seems best suited to each individual problem.

EXERCISES

1. Find two numbers whose sum is 29, and difference 13.

2. Find two numbers whose sum is 9, and such that one of them exceeds twice the other by 27.

3. The difference between two numbers is 72, and one of them is 4 times the other. What are the numbers?

4. The sum of two numbers is 64, and their quotient 7. What are the numbers?

5. The difference between two numbers is 21, and if the larger is divided by the smaller, the quotient is 4 and the remainder 3. Find the numbers.

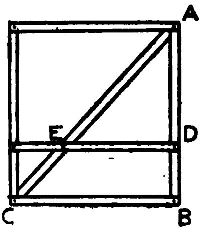
6. The value of a fraction is $\frac{2}{3}$, and if 7 be added to each term, the value of the resulting fraction is $\frac{3}{4}$. Find the fraction.

7. A board 10 ft. long is to be cut into two pieces whose lengths are as 3 and 4. How long are the pieces?

8. A surveyor wishes to set a stake in a line 320 ft. long, so that its distances from the ends of the line are as 7 is to 19. Where must he place it?

9. In building a bridge a steel bar 12 ft. long is to be bolted to another bar at a point which divides the 12 ft. bar into parts whose lengths are as 7 and 17. Where must the hole be bored for the bolt?

10. In the steel frame in the figure the brace AC is 15 ft., AD is 7 ft., and DB is 3 ft. It is desired to find where to bore the hole for the bolt E in AC. By proportion, $\frac{AE}{EC} = \frac{AD}{DB}$. Find where to locate E.



11. Two boys desired to weigh themselves. They had no scales, but had a 10 lb. weight. They placed a board across a support, as in playing teeter. When they sat, one 5 ft. from the fulcrum and the other 4 ft. from it, they balanced. When the smaller boy took the 10 lb. weight in his hands, he had to sit only 4 ft. 4 in. from the fulcrum to balance. What were their weights?

12. Two weights balance when the larger is 3 ft. and the smaller 4 ft. from the fulcrum. If the smaller weight, with 3 lb. added to it, is moved to a position 3 ft. $5\frac{1}{7}$ in. from the fulcrum, they still balance. Find the weights.

13. Two weights balance when one is 4 in. and the other 6 in. from the fulcrum. If the first weight is decreased 12 lb., the other must be moved 1 in. nearer the fulcrum to balance. Find the weights.

14. Two weights balance when one is 3 ft. and the other 2 ft. from the fulcrum. The larger weight is decreased 12 oz., and the smaller weight is decreased 6 oz. and moved $2\frac{2}{5}$ in. nearer the fulcrum. Again they balance. Find the weights.

15. Two partners in business, A and B, are to divide profits in the ratio of 2 to 5. In dividing a profit of $6300, how much should each receive?

16. A man invests part of $12,000 at 5% and the balance at 6%. His annual income from both investments is $645. Find the number of dollars in each investment.

17. A man invests $2000, part at 5% and the rest at 4%. The annual income from the 5% investment exceeds that of the 4% investment by $2.80. Find the amount of each investment.

18. An investor purchased two kinds of bonds. One kind yielded 3% and the other 4%. His yearly income from both was $558. Had he invested as much in 4% bonds as in 3% bonds, and *vice versa*, his yearly income would have been $576. How much did he invest in each kind of bonds?

19. A man invested in $22,500 worth of bonds, part of them New York City 4s and part U. S. Steel 5s. His annual income was $1045. Had he purchased as many New York City 4s as he did U. S. Steel 5s, and *vice versa,* his annual income would have been only $980. What was the value of each kind of bonds bought?

20. I change $3 into dimes and nickels. There are 50 coins in all. How many dimes and how many nickels are there?

21. I got $100 changed into $5 bills and $10 bills. I received 16 bills in all. How many bills of each kind did I get?

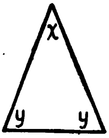
22. If a merchant blends 25-cent coffee and 32-cent coffee to sell at 30 cents a pound, what quantities of each grade of coffee must he take to make 50 lb. of the blend?

23. How much each of copper, specific gravity 8.9, and zinc, specific gravity 6.9, must be combined to produce 1000 cu. cm. of brass, specific gravity 8.4? (See Problem 7, page 112.)

24. In a number of two digits the sum of the digits is 11, and when the digits are interchanged, the number is diminished by 45. Find the number.

25. Two angles are called supplementary when their sum is 180°. If one of two supplementary angles is 20° more than twice the other, how many degrees in each?

26. The sum of the angles of a triangle is always 180°. If each of the angles at the base of the triangle is twice the angle at the vertex, how many degrees in each angle of the triangle?



27. The altitude of a trapezoid is 6 ft. and its area 84 sq. ft. One base is 8 ft. longer than the other. Find the lengths of the bases.

28. If the length of a rectangle is diminished 3 in. and its width increased 2 in., its area is unchanged; and if its length is increased 5 in. and its width diminished 2 in., its area is unchanged. Find the length and width.

29. Two groups of students of surveying were sent to find the area of a rectangular field. One group got the length 4 ft. too small and the width 6 ft. too large, which gave the area 2960 sq.

ft. too large. The other group got the length 8 ft. too large and the width 3 ft. too small, which gave the area 741 sq. ft. too small. Find the dimensions of the field.

30. The circumference of the fore wheel of a carriage is 3 ft. less than that of the rear wheel. One goes the same distance in 40 revolutions that the other does in 30. Find the circumference of each.

31. A belt runs over two pulleys. One makes 10 revolutions while the other makes 3. If the larger pulley were replaced by one whose circumference was 20 in. less, it would only make 10 revolutions while the other made 4. Find the circumference of each pulley.

32. A man who can row 6 mi. an hour downstream can row 2 mi. an hour upstream. What is the speed of the current?

33. A crew rows downstream $6\frac{1}{2}$ mi. in an hour, and returns in 4 hr. 20 min. What is the speed of the current, and at what rate could the crew row in still water?

34. An aviator flying against the wind makes 20 mi. an hour. On returning with the wind he makes 50 mi. an hour. What is the speed of the wind, and what would be his speed if there were no wind?

35. If an aviator flies to a point 24 mi. distant, against the wind, in 40 min., and returns in 30 min., what is the speed of the wind, and what would be his speed if there were no wind?

36. Two trains, each 300 ft. long, run on parallel tracks. If running in the same direction, it requires 20 sec. for one to pass the other. If running in opposite directions, it requires only 4 sec. for them to pass. What are the speeds of the trains?

37. A passenger train and a freight train run in opposite directions on parallel tracks. The speed of the passenger train is 60 ft. per second, and the speed of the freight train is 40 ft. per second. The freight train is 180 ft. longer than the passenger train. It takes the trains 12.6 sec. to pass. Find the length of each train

38. A passenger and a freight train run on parallel tracks and in the same direction. The speed of the passenger train is 45 mi. an hour, and the speed of the freight train is 25 mi. an hour. The freight train is 96 ft. longer than the passenger train. It takes the passenger train 36 sec. to pass the freight. Find the length of each train.

39. How much milk testing 4 % butter fat and cream testing 24 % butter fat must be mixed to make 20 gal. that test 20 % butter fat?

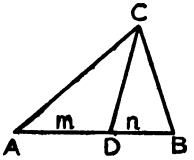
40. How much milk testing 3.8 % butter fat and cream testing 26.6 % butter fat must be mixed to make 16 gal. that test 18 % butter fat?

41. Fifteen pounds of tin weigh 13 lb. in water and 15 lb. of zinc weigh 13.5 lb. in water. How much tin and how much zinc in an alloy which weighs 56 lb. in air and 49 lb. in water?

42. Some authorities claim that the daily ration for the average adult workingman should contain 4 oz. of protein and an equal amount of fat. White bread contains 9 % protein and 1 % fat. Mutton contains 14 % protein and 37 % fat. Find how many ounces each of bread and mutton would be required to make a daily ration.

43. Eggs contain 13 % protein and 9 % fat. From the facts given in Problem 42, can bread and eggs be used to make a standard ration?

44. A society that wished to raise $ 100 gave an entertainment. They estimated that enough children and adults would attend that they could raise this amount by charging children 10 cents and adults 25 cents admission, and that if they charged children 15 cents and adults 30 cents admission, they would raise $ 26 more than the amount desired. How many children and how many adults did they expect would attend?



45. In the triangle *ABC*, the line *CD* bisects (divides into two equal parts) the angle at *C*. It is known from geometry that *CD* divides *AB*

into two parts, m and n, which are proportional to the adjacent sides. If $AC = 15$ in., $BC = 12$ in., and $AB = 10$ in., find the lengths of m and n.

130. Systems of Literal Equations. — Systems of linear equations in which one or more of the known numbers are literal are solved by the methods of §§ 126, 127, 128, the solutions being expressed in terms of these literal quantities.

EXAMPLE. — Solve for x and y:

$$\begin{cases} ax - by = a^3, & (1) \\ bx - ay = b^3. & (2) \end{cases}$$

Multiplying (1) by b, $abx - b^2y = a^3b$. (3)

Multiplying (2) by a, $abx - a^2y = ab^3$. (4)

Subtracting (4) from (3), $a^2y - b^2y = a^3b - ab^3$.

Factoring (combining terms), $(a^2 - b^2)y = ab(a^2 - b^2)$.

Dividing by $a^2 - b^2$, $y = ab$.

Substituting ab for y in (1), $ax - ab^2 = a^3$.

$$ax = a^3 + ab^2.$$

$$x = a^2 + b^2.$$

Hence, the solution is $x = a^2 + b^2,\ y = ab$.

EXERCISES

Solve for x and y:

1. $\begin{cases} ax + by = a^2 + b^2, \\ bx + ay = 2ab. \end{cases}$

2. $\begin{cases} ax + by = a^2 - b^2, \\ ay + bx = a^2 - b^2. \end{cases}$

3. $\begin{cases} my + x = m + n, \\ mx + my = m^2 + n. \end{cases}$

4. $\begin{cases} x + Ry = S, \\ 2x - Sy = R. \end{cases}$

5. $\begin{cases} Ax + y = B, \\ Bx + y = A. \end{cases}$

6. $\begin{cases} x - p^2y = 0, \\ x + q^2y = 1. \end{cases}$

7. $\begin{cases} r_1x + r_2y = 1, \\ r_2x + r_1y = 1. \end{cases}$

8. $\begin{cases} ax - b^2y = a^3 - b^3, \\ a^2y - bx = 0. \end{cases}$

9. $\begin{cases} 2px + 2qy = 4p^2 + q^2, \\ x - 2y = 2p - q. \end{cases}$

10. $\begin{cases} 12x - 11y = R_1 + 12R_2, \\ x + y = 2R_1 + R_2. \end{cases}$

11. $\begin{cases} (m + n)x = 1 - py, \\ (m + n)y + px = 1. \end{cases}$

12. $\begin{cases} vx - ty = 0, \\ x + y = w. \end{cases}$

13. $\begin{cases} Px - Qy = R, \\ x + y = Q. \end{cases}$

14. $\begin{cases} \dfrac{x}{2} + \dfrac{y}{3} = a, \\ \dfrac{x}{3} + \dfrac{y}{4} = b. \end{cases}$

15. $\begin{cases} \dfrac{x}{a-b} + \dfrac{y}{a+b} = 2, \\ x + y = 2a. \end{cases}$

16. $\begin{cases} \dfrac{x}{2} - 3y = \dfrac{m - 12n}{4}, \\ 8x + 9y = 4m + 9n. \end{cases}$

17. $\begin{cases} \dfrac{x}{v} + 5 = \dfrac{2y}{v}, \\ \dfrac{3x}{v} - \dfrac{7y}{4v} = \dfrac{3}{4}. \end{cases}$

18. If m be added to the numerator of a certain fraction, the value of the resulting fraction is 4. But if n be added to the denominator, the value of the resulting fraction is 3. What is the fraction?

19. The sum of two numbers is s, and the quotient of the first divided by the second is q. Find the numbers.

20. If a boy who weighs W pounds and one who weighs P pounds balance at the ends of a teeter board l feet long, find the lengths of the two parts into which the board is divided at the fulcrum.

21. Part of \$12,000 is invested at $x\%$ and the rest at $y\%$. The annual income from both investments is d dollars. Find the number of dollars in each investment.

22. If milk tests $m\%$ butter fat and cream $n\%$ butter fat, how much of each must be taken to make w pounds of cream testing $a\%$ butter fat?

131. Number of Solutions of a System. — It has been seen that each of the problems in the preceding sections has one and only one solution. That this is true in general of a system of two linear equations with two unknown numbers may be shown as follows:

Any two linear equations whose unknown numbers are x and y may be written in the forms

$$\begin{cases} ax + by = c, \\ dx + ey = f, \end{cases}$$

where a, b, c, d, e, and f represent any numbers whatever. If this general system be solved,

$$x = \frac{ce - bf}{ae - bd},\quad y = \frac{af - cd}{ae - bd}.$$

Hence, in general, the system has *one and only one solution*, depending upon the values of a, b, c, d, e, and f. But these numbers may be so related that there is *no solution*, or that there is an *infinitely great number of solutions*.

EXAMPLE 1. — Solve $\begin{cases} 2x - y = 12, & (1) \\ 6x - 3y = 4. & (2) \end{cases}$

Multiplying (1) by 3, $6x - 3y = 36.$ (3)

Subtracting (2) from (3), $0 = 32.$

This is absurd. It is impossible to eliminate one of the unknown numbers without at the same time eliminating the other, and hence no solution can be found. Or, looking at the matter in a different way, the supposition that both equations can be true simultaneously, that is, for the same values of x and y, leads to the conclusion that $0 = 32$, which is false. Hence the supposition is false, and the equations have no common solution.

Two linear equations having no common solution are called **inconsistent equations.**

EXAMPLE 2. — Solve $\begin{cases} 2x - y = 3, & (1) \\ 4x = 6 + 2y. & (2) \end{cases}$

Transposing (2) gives $4x - 2y = 6.$ (3)

Dividing (3) by 2, $2x - y = 3.$ (4)

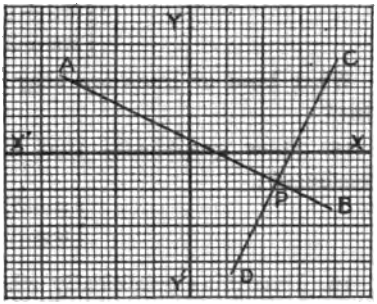
Equation (1) is the same as equation (4). Hence all of their solutions are common.

Two equations, such as (1) and (2) above, that have all of their solutions common are called **equivalent equations.** It is evident that one of the equations may always be derived from the other by transposition, etc.

132. Systems solved Graphically. — The solution of a system of two equations in two unknown numbers may be found graphically, without performing the process of elimination.

EXAMPLE. — Solve graphically $\begin{cases} x + 2y = 4, \\ 2x - y = 28. \end{cases}$

Upon the same pair of axes draw the graphs of both equations, as in § 124. Only two points of each graph need be located in order to draw it. The graph of $x+2y=4$ is AB, and the graph of $2x-y=28$ is CD. The graphs AB and CD meet at a point P. Since P is on AB, the values of x and y corresponding to the point P must satisfy $x+2y=4$. And since P is also on CD, the values of x and y corresponding to the point P must also satisfy $2x-y=28$. Hence the values of x and y corresponding to the point P, which are seen to be $x=12$, $y=-4$, must be the solution of the system. See if $x=12$, $y=-4$ satisfies both equations.



To solve a system graphically, draw the graphs of both equations upon the same axes. The set of values of the unknown numbers corresponding to the point where the graphs meet is the solution of the system.

EXERCISES

Solve graphically:

1. $\begin{cases} x+y=10, \\ x-y=4. \end{cases}$

2. $\begin{cases} 3x-y=2, \\ 7x-2y=8. \end{cases}$

3. $\begin{cases} 5x+4y=42, \\ 6x-y=4. \end{cases}$

4. $\begin{cases} x+4y=10, \\ 3x-y=12. \end{cases}$

5. $\begin{cases} 4x+y=3, \\ 2x+3y=4. \end{cases}$

6. $\begin{cases} x+y=5, \\ 2x+3y=12. \end{cases}$

7. $\begin{cases} 3x+2y=12, \\ 4x-3y+1=0. \end{cases}$

8. $\begin{cases} 7x-2y+15=0, \\ y-2x=6. \end{cases}$

9. $\begin{cases} 2x+5y=15, \\ 3x-4y=11. \end{cases}$

10. $\begin{cases} 4x-3y=1, \\ 3x-4y=6. \end{cases}$

11. $\begin{cases} 2x-y=5, \\ x+2y=25. \end{cases}$

12. $\begin{cases} 2x-y=5, \\ 5x-2y=14. \end{cases}$

13. Show by graphs that in general a system of two equations in two unknown numbers has one and only one solution.

14. Draw the graphs of the inconsistent equations of the system

$$\begin{cases} 2x + y = 11, \\ 4x + 2y = 43. \end{cases}$$

What kind of lines are they? Would you naturally expect this?

15. Show graphically that the following equations are inconsistent:

$$\begin{cases} x - 3y = 6, \\ 3x = 4 + 9y. \end{cases}$$

16. Determine graphically whether or not the following equations are consistent:

$$\begin{cases} 5x - y = 0, \\ 3y = 40 + 15x. \end{cases}$$

17. By drawing their graphs, show that $3x - 2y = 6$, $x + y = 7$, and $4x - y = 13$ have a common solution. What is it?

18. Show by their graphs that $3x - 2y = 12$, $4x - 3y + 1 = 0$, and $2x - y = 11$ have no common solution.

19. Determine graphically whether or not $3x + 2y = 18$, $x - y = 8$, and $4x + y = 4$ have a common solution.

20. Show graphically that in general three or more linear equations containing the same two unknown numbers cannot have a common solution. When would they have a common solution?

133. Systems of Fractional Equations. — As a general rule, in solving a system of fractional equations, it is best first to clear the equations of fractions. *But in certain cases, as when the unknown numbers occur only in monomial denominators, it is best not to clear of fractions before eliminating.* If such equations were cleared of fractions, they usually would not be linear, and also would give solutions which would not satisfy the original system.

EXAMPLE 1. — Solve
$$\begin{cases} \dfrac{9}{a} - \dfrac{4}{b} = 2, & (1) \\ \dfrac{9}{a} + \dfrac{9}{b} = 5. & (2) \end{cases}$$

Subtracting (1) from (2), $\frac{13}{b} = 3.$

Solving, $b = 4\tfrac{1}{3}.$

Substituting $4\tfrac{1}{3}$ for b in (1), $\frac{9}{a} - \frac{4}{4\frac{1}{3}} = 2.$

Solving, $a = 3\tfrac{3}{38}.$

Example 2. — Solve

$$\begin{cases} \dfrac{2}{3x} + \dfrac{3}{2y} = 5, & (1) \\[2ex] \dfrac{3}{2x} - \dfrac{2}{3y} = 3. & (2) \end{cases}$$

Multiplying (1) by $\frac{9}{4}$, to make the fraction in x the same as in (2),

$$\frac{3}{2x} + \frac{27}{8y} = \frac{45}{4}. \qquad (3)$$

Subtracting (2) from (3), $\frac{27}{8y} + \frac{2}{3y} = \frac{45}{4} - 3.$

Solving, $y = \tfrac{97}{198}.$

Substituting $\frac{97}{198}$ for y in (1), $\frac{2}{3x} + \frac{594}{194} = 5.$

Solving, $x = \tfrac{97}{282}.$

EXERCISES

Solve:

1. $\begin{cases} \dfrac{1}{m} + \dfrac{2}{n} = 2, \\[2ex] \dfrac{2}{m} - \dfrac{2}{n} = 1. \end{cases}$

2. $\begin{cases} \dfrac{4}{p} + \dfrac{9}{q} = 1, \\[2ex] \dfrac{3}{p} + \dfrac{6}{q} = \dfrac{1}{2}. \end{cases}$

3. $\begin{cases} \dfrac{1}{A} + \dfrac{1}{B} = 7, \\[2ex] \dfrac{1}{A} - \dfrac{1}{B} = 3. \end{cases}$

4. $\begin{cases} \dfrac{3}{w} + \dfrac{8}{v} = 3, \\[2ex] \dfrac{15}{w} - \dfrac{4}{v} = 4. \end{cases}$

5. $\begin{cases} \dfrac{3}{r_1} + \dfrac{4}{r_2} = 3, \\[2ex] \dfrac{6}{r_1} - \dfrac{2}{r_2} = 1. \end{cases}$

6. $\begin{cases} \dfrac{10}{P} - \dfrac{6}{Q} + 1 = 0, \\[2ex] \dfrac{15}{P} + \dfrac{2}{Q} = 4. \end{cases}$

7. $\begin{cases} \dfrac{6}{R} + \dfrac{8}{r} = 3, \\[2ex] \dfrac{12}{R} - \dfrac{20}{r} + 3 = 0. \end{cases}$

8. $\begin{cases} \dfrac{3}{x_1} + \dfrac{2}{x_2} + 12 = 0, \\[2ex] \dfrac{4}{x_1} - \dfrac{3}{x_2} = 1. \end{cases}$

9. $\begin{cases} \dfrac{1}{2a} - \dfrac{2}{5b} = 4, \\ \dfrac{5}{a} + \dfrac{6}{b} = 7. \end{cases}$

11. $\begin{cases} \dfrac{1}{4A} + \dfrac{7}{2B} = \dfrac{5}{8}, \\ \dfrac{1}{2A} - \dfrac{3}{B} + \dfrac{5}{28} = 0. \end{cases}$

10. $\begin{cases} \dfrac{2}{5t} + \dfrac{5}{2T} = 7, \\ \dfrac{5}{t} + \dfrac{2}{T} = 29. \end{cases}$

12. $\begin{cases} \dfrac{2}{9x} - \dfrac{1}{y} = 4, \\ \dfrac{3}{5x} + \dfrac{1}{2y} = 7. \end{cases}$

Solve for x and y:

13. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = c, \\ \dfrac{a'}{x} + \dfrac{b'}{y} = c'. \end{cases}$

14. $\begin{cases} \dfrac{1}{mx} + \dfrac{1}{ny} = 1, \\ \dfrac{2}{mx} - \dfrac{3}{ny} = 1. \end{cases}$

15. $\begin{cases} \dfrac{m}{nx} + \dfrac{n}{my} = \dfrac{m+n}{mn}, \\ \dfrac{n}{mx} - \dfrac{m}{ny} = \dfrac{n^3 - m^3}{m^2n^2}. \end{cases}$

16. A water tank can be filled by two pipes in $8\frac{8}{9}$ minutes. If the first is left open 10 minutes and the second 8 minutes, the tank will be filled. In what time can each pipe alone fill the tank?

17. Two steam pumps together can fill a reservoir with water in $6\frac{3}{7}$ hr. If one pump works 8 hr. and the other 3 hr. 48 min., the reservoir will be filled. How long would it take each pump alone to fill it?

18. In a factory which operates two sizes of machines it is found that 2 large machines and 5 small ones can turn out a certain quantity of goods in 12 hr., while 4 large ones and 3 small ones can do it in 10 hr. Find how long it would require one machine of each size alone to turn out the goods.

19. A and B together can do a piece of work in $13\frac{1}{3}$ days. After they have worked 6 days B leaves, and A finishes the work in $16\frac{1}{2}$ days more. In how many days could each of them alone do the work?

20. A current of electricity flows through two branches of a circuit. The total resistance to the current is found to be 40 ohms. When the resistance of one branch is made three times as great, the resistance of the whole circuit is doubled. If r_1 and r_2 are the resistances of the branches, it is known that $\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{40}$ and $\frac{1}{3r_1}+\frac{1}{r_2}=\frac{1}{80}$. Find the resistance of each branch.

134. Systems involving Three Unknown Numbers. — Some problems involving three unknown quantities may be solved by first expressing them by means of a system of three linear equations containing three unknown numbers. To solve such a system we first eliminate one of the unknown numbers from any two of the equations, then eliminate the same number from one of these two equations and the third equation. This will give rise to two new equations which contain only two unknown numbers. These two equations may then be solved as a new system by the methods of the preceding sections.

EXAMPLE. — Solve

$$\begin{cases} x+2y+2z=11, & (1)\\ 2x+y+z=7, & (2)\\ 3x+4y+z=14. & (3)\end{cases}$$

By subtraction, eliminate x between (1) and (2).

This gives $$3y+3z=15. \qquad (4)$$

By subtraction, eliminate x between (1) and (3).

This gives $$2y+5z=19. \qquad (5)$$

Eliminating y between (4) and (5),

$$z=3.$$

By substituting 3 for z in (5) we get

$$y=2.$$

Now, substituting the values of both y and z in (1), we get

$$x=1.$$

From a system of four linear equations with four unknown numbers we can, in like manner, derive a new system of three equations with three unknown numbers; and so on.

EXERCISES

Solve:

1. $$\begin{cases} a+b+c=9, \\ 2a+b-c=0, \\ 3a-b+c=5. \end{cases}$$

2. $$\begin{cases} P-2Q+R=6, \\ P+3Q+2R=13, \\ 2P-Q+R=13. \end{cases}$$

3. $$\begin{cases} 2u+2v+3w=4, \\ 3u+4v+6w=7, \\ u+2v+6w=4. \end{cases}$$

4. $$\begin{cases} 5r_1+6r_2+7r_3=3, \\ 10r_1-12r_2+21r_3=3, \\ 15r_1-6r_2+14r_3=4. \end{cases}$$

5. $$\begin{cases} A+B=1, \\ B+C=9, \\ A+C+6=0. \end{cases}$$

6. $$\begin{cases} 2x+y-z=2, \\ x-2y-3z=6, \\ y-z+1=0. \end{cases}$$

7. $$\begin{cases} 2h+k=7+l, \\ k=1+h, \\ l-k=1. \end{cases}$$

8. $$\begin{cases} 2p-2q+3r=10, \\ 3p+q-r=5, \\ p-q+2r=7. \end{cases}$$

9. $$\begin{cases} \frac{a}{5}+\frac{b}{2}+\frac{c}{3}=19, \\ \frac{a}{10}-\frac{b}{10}+\frac{c}{6}=6, \\ \frac{a}{4}+\frac{b}{5}-\frac{c}{15}=5. \end{cases}$$

10. $$\begin{cases} \frac{1}{x}+\frac{1}{y}=2, \\ \frac{1}{x}+\frac{1}{z}=3, \\ \frac{1}{y}+\frac{1}{z}=4. \end{cases}$$

11. $$\begin{cases} \frac{2}{3m}+\frac{1}{n}+\frac{3}{10}=0, \\ \frac{1}{m}+\frac{4}{5p}=\frac{1}{12}, \\ \frac{1}{p}+\frac{7}{30}=\frac{3}{4n}. \end{cases}$$

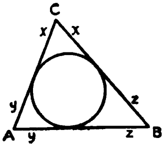
12. The perimeter of a triangle is 28 in. Two of the sides are equal, and their sum exceeds the third side by 4 in. Find the lengths of the sides.

13. The sum of the two sides of a triangle which meet at one vertex is 44 in., at another vertex 40 in., and at the third vertex 36 in. Find the lengths of the three sides.

14. The sum of the three angles of any triangle is 180°. If two angles of a triangle are equal, and each is 5° more than the third, find the size of each.

15. In triangle PQR, angle P is 16° less than angle Q, and angle Q is 4° less than angle R. How many degrees in each?

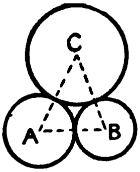
16. It is known in geometry that two tangents drawn from the same point to a circle are equal. Hence, if a circle is inscribed in a triangle, as in the figure, the points of contact divide the sides of the triangle into three sets of equal tangents.



If $AB = 12$ in., $BC = 16$ in., and $AC = 14$ in., show that $x + y = 14$, $x + z = 16$, and $y + z = 12$. Find the distances of the points of contact of the circle with the three sides from the vertices.

17. The sides of a triangle are 24 in., 32 in., and 38 in. Find the points where the inscribed circle touches the sides.

18. Three circles are to be drawn tangent to each other, and with their centers at three given points A, B, and C, respectively. If the distance between A and B is 10 in., between B and C 13 in., and between A and C 15 in., find the radii of the circles.



19. In accurate tool work where holes are to be bored close together in a metal plate, the centers of the holes are first marked carefully to thousandths of an inch. This may be done by first turning out disks on a lathe of such sizes that when placed tangent to each other their centers mark the positions of the centers of the required holes. These disks are then fastened on the metal plate in tangent positions, and the holes bored at their centers.

Three holes are to be bored, the distances between whose centers shall be 0.560 in., 0.620 in., and 0.844 in., respectively. Find the radii of the required disks. See Problem 18.

20. A certain number consists of three digits whose sum is 9. If 198 be subtracted from the number, the remainder will consist of the same digits in a reverse order; and if the number be divided by the hundreds' digit, the quotient will be 108. What is the number?

21. It is given that the standard daily ration for an adult laboring man should include 16 oz. starch, 4 oz. fat, and 4 oz. albumen. The amount of these materials in bread, butter, and beef are as follows:

FOOD	STARCH	FAT	ALBUMEN
Bread	54%	.1%	9%
Butter	0%	83%	1%
Beef	0%	15%	15%

Find the quantities of these foods required to make a daily ration.

22. Potatoes contain 20% starch, 1% fat, 2% albumen; pork contains no starch, 26% fat, 13% albumen; and cream contains 4% starch, 18% fat, 2% albumen. Find the quantities of these foods required to make a standard daily ration.

23. There are three compounds composed of different metals. The first contains 7 parts (in weight) silver, 3 parts copper, and 6 parts tin; the second contains 12 parts silver, 3 parts copper, and 1 part tin; the third contains 4 parts silver, 7 parts copper, and 5 parts tin. How much of each of these three compounds must be taken in order to form a fourth which shall contain 8 oz. of silver, $3\frac{3}{4}$ oz. of copper, and $4\frac{1}{2}$ oz. of tin?

24. In an athletic contest team A won with a total of 35 points, team B got second place with a total of 32 points, and team C got third place with a total of 31 points, as shown in the table:

TEAM	1STS	2DS	3DS	TOTAL
A	7	3	1	35
B	5	4	4	32
C	2	7	9	31

How many points does each place count?

SUPPLEMENTARY EXERCISES

Solve:

1. $$\begin{cases} \dfrac{5M}{6}+\dfrac{N}{4}=7, \\ \dfrac{2M}{3}-\dfrac{N}{8}=3. \end{cases}$$

2. $$\begin{cases} \dfrac{4W+w}{6W+w}=\dfrac{2}{5}, \\ 5+w=\dfrac{2W}{3}. \end{cases}$$

3. $$\begin{cases} \dfrac{9r_1+4}{r_1+r_2+1}=4, \\ \dfrac{r_1+3}{r_1+1}=\dfrac{r_2+1}{r_2-1}. \end{cases}$$

4. $$\begin{cases} \dfrac{1}{a+b}+\dfrac{1}{a-b}=0, \\ \dfrac{a-3b}{b-2a}=2. \end{cases}$$

5. $$\begin{cases} \dfrac{1}{1+x}+\dfrac{1}{1+y}=1, \\ \dfrac{2}{1+x}-\dfrac{1}{1+y}=\dfrac{1}{2}. \end{cases}$$

6. $$\begin{cases} \dfrac{4}{x-2}+\dfrac{3}{y+1}=1, \\ \dfrac{1}{x-2}+\dfrac{6}{y+1}=\dfrac{9}{8}. \end{cases}$$

Solve for x and y:

7. $$\begin{cases} \dfrac{a}{x}-\dfrac{b}{y}=c, \\ \dfrac{2a}{3x}+\dfrac{3b}{4y}=2c. \end{cases}$$

8. $$\begin{cases} \dfrac{3n}{x}+\dfrac{n-1}{y}=m, \\ \dfrac{m}{2x}-\dfrac{m+1}{y}=n. \end{cases}$$

Solve:

9. $$\begin{cases} a+b+c=2, \\ b+c+d=4, \\ c+d+a=6, \\ a+b+d=3. \end{cases}$$

10. $$\begin{cases} r+s+t=18, \\ r+s+w=18, \\ s+w+t=18, \\ r+w+t=18. \end{cases}$$

11. $$\begin{cases} x+y+z+w=10, \\ x-y-z+w=0, \\ 2x+y+3z-w=9, \\ 3x-y+z-2w+4=0. \end{cases}$$

12. $$\begin{cases} 2p-q-r+s=13, \\ p+q-r-s+1=0, \\ p-q+r-s+5=0, \\ 3p+2q-r+2s=17. \end{cases}$$

13. Find a number of three digits, such that the sum of the digits shall be 15; the sum of the hundreds' digit and the ones' digit shall be 1 less than the tens' digit; and the ones' digit subtracted from 4 times the hundreds' digit shall equal the tens' digit.

14. A merchant has two kinds of tea. If he mix 3 pounds of the poorer with 7 pounds of the better, the mixture will be worth $76\frac{1}{2}$¢ a pound; but if he mix 7 pounds of the poorer with 3 pounds of the better, the mixture will be worth $68\frac{1}{2}$¢ a pound. What is the price of each kind of tea?

15. There are two alloys of copper and silver, of which one contains 3 times as much copper as silver, and the other contains 5 times as much silver as copper. How much must be taken of each alloy to make 7 pounds, of which half shall be silver and the other half copper?

16. If the altitude of a rectangle be increased 4 inches, and its base diminished 2 inches, the area will be increased 22 square inches; and if the altitude be increased 1 inch, and the base diminished 1 inch, the area will be increased 2 square inches. Find the base and the altitude of the rectangle.

17. A company of men rented a yacht. When they paid their rental they found that if there had been 2 more persons to pay the same bill, each would have paid 50 cents less than he did; and if there had been 2 fewer persons, each would have paid $1 more than he did. Find the number of persons and the amount that each paid.

CHAPTER XIII

SQUARE ROOT. QUADRATIC SURDS

135. Square Roots of Polynomials.—Since $(a+b)^2=a^2+2ab+b^2$,

$$\sqrt{a^2+2ab+b^2}=a+b.$$

NOTE.—As shown in § 72, every quantity has two square roots, differing only in sign. Thus, the other square root of $a^2+2ab+b^2$ is $-(a+b)$, or $-a-b$. In this chapter we shall consider only one of the square roots of any expression, viz. the one of which the first term is positive.

A study of the above identity will reveal the process of finding the square root of any polynomial which is a perfect square.

If the polynomial has three terms arranged according to the powers of one letter, these correspond to the terms of $a^2+2ab+b^2$. Evidently, the first term a of the root is a square root of a^2. If a^2 be subtracted from the trinomial, the remainder is $2ab+b^2$. The second term of the root, b, may be found by dividing $2ab$, the first term of the remainder, by $2a$, or twice the term of the root already found.

The work is usually arranged as follows:

$$\begin{array}{l|l}
\multicolumn{2}{r}{a^2+2ab+b^2\,\big|\,\underline{a+b}} \\
\multicolumn{2}{r}{a^2 \qquad\qquad\quad} \\ \hline
2a & 2ab+b^2 \\
2a+b & \underline{2ab+b^2}
\end{array}$$

The divisor $2a$, used in finding the second term, is called the **trial divisor.** When the second term b is added to the trial divisor, it gives the **true divisor,** $2a+b$, because when the latter is multiplied by b it gives the entire remainder $2ab+b^2$.

Example 1. — Find the square root of $9x^2 + 4y^2 - 12xy$.

Writing the expression in descending powers of x, the work is as follows

$$\begin{array}{ll}
 & 9x^2 - 12xy + 4y^2 \,\big|\, \underline{3x - 2y} \\
 & 9x^2 \\ \hline
6x & \big|\, -12xy + 4y^2 \\
6x - 2y & \big|\, \underline{-12xy + 4y^2}
\end{array}$$

The first term of the root is $\sqrt{9x^2}$, or $3x$. Subtracting $9x^2$ leaves $-12xy + 4y^2$. The trial divisor is $2(3x)$, or $6x$. Dividing $-12xy$ by $6x$ gives $-2y$, the second term of the root. Adding $-2y$ to $6x$, the trial divisor, gives the true divisor $6x - 2y$. Multiplying this by $-2y$ gives $-12xy + 4y^2$ exactly, which shows that $6x - 2y$ is the entire root.

If the square root of a polynomial has three or more terms, the first two may be found as above; then by grouping terms, these two may be used as one, and the third term obtained by a repetition of the process used to obtain the second. Similarly, by grouping the first three terms of the root, the fourth may be found, etc.

Example 2. — Find the square root of $8n - 4n^3 + n^4 + 4$.

First, arrange the terms in descending powers of n.

$$\begin{array}{ll}
 & n^4 - 4n^3 \qquad + 8n + 4 \,\big|\, \underline{n^2 - 2n - 2} \\
 & n^4 \\ \hline
2n^2 & \big|\, -4n^3 \qquad + 8n + 4 \\
2n^2 - 2n & \big|\, \underline{-4n^3 + 4n^2} \\
2n^2 - 4n & \qquad \big|\, -4n^2 + 8n + 4 \\
2n^2 - 4n - 2 & \qquad \big|\, \underline{-4n^2 + 8n + 4}
\end{array}$$

The first term of the root is $\sqrt{n^4}$, or n^2. Hence the *first trial divisor* is $2n^2$. Dividing $-4n^3$, first term of the remainder, by $2n^2$ gives $-2n$, second term of the root. Adding this to $2n^2$ gives $2n^2 - 2n$, the *first true divisor*. Multiplying this by $-2n$ gives $-4n^3 + 4n^2$. Subtracting this product from the first remainder leaves $-4n^2 + 8n + 4$, the second remainder. Now using the root found, $n^2 - 2n$, as one term, the *second trial divisor* becomes $2(n^2 - 2n)$ or $2n^2 - 4n$. Dividing the first term of this into the first term of the remainder gives -2, the third term of the root. Adding -2 to the trial divisor gives $2n^2 - 4n - 2$, the *second true divisor*. This multiplied by -2 gives the second remainder, which shows that $n^2 - 2n - 2$ is the entire root.

Note. — Care must be taken to first arrange the terms in any problem in descending or ascending powers of some letter. Each remainder and each divisor must also be arranged like the original expression.

The above examples illustrate the general rule:

(1) *Write the given polynomial in descending or ascending powers of some letter.*

(2) *Take the square root of the first term for the first term of the root, and subtract the first term from the polynomial.*

(3) *Double the root found for the first trial divisor, divide the first term of the remainder by this, and write the quotient as the second term of the root.*

(4) *Add this quotient to the trial divisor to obtain the true divisor, multiply the true divisor by the second term of the root, and subtract the product from the preceding remainder.*

(5) *If there is still a remainder, double all of the root already found, for a new trial divisor, and proceed as before. Continue this process until all terms of the root are found.*

EXERCISES

Find the square roots of the following:

1. $4a^2 + 20a + 25$.
2. $1 - 16y + 64y^2$.
3. $m^4 + 25n^2 - 10m^2n$.
4. $5x^2 + x^4 - 2x^3 + 4 - 4x$.
5. $4t^4 + 49 - 3t^2 - 70t + 20t^3$.
6. $1 + 2b - b^2 + 3b^4 - 2b^5 + b^6$.
7. $49A^8 + 42A^6 - 19A^4 - 12A^2 + 4$.
8. $r^4 + 21r^2 + 4 - 10r^3 + 20r$.
9. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.
10. $4w^4 - 4w^3 + 17w^2 - 8w + 16$.
11. $13M^2N^2 + M^4 - 6M^3N - 12MN^3 + 4N^4$
12. $x^4 - 2x^3y - 2xy^3 + 3x^2y^2 + y^4$.
13. $9p^4 - 12p^3q - 26p^2q^2 + 20pq^3 + 25q^4$.

14. $49 - 42\,V + 37\,V^2 + 4\,V^4 - 12\,V^3$.

15. $x^6 - 2\,x^5 + 3\,x^4 - 4\,x^3 + 3\,x^2 - 2\,x + 1$.

16. $m^6 - 6\,m^5n + 15\,m^4n^2 - 20\,m^3n^3 + 15\,m^2n^4 - 6\,mn^5 + n^6$.

17. $v^6 + 16\,w^6 - 4\,v^5w + 10\,v^4w^2 - 24\,vw^5 - 20\,v^3w^3 + 25\,v^2w^4$.

18. $4\,P^6 - 20\,P^5 + 46\,P^2 + 41\,P^4 + 9 - 52\,P^3 - 24\,P$.

136. Square Roots of Arithmetical Numbers. — Any arithmetical number of two or more figures is in nature a polynomial.

Thus, $5263 = 5000 + 200 + 60 + 3$
$= 5 \times 10^3 + 2 \times 10^2 + 6 \times 10 + 3$.

Hence, the square root of an arithmetical number may be obtained in practically the same manner as that of a polynomial.

The square root of 1 is 1; of 100 is 10; of 10000 is 100; etc. Hence the square root of a number between 1 and 100 is between 1 and 10; of a number between 100 and 10000 it is between 10 and 100; etc. That is, the integral part of the square root of an integer of *one* or *two* figures contains *one* figure; of an integer of *three* or *four* figures it contains *two* figures; etc. Therefore,

If the figures of an integer are marked off from right to left into groups of two, the number of figures in the integral part of the square root will be equal to the number of groups, any one figure that remains on the left being counted as a whole group.

Thus, marked off into groups, 20684 will become 2′06′84′. Since there are three groups, the integral part of the square root will contain three figures.

If the square root of a number be a number of two figures, the *tens* of the root may be denoted by a and the *ones* by b. Then the root will be expressed by $a + b$, and hence the number by $a^2 + 2\,ab + b^2$. The use of the fact that $\sqrt{a^2 + 2\,ab + b^2} = a + b$ is best shown by examples.

Example 1. — Find the square root of 3969.

Pointing off, we have 39′69′. Since there are two groups, there must be two figures in the root. The root lies between 60 and 70, because 60^2 is less than 3969, and 70^2 is greater than 3969. That is, the tens' figure of the root is 6, *the square root of the largest square in the left-hand group of the given number.*

The work is indicated thus:

$$\begin{array}{rr|l}
 & & a+b \\
 & 39'69 & \underline{60+3=63} \\
a^2 = & 36\ 00 & \\
\cline{2-2}
2\,a=120 & 3\ 69 & \\
2\,a+b=123 & \underline{3\ 69} &
\end{array}$$

Since $a=60$, $a^2=3600$. Subtracting, the remainder is 369. The trial divisor $2\,a$ becomes 2×60, or 120. Dividing 369 by 120 gives approximately 3. Hence b is probably 3. Hence the trial divisor $2\,a+b$ becomes 123. Multiplying 123 by 3 gives 369. Taking this from the first remainder leaves zero. Hence, $60+3$, or 63, is the exact root.

EXAMPLE 2. — Find the square root of 203401.

By pointing off it is seen that there are three figures in the root.

$$\begin{array}{rr|l}
 & & a+b+c \\
 & 20'34'01' & \underline{400+50+1=451} \\
a^2 = & 16\ 00\ 00 & \\
\cline{2-2}
2\,a=800 & 4\ 34\ 01 & \\
2\,a+b=850 & \underline{4\ 25\ 00} & \\
2\,(a+b)=900 & 9\ 01 & \\
2\,(a+b)+c=901 & \underline{9\ 01} &
\end{array}$$

The largest square in 20 is 16. Hence $a^2=160000$, and $a=400$. The first remainder is 43401. The first trial divisor $2\,a$ is 2×400, or 800. Dividing 43401 by 800 gives approximately 50, the value of b. The first true divisor is $800+50$, or 850. Multiplying 850 by 50 gives 42500. Subtracting, the second remainder is 901. The second trial divisor $2(a+b)$ is 2×450, or 900. 901 divided by 900 gives approximately 1, the value of c. The second true divisor is $900+1$, or 901. Multiplying 901 by 1 gives 901. Subtracting this from the second remainder leaves zero. Hence the exact root is $400+50+1$, or 451.

When the square root of a number has decimal places, the number itself has twice as many.

Thus, $0.23^2=0.0529.$

Hence, *to mark off a number which contains a decimal, begin at the decimal point and mark to the left and to the right, putting two figures in each group.*

If a decimal has an odd number of figures, add a zero to make it even.

Thus, 723.618 will become 7 23'.61'80'.

Example 3. — Find the square root of 50.9796.

$$\begin{array}{r|rl}
\multicolumn{1}{r}{} & 50.'97'96' & \underline{|\,7.00+.10+.04=7.14} \\
\multicolumn{1}{r}{} & 49.\;00\;00 & \\ \cline{2-2}
14.00 & 1.\;97\;96 & \\
14.10 & 1.\;41\;00 & \\ \cline{2-2}
14.20 & .\;56\;96 & \\
14.24 & .\;56\;96 & \\ \cline{2-2}
\end{array}$$

An approximation to the value of the square root of a number that is not a perfect square may be found to any degree of accuracy desired.

Example 4. — Find to three decimal places the square root of 2.

Since three decimal places are desired in the root, we annex 6 zeros.

$$\begin{array}{r|rl}
\multicolumn{1}{r}{} & 2.'00'00'00' & \underline{|\,1+.4+.01+.004=1.414} \\
\multicolumn{1}{r}{} & 1.\;00\,00\,00 & \\ \cline{2-2}
2 & 1.\;00\,00\,00 & \\
2.4 & .\;96\,00\,00 & \\ \cline{2-2}
2.8 & .\;04\,00\,00 & \\
2.81 & .\;02\,81\,00 & \\ \cline{2-2}
2.82 & .\;01\,19\,00 & \\
2.824 & .\;01\,12\,96 & \\ \cline{2-2}
\end{array}$$

EXERCISES

Find the square root of:

1. 7396.
2. 1849.
3. 5776.
4. 26244.
5. 41209.
6. 17424.
7. 16129.
8. 96.4324.
9. 12.25.
10. 146.41.
11. 125.44.
12. 4.6225.

Find to three decimal places the square root of:

13. 5.
14. 3.
15. 0.6.
16. 1.2.
17. 28.
18. 0.45.
19. 37.1.
20. 0.75.

21. The area of a triangle whose sides are a, b, and c equals $\sqrt{s(s-a)(s-b)(s-c)}$, where s equals one half of the sum of the three sides. The sides of a triangular field are 42 rd., 38 rd., and 46 rd., respectively. Find its area to hundredths of a square rod.

NOTE. — The use of square root in solving equations will be shown in the next chapter.

137. Real and Imaginary Numbers. — Since any even power of either a positive or a negative number is positive, any *even root of a negative number* is neither a positive nor a negative number. The indicated even root of a negative number is called an **imaginary number.**

Thus, $\sqrt{-4}$, $\sqrt[4]{-9}$, $\sqrt[6]{-1}$, etc., are imaginary numbers.

A number that is not imaginary is called a **real number.** Hence, all numbers are classified as either real or imaginary. All numbers with which we have had to deal up to the present time are *real numbers.*

NOTE. — In the chapter following this some imaginary numbers will be encountered in solving equations. But no knowledge of the operations with these numbers will be needed in the *First Course.* A thorough treatment of imaginary numbers will be found in the *Second Course.*

138. Surds. — An indicated root which is real but the value of which cannot be computed exactly is called a **surd.**

Thus, $\sqrt{3}$, $\sqrt[3]{5}$, $\sqrt{m+n}$, etc., are surds.

A surd whose index is 2 is called a **quadratic surd.**

NOTE. — In this chapter are given only a few of the important processes applied to quadratic surds. A fuller treatment of surds will be found in the *Second Course.*

Every positive number has two square roots, differing only in sign. But in dealing with quadratic surds only the positive value of each will be considered.

139. Reducing a Surd to its Simplest Form. — A quadratic surd is in its **simplest form** when the expression under the radical sign is integral and contains no factor of which the indicated root can be found. A quadratic surd that is not in its simplest form can be reduced to its simplest form by use of the principle that

The square root of the product of two numbers equals the product of the square roots of the numbers.

Thus, $\sqrt{36} = \sqrt{4} \times \sqrt{9}$, because each equals 6.

$\sqrt{25\,v^2(v-w)^4} = \sqrt{25\,v^2} \times \sqrt{(v-w)^4}$, because each equals $5\,v(v-w)^2$.

The application of this principle in simplifying surds is shown in the following examples:

Example 1. — Reduce to its simplest form $\sqrt{63}$.

Factoring 63 into two factors one of which is the largest perfect square possible, we have

$$\sqrt{63} = \sqrt{9 \times 7} = \sqrt{9} \times \sqrt{7} = 3\sqrt{7}.$$

Example 2. — Reduce to its simplest form $\sqrt{\frac{4}{5}}$.

Multiplying both terms of $\frac{4}{5}$ by 5 to make the denominator a perfect square, then factoring, we get

$$\sqrt{\tfrac{4}{5}} = \sqrt{\tfrac{20}{25}} = \sqrt{\tfrac{4}{25}} \times \sqrt{5} = \tfrac{2}{5}\sqrt{5}.$$

Example 3. — Reduce to its simplest form $\sqrt{32\,a^2b^4c^3}$.

$$\sqrt{32\,a^2b^4c^3} = \sqrt{16\,a^2b^4c^2 \times 2\,c} = \sqrt{16\,a^2b^4c^2} \times \sqrt{2\,c} = 4\,ab^2c\sqrt{2\,c}.$$

If the expression under the radical sign is integral, separate it into two factors of which one is the largest perfect square possible. The square root of the square factor becomes the coefficient of the radical.

If the expression under the radical sign is a fraction, multiply both terms of the fraction by such a quantity as will make the resulting denominator a perfect square. Then separate into two factors and proceed as above.

Note. — The importance of knowing how to simplify a quadratic surd is seen when the approximate value of a surd is to be found. Thus, if one knows that $\sqrt{2} = 1.414$, then $\sqrt{200} = 10\sqrt{2} = 14.14$. Also if it is known that $\sqrt{3} = 1.732$, then $\sqrt{27} = 3\sqrt{3} = 5.196$.

EXERCISES

Reduce to the simplest form:

1. $\sqrt{8}$.
2. $\sqrt{12}$.
3. $\sqrt{24}$.
4. $\sqrt{27}$.
5. $\sqrt{18}$.
6. $\sqrt{50}$.
7. $\sqrt{28}$.
8. $\sqrt{48}$.
9. $\sqrt{75}$.
10. $\sqrt{54}$.
11. $\sqrt{72}$.
12. $\sqrt{125}$.
13. $\sqrt{32}$.
14. $\sqrt{80}$.
15. $\sqrt{242}$.

16. $\sqrt{338}$.
17. $\sqrt{432}$.
18. $\sqrt{300}$.
19. $\sqrt{\frac{1}{2}}$.
20. $\sqrt{\frac{2}{3}}$.
21. $\sqrt{\frac{4}{7}}$.
22. $\sqrt{\frac{9}{20}}$.
23. $\sqrt{\frac{16}{63}}$.
24. $\sqrt{\frac{1}{24}}$.
25. $\sqrt{\frac{25}{27}}$.
26. $\sqrt{\frac{32}{40}}$.
27. $\sqrt{\frac{7}{32}}$.
28. $\sqrt{\frac{5}{21}}$.
29. $\sqrt{\frac{8}{75}}$.
30. $\sqrt{\frac{11}{96}}$.
31. $\sqrt{a^2b}$.
32. $\sqrt{4\,ab^4}$.
33. $\sqrt{25\,m^3n}$.
34. $\sqrt{49\,x^5y}$.
35. $\sqrt{8\,t^6v^3}$.
36. $\sqrt{50\,P^4t}$.
37. $\sqrt{27\,W^3g^2}$.
38. $\sqrt{5\,k^8v^5}$.
39. $\sqrt{45\,AB^7}$.
40. $\sqrt{a^3(a-b)^2}$.
41. $\sqrt{\frac{a^2b^4}{cd^4}}$.

42. If $\sqrt{2} = 1.414$, find the values of the surds in Exercises 1, 5, 6, 11, 13, 15, 16, and 19.

43. If $\sqrt{3} = 1.732$, find the values of the surds in Exercises 2, 4, 8, and 9.

140. Addition and Subtraction of Surds. — Only quadratic surds that have the same number under the radical sign can be added or subtracted. They are added or subtracted just as any two similar terms.

Thus, just as $3\,n + 2\,n = 5\,n$, so $3\sqrt{7} + 2\sqrt{7} = 5\sqrt{7}$.

It usually is best first to reduce surds to be added or subtracted to their simplest forms before trying to unite them.

Thus, $\sqrt{12} + \sqrt{27} + \sqrt{75} = 2\sqrt{3} + 3\sqrt{3} + 5\sqrt{3} = 10\sqrt{3}$.

To add or subtract quadratic surds having the same radical part, add or subtract the coefficients of the radical part.

EXERCISES

Simplify:

1. $\sqrt{2} + \sqrt{8}$.
2. $\sqrt{3} + \sqrt{12}$.
3. $\sqrt{18} + \sqrt{27}$.
4. $\sqrt{45} - \sqrt{20}$.
5. $\sqrt{32} - \sqrt{8}$.
6. $\sqrt{50} - \sqrt{18}$.
7. $2\sqrt{2} + \sqrt{32}$
8. $5\sqrt{5} - \sqrt{45}$.
9. $2\sqrt{63} - \sqrt{28}$.
10. $\sqrt{99} + \sqrt{44}$.
11. $\sqrt{18} - \sqrt{8} + \sqrt{32}$.
12. $\sqrt{75} - \sqrt{3} - \sqrt{12}$.
13. $\sqrt{20} + \sqrt{45} - \sqrt{80}$.

14. $\sqrt{6}+\sqrt{294}+\sqrt{24}$.

15. $2\sqrt{2}+3\sqrt{18}-\sqrt{50}$.

16. $\sqrt{\frac{1}{2}}+3\sqrt{\frac{2}{9}}$.

17. $3\sqrt{48}-\sqrt{\frac{1}{3}}-2\sqrt{\frac{4}{3}}$.

18. $\sqrt{\frac{4}{5}}+5\sqrt{\frac{1}{125}}-2\sqrt{\frac{5}{4}}$.

19. $\sqrt{20}+3\sqrt{5}+\sqrt{\frac{1}{5}}$.

20. $\sqrt{\frac{1}{3}}-4\sqrt{3}-6\sqrt{\frac{3}{4}}$.

21. $2a\sqrt{a^3}+3\sqrt{a^5}$.

22. $\sqrt{x^3y^2}-3y\sqrt{x^3}$.

23. $\sqrt{9a^4b^3}+5ab\sqrt{a^2b}$.

24. $\sqrt{mn^3}+n\sqrt{9mn}$.

25. $8\sqrt{2v^5t}-v\sqrt{50v^3t}$.

26. $18p\sqrt{3pq^3}+2q\sqrt{27p^3q}$.

27. $r\sqrt{t}+\sqrt{r^2t}+\sqrt{t^3}$.

28. $\sqrt{u^3vw}+\sqrt{uv^3w}+\sqrt{uvw^3}$.

29. $\sqrt{a^2bc}+\sqrt{a^2b^3c}+\sqrt{a^2bc^3}$.

141. Multiplication of Surds. — The product of two quadratic surds is obtained by use of the principle in § 139, *i.e.*

The product of the square roots of two numbers equals the square root of the product of the numbers.

Thus, $\sqrt{2}\times\sqrt{3}=\sqrt{2\times 3}$ or $\sqrt{6}$; $\sqrt{15}\times\sqrt{5}=\sqrt{15\times 5}=\sqrt{75}=5\sqrt{3}$.
Similarly, $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$; $\sqrt{m}\times\sqrt{mn}=\sqrt{m^2n}=m\sqrt{n}$.

The product of two expressions involving surds is obtained just as the product of any two polynomials, *i.e.* by multiplying each term of one by each term of the other, and adding the similar terms obtained.

EXAMPLE. — Multiply $3+\sqrt{2}$ by $2-\sqrt{2}$.

$$\begin{array}{l}
3+\sqrt{2} \\
\underline{2-\sqrt{2}} \\
6+2\sqrt{2} \\
\underline{-3\sqrt{2}-2} \\
6-\sqrt{2}-2=4-\sqrt{2}
\end{array}$$

EXERCISES

Multiply:

1. $\sqrt{3}$ by $\sqrt{5}$.
2. $\sqrt{2}$ by $\sqrt{7}$.
3. $\sqrt{3}$ by $\sqrt{11}$.
4. $\sqrt{5}$ by $\sqrt{13}$.
5. $\sqrt{2}$ by $\sqrt{6}$.
6. $\sqrt{3}$ by $\sqrt{15}$.
7. $\sqrt{21}$ by $\sqrt{3}$.
8. $\sqrt{10}$ by $\sqrt{30}$.
9. $\sqrt{2}\times\sqrt{14}$.
10. $\sqrt{5}\times\sqrt{35}\times\sqrt{21}$.
11. $\sqrt{3}\times\sqrt{6}\times\sqrt{2}$.
12. $\sqrt{8}\times\sqrt{22}\times\sqrt{33}$.

13. $2-\sqrt{5}$ by $4+\sqrt{5}$.

14. $1+\sqrt{3}$ by $3+\sqrt{3}$.

15. $7-\sqrt{6}$ by $2-\sqrt{6}$.

16. $9+\sqrt{11}$ by $3-\sqrt{11}$.

17. $2+\sqrt{3}$ by $3+\sqrt{2}$.

18. $\sqrt{15}-4$ by $\sqrt{5}+1$.

19. $\sqrt{7}+1$ by $3-\sqrt{7}$.

20. $\sqrt{12}+2$ by $\sqrt{3}-5$.

21. $\sqrt{3}+\sqrt{2}$ by $\sqrt{3}-\sqrt{2}$.

22. $\sqrt{5}+\sqrt{6}$ by $\sqrt{5}+\sqrt{6}$.

23. $\sqrt{7}-2\sqrt{2}$ by $\sqrt{7}-\sqrt{2}$.

24. $\sqrt{3}+4\sqrt{11}$ by $\sqrt{2}+\sqrt{11}$.

25. $2\sqrt{7}+3\sqrt{2}$ by $3\sqrt{7}-5\sqrt{2}$.

26. $1-\sqrt{2}$ by $2\sqrt{3}+3\sqrt{2}$.

27. $\sqrt{r}$ by $\sqrt{rs}$.

28. $a\sqrt{b}$ by $b\sqrt{a}$.

29. $2x\sqrt{xy}$ by $x^2\sqrt{x}$.

30. $p^2\sqrt{p}$ by $pq\sqrt{pq}$.

31. $5w\sqrt{wv}$ by $2v\sqrt{2w}$.

32. $\sqrt{3mn}$ by $\sqrt{7mn}$.

33. $x+\sqrt{y}$ by $x-\sqrt{y}$.

34. $r-\sqrt{s}$ by $r-\sqrt{s}$.

35. $2a-3\sqrt{b}$ by $a+2\sqrt{b}$.

36. $\sqrt{m}+\sqrt{n}$ by $\sqrt{m}-\sqrt{n}$.

142. Division of Surds. — In the division of expressions involving surds it is best to first free the divisor of surds. This is accomplished by use of the principle that if the dividend and divisor are both multiplied by the same expression, the value of the quotient is unchanged.

EXAMPLE 1. — Divide $\sqrt{3}$ by $\sqrt{2}$.

Multiplying both dividend and divisor by $\sqrt{2}$,

$$\frac{\sqrt{3}}{\sqrt{2}}=\frac{\sqrt{3}\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}=\frac{\sqrt{6}}{2}=\frac{1}{2}\sqrt{6}.$$

If the divisor is a binomial, in order to free it of surds it must be multiplied by the same binomial with the sign between the terms changed. This gives the product of the sum and difference of two numbers, which is the difference between their squares.

EXAMPLE 2. — Divide $\sqrt{7}+2\sqrt{5}$ by $\sqrt{7}+\sqrt{5}$.

Multiplying both dividend and divisor by $\sqrt{7}-\sqrt{5}$,

$$\frac{\sqrt{7}+2\sqrt{5}}{\sqrt{7}+\sqrt{5}}=\frac{(\sqrt{7}+2\sqrt{5})(\sqrt{7}-\sqrt{5})}{(\sqrt{7}+\sqrt{5})(\sqrt{7}-\sqrt{5})}=\frac{\sqrt{35}-3}{2}=\tfrac{1}{2}\sqrt{35}-1\tfrac{1}{2}.$$

EXERCISES

Perform the indicated divisions:

1. $\frac{1}{\sqrt{2}}$.
2. $\frac{3}{\sqrt{3}}$.
3. $\frac{10}{\sqrt{5}}$.
4. $\frac{\sqrt{6}}{\sqrt{5}}$.
5. $\frac{\sqrt{6}}{\sqrt{2}}$.
6. $\frac{\sqrt{10}}{\sqrt{5}}$.
7. $\frac{8}{\sqrt{6}}$.
8. $\frac{12}{\sqrt{18}}$.
9. $\frac{\sqrt{20}}{\sqrt{8}}$.
10. $\frac{a}{\sqrt{a}}$.
11. $\frac{\sqrt{n}}{\sqrt{m}}$.
12. $\frac{\sqrt{x+1}}{\sqrt{x}}$.
13. $\frac{4}{\sqrt{2p}}$.
14. $\frac{3v}{\sqrt{6v}}$.
15. $\frac{20r^2}{\sqrt{2r}}$.
16. $\frac{2}{2+\sqrt{2}}$.
17. $\frac{4}{\sqrt{3}-1}$.
18. $\frac{6}{\sqrt{3}+\sqrt{2}}$.
19. $\frac{16}{\sqrt{5}-\sqrt{3}}$.
20. $\frac{12}{\sqrt{7}-\sqrt{6}}$.
21. $\frac{4}{\sqrt{21}-\sqrt{19}}$.
22. $\frac{\sqrt{3}+1}{\sqrt{3}-1}$.
23. $\frac{\sqrt{5}+2}{\sqrt{5}-2}$.
24. $\frac{\sqrt{11}-\sqrt{7}}{\sqrt{11}+\sqrt{7}}$.
25. $\frac{\sqrt{6}+2\sqrt{2}}{\sqrt{6}+\sqrt{2}}$.
26. $\frac{2\sqrt{5}-3\sqrt{3}}{3\sqrt{5}+\sqrt{3}}$.
27. $\frac{4\sqrt{7}+3\sqrt{2}}{3\sqrt{7}-7\sqrt{2}}$.
28. $\frac{1}{a+\sqrt{b}}$.
29. $\frac{x}{\sqrt{x}-\sqrt{y}}$.
30. $\frac{m-n}{\sqrt{m}-\sqrt{n}}$.

143. Graphs of Quadratic Surds. — Although the exact value of a quadratic surd cannot be computed decimally, its value can be represented exactly by the length of a line. The principle involved is that the square of the hypotenuse of any right triangle equals the sum of the squares of the other two sides.

Thus, if a right triangle ABC be drawn with AB equal to 1 and BC equal to 1, then AC will equal $\sqrt{2}$. Now, if DC be drawn from C, perpendicular to CA, and equal to 1, the side DA of the right triangle ACD will equal $\sqrt{3}$.

By constructing another right triangle with AD as a side and the new side equal to 1, and continuing this process, the hypotenuse of a right triangle can be obtained that will represent any quadratic surd.

Many quadratic surds can be represented by drawing a single triangle. Thus, if $AB = 3$ and $BC = 1$, $AC = \sqrt{10}$. If $AB = 3$ and $BC = 2$, $AC = \sqrt{13}$.

EXERCISES

By drawing right triangles, represent the following:

1. $\sqrt{5}$.
2. $\sqrt{6}$.
3. $\sqrt{7}$.
4. $\sqrt{8}$.
5. $\sqrt{11}$.
6. $\sqrt{15}$.
7. $\sqrt{34}$.
8. $\sqrt{53}$.
9. $\sqrt{45}$.
10. $\sqrt{41}$.
11. $\sqrt{40}$.
12. $\sqrt{65}$.

SUPPLEMENTARY EXERCISES

Find the square root of:

1. $y^4 + 4y^3 + 4y^2 + 2y + 4 + \frac{1}{y^2}$.
2. $R^2 + 8R + 12 - \frac{16}{R} + \frac{4}{R^2}$.
3. $2 + \frac{a^2}{b^2} + 2ab + b^4 + \frac{2b^3}{a} + \frac{b^2}{a^2}$.
4. $n^2 - tn - \frac{11t^2}{4} + \frac{3t^3}{2n} + \frac{9t^4}{4n^2}$.

Find the square root to hundredths of:

5. $\frac{5}{7}$.

Suggestion. — In finding the square root of a common fraction it is best to first change the fraction to a decimal. Since we want two places of decimals in the root, write the first four decimals of the number. Thus, $\frac{5}{7} = 0.7142$.

6. $\frac{1}{2}$. 7. $\frac{6}{7}$. 8. $\frac{11}{15}$. 9. $\frac{3}{4}$. 10. $\frac{7}{8}$.

11. A method used by the Arabs, and as late as the Middle Ages, of finding the approximate value of the square root of a number was to substitute in the formulæ of approximation:

$$(a)\ \sqrt{n^2 + a} = n + \frac{a}{2n},\quad (b)\ \sqrt{n^2 + a} = n + \frac{a}{2n + 1}.$$

The true value of the root was found to be between the two values obtained. For example, to find $\sqrt{82}$, separate 82 into $81+1$. Then from

$$(a)\quad \sqrt{82}=\sqrt{81}+1=\sqrt{9^2+1}=9+\tfrac{1}{18}=9\tfrac{1}{18};\text{ from}$$

$$(b)\quad \sqrt{82}=\sqrt{81+1}=\sqrt{9^2+1}=9+\tfrac{1}{19}=9\tfrac{1}{19}.$$

Find the decimal values of these results, which are the same to the hundredths figure, and hence find the true square root to hundredths.

Find by the method of Problem 11 between what values the square roots of the following lie:

12. 17. **13.** 50. **14.** 65. **15.** 102. **16.** 147.

Reduce to the simplest form:

17. $\sqrt{\frac{1}{x^2}+\frac{2}{x}+3}$. **18.** $\sqrt{\frac{m}{n^2}+\frac{n}{m^2}}$. **19.** $\sqrt{\frac{B^2}{4A^2}-\frac{C}{A}}$.

20. Divide $1+\sqrt{2}$ by $1+\sqrt{2}+\sqrt{3}$.

Suggestion. — To free the divisor of surds it will be necessary to multiply both dividend and divisor twice when the divisor is a trinomial such as this. First multiply by the expression obtained by changing one of the signs in the divisor. Thus,

$$\frac{1+\sqrt{2}}{1+\sqrt{2}+\sqrt{3}}=\frac{(1+\sqrt{2})(1+\sqrt{2}-\sqrt{3})}{(1+\sqrt{2}+\sqrt{3})(1+\sqrt{2}-\sqrt{3})}=\frac{3+\sqrt{2}-\sqrt{3}-\sqrt{6}}{2\sqrt{2}}=\text{etc.}$$

Divide:

21. $\sqrt{3}+\sqrt{5}$ by $1+\sqrt{3}+\sqrt{5}$.

22. $1+\sqrt{2}$ by $1-\sqrt{2}+\sqrt{5}$.

23. 2 by $\sqrt{2}+\sqrt{3}+\sqrt{7}$.

24. $\sqrt{2}$ by $\sqrt{2}+\sqrt{3}+\sqrt{5}$.

25. $\sqrt{5}-\sqrt{3}$ by $\sqrt{5}-\sqrt{3}-1$.

26. $\sqrt{x}$ by $\sqrt{x}+\sqrt{y}+\sqrt{z}$.

27. $\sqrt{a}-\sqrt{b}$ by $\sqrt{a}+\sqrt{b}-\sqrt{c}$.

28. $1-\sqrt{m}$ by $1+\sqrt{m}+\sqrt{n}$.

CHAPTER XIV

QUADRATIC EQUATIONS

144. Quadratic Equations. — As shown in § 73 and § 96, some problems may be expressed and solved by use of equations of the **second degree, or quadratic equations** (see § 57 for definitions). The equations discussed in § 73 and § 96 were of very special types, and the methods of solution there shown are not applicable to all quadratic equations. We shall now discover how to solve any quadratic equation whatever.

A quadratic equation that contains a term of the first degree in the unknown number is called a **complete quadratic equation.** One that does not contain a term of the first degree in the unknown number is called a **pure quadratic equation.**

Thus, $3n^2 - 4n = 5$ is a complete quadratic equation. And $7t^2 - 9 = 0$ is a pure quadratic equation.

By simplifying and combining similar terms, if necessary, any quadratic equation whose unknown number is denoted by x can be written in the form $Ax^2 + Bx + C = 0$, where A, B, and C are known numbers.

145. Quadratics solved by Factoring. — Some quadratics can be solved by *factoring,* as shown in § 96. The rule there given should now be reviewed and applied in the following exercises.

NOTE. — It must be remembered that every quadratic equation has *two roots. Both roots* should be checked by seeing if they satisfy the given equation.

EXERCISES

Solve by factoring, and check:

1. $a^2 = 7a - 12$.
2. $P^2 + 3P = 10$.
3. $w^2 + 9w + 14 = 0$.
4. $2x^2 + 5x = 3$.
5. $6V^2 + 13V + 6 = 0$.
6. $10t^2 + 33t = 7$.
7. $6y^2 = 23y + 4$.
8. $2 - R = 21R^2$.
9. $54z = 9z^2 + 72$.
10. $4A = A^2 - 77$.
11. $\frac{m}{2} + \frac{1}{2} = \frac{1}{m}$.
12. $1 + \frac{28}{D^2} = \frac{11}{D}$.
13. $15K + 4 = \frac{3}{K}$.
14. $5 = \frac{4}{r} + \frac{33}{r^2}$.
15. $a^2 + \frac{1}{8}a = 3\frac{3}{4}$.
16. $1 + \frac{22}{P} - 6P = 0$.
17. $6 - \frac{50}{s^2} + \frac{55}{s} = 0$.
18. $5r - 8 = \frac{21}{r}$.
19. $\frac{3z^2 - 4}{z + 2} = 2z + 1$.
20. $\frac{y + 15}{y + 3} = y + 8$.
21. $\frac{3b - 10}{14} = \frac{b + 120}{b}$.
22. $\frac{1}{2x - 2} - \frac{x - 7}{x + 1} + \frac{x + 24}{5x^2 - 5} = 0$
23. $\frac{N}{N - 3} - \frac{N - 1}{N + 3} = 2$.
24. $\frac{s + 3}{2} = \frac{10}{3} - \frac{2}{s + 3}$.
25. $\frac{3(t - 1)}{t + 1} = 5 + \frac{2(t + 1)}{t - 1}$.
26. $\frac{r^2 + r + 1}{r - 1} - 6 = \frac{r^2 - r + 1}{r + 1}$.
27. $3a = a^2 - 18$.

146. Quadratics solved by Square Root. — Any pure quadratic equation can be solved by the method of § 73.

To solve a pure quadratic equation first, by clearing of fractions, transposing terms, etc., reduce the equation to the form $x^2 = A$, where x is the unknown number and A is known. Take the square root of both members of the resulting equation, attaching the double sign ± to the second member.

EXAMPLE 1. — Solve $8t^2 = 2t(t - 5) + 10(t + 2)$.

Removing parentheses, transposing, combining terms, and dividing by the coefficient of t,

$$t^2 = \tfrac{10}{3}.$$

Taking square root, $$t = \pm\sqrt{\tfrac{10}{3}}$$
$$= \pm\tfrac{1}{3}\sqrt{30}.$$

These roots may be expressed decimally, if desired. Thus, $\sqrt{30} = 5.477$, approximately. Hence, $t = \pm 1.825$, approximately.

Note. — In some problems in which the roots involve surds it is sufficient to express the roots with radical signs. But in many applied problems it is desirable to compute the approximate values of such roots decimally.

The student should be able to compute the roots in either form when desirable.

Example 2. — Solve $D^2 + 5 = 0$.

Transposing, $$D^2 = -5.$$

Taking square root, $$D = \pm\sqrt{-5},\ \textit{imaginary numbers.}$$

EXERCISES

Solve:

1. $7y^2 - 28 = 0.$
2. $10V^2 - 150 = 4V^2.$
3. $r^2 = 845 - 4r^2.$
4. $a(a+4) = 4a + 49.$
5. $(M+2)(2M+3) = 7(M+3).$
6. $8t^2 + 72 = 0.$
7. $5n^2 + 20 = 0.$
8. $4A^2 = 7.$
9. $6R^2 - 8 = 0.$
10. $5d^2 + 10 = 0.$
11. $20P^2 + 8 = 0.$
12. $\dfrac{x}{125} = \dfrac{5}{x}.$
13. $\dfrac{3t}{4} = \dfrac{4}{3t}.$
14. $\dfrac{1}{y+1} = \dfrac{y-1}{6}.$
15. $\dfrac{a^2+4}{8} = 1.$
16. $\dfrac{V^2+8}{2} = 2.$
17. $\dfrac{9s^2-1}{3} = \dfrac{3}{4}.$
18. $\dfrac{5}{v^2-1} = 4.$
19. $\dfrac{3}{2k^2} - \dfrac{1}{k^2} = \dfrac{9}{4}.$
20. $\dfrac{1-b^2}{6} - \dfrac{b^2-1}{5} = 1.$
21. $\dfrac{n-1}{n+1} + \dfrac{n+1}{n-1} = 3.$

22. $\frac{3A^2-7}{A^2-1}-\frac{A-1}{A+1}=\frac{A+1}{A-1}.$

23. $\frac{Q+5}{6}-\frac{8}{Q+1}=1.$

24. $\frac{z+6}{6z+1}-\frac{6z+1}{z+6}=0.$

25. $\frac{4p+1}{3p-4}=\frac{8p-19}{7p-24}.$

26. $\frac{3s+2}{2+s}-\frac{13}{2s+3}=0.$

27. $2T+1+\frac{(3T+1)(T-2)}{T+2}=0.$

28. $\frac{2G}{5}-\frac{1+G^2}{2G+3}=\frac{3}{5}.$

29. $\frac{x^2+x+1}{x-1}-6=\frac{x^2-x+1}{x+1}.$

Find approximate to hundredths the roots of:

30. $x^2=7.$

31. $4a^2-10=0.$

32. $8t^2-12=0.$

33. $15n^2-4=0.$

34. $7V^2=26.$

35. $2D^2=17.$

36. $5B^2-14=0.$

37. $3t^2=28.$

38. $(p+4)(p-4)=4.$

39. $\frac{2}{r+3}=\frac{r-3}{4}.$

40. $\frac{K+5}{4}-\frac{3}{K-1}=1.$

147. Quadratics solved by completing the Square. — As shown in § 86, any trinomial that is a perfect square may be written in either the form $a^2+2ab+b^2$ or $a^2-2ab+b^2$. Hence, if we have given either the binomial a^2+2ab or a^2-2ab, to make it a perfect square we must add b^2, *i.e. the square of one half of the coefficient of a in the term 2 ab.* This process is called **completing the square.**

Thus, to complete the square whose first two terms are n^2-10n, we add the square of $\frac{1}{2}$ of 10, or 25. This gives $n^2-10n+25$, which is the square of $n-5$.

By use of this process of completing the square any complete quadratic equation whatever may be solved, as shown in the following examples.

EXAMPLE 1. — Solve $v^2+3v-10=0.$

Transposing, $v^2+3v=10.$

Adding the square of $\frac{1}{2}$ of 3 to both members, to complete the square on the left,

$$v^2+3v+\tfrac{9}{4}=\tfrac{49}{4}.$$

Taking square root, $v + \frac{3}{2} = \pm \frac{7}{2}$.

Transposing, $v = -\frac{3}{2} \pm \frac{7}{2}$, *i.e.* $-\frac{3}{2} + \frac{7}{2}$ or $-\frac{3}{2} - \frac{7}{2}$.

Simplifying, $v = 2$ or -5

Example 2. — Solve $5P^2 + 12P = 2$.

Dividing by 5, $P^2 + \frac{12}{5}P = \frac{2}{5}$.

Adding square of $\frac{1}{2}$ of $\frac{12}{5}$, $P^2 + \frac{12}{5}P + \frac{36}{25} = \frac{46}{25}$.

Taking square root, $P + \frac{6}{5} = \pm \frac{1}{5}\sqrt{46}$.

Transposing, $P = -\frac{6}{5} \pm \frac{1}{5}\sqrt{46}$.

If the approximate values of these roots, expressed decimally, are desired, they may be found by taking the square root of 46 and simplifying. Thus, $\sqrt{46} = 6.782$, approximately. Hence, $P = 0.15$ or -2.55, approximately.

Example 3. — Solve $y^2 = 4y - 7$.

Transposing, $y^2 - 4y = -7$.

Adding square of $\frac{1}{2}$ of 4, $y^2 - 4y + 4 = -3$.

Taking square root, $y - 2 = \pm\sqrt{-3}$.

Transposing, $y = 2 \pm \sqrt{-3}$.

These roots involve *imaginary numbers.*

Evidently, any complete quadratic equation may be solved by the following rule:

(1) *Reduce the equation to the form* $x^2 + px = q$, *where* x *is the unknown number.*

(2) *Add to each member the square of one half of the coefficient of the term in the first power of the unknown number.*

(3) *Take the square root of both members, attaching the double sign* ± *to the second member.*

(4) *Solve the resulting linear equations.*

EXERCISES

Solve by completing the square:

1. $y^2 - 6y + 8 = 0$.
2. $n^2 + 4n = 12$.
3. $A^2 - 8A = 9$.
4. $t^2 = 22 + 9t$.
5. $v^2 + 14v + 40 = 0$.
6. $x^2 + x = 6$.
7. $V^2 - 11V + 30 = 0$.
8. $d^2 = 14 - 5d$.
9. $M^2 + 12M + 15 = 0$.
10. $2k^2 + k = 4$.

11. $2P^2-5P=3.$

12. $6r^2+6=13r.$

13. $4W^2-11W=3.$

14. $10n^2=29n-10.$

15. $3R^2=17R+28.$

16. $2b^2+3b=4.$

17. $5k^2=k+7.$

18. $12G^2+3=14G.$

19. $9E^2=6E+26.$

20. $16a^2-96a=1792.$

21. $N^2-7N+15=0.$

22. $\dfrac{1}{t+1}-\dfrac{2}{1-t}=\dfrac{13}{4t-1}.$

23. $\dfrac{5}{4+r'}+\dfrac{5}{4-r'}=\dfrac{8}{3}.$

24. $\dfrac{S^2}{2}-\dfrac{S}{3}+20\frac{1}{2}=42\frac{2}{3}.$

25. $6H+\dfrac{35-3H}{H}=42.$

26. $x^2+7x=7(x+3)+4.$

27. $\dfrac{w-3}{w+5}-\dfrac{w+4}{w-7}=2.$

28. $\dfrac{3}{p^2-1}+\dfrac{1}{p+1}=1+\dfrac{2}{p-1}.$

29. $\dfrac{1}{2V-3}+\dfrac{2}{V+1}+3=0.$

30. $3-\dfrac{1}{y+2}=\dfrac{1}{y-2}.$

31. $\dfrac{1}{U-1}+\dfrac{1}{U-2}+\dfrac{1}{U-3}=0$

32. $\dfrac{c+2}{c-1}-\dfrac{4-c}{2c}=2\frac{1}{3}.$

33. $(z-5)^2+(z-10)^2=37.$

34. $\dfrac{r_1+3}{r_1+9}+\dfrac{r_1-4}{r_1-1}=0.$

35. $\dfrac{3}{B^2-1}+\dfrac{1}{2B-2}=\dfrac{1}{4}.$

36. $\dfrac{1}{m+2}-\dfrac{1}{m-2}=\dfrac{1}{1-m}.$

37. $\dfrac{2Y-1}{Y-1}+\dfrac{Y-1}{Y-2}=\dfrac{3Y-10}{Y-3}.$

38. $\dfrac{d-4}{2d-12}-\dfrac{d-4}{3d-16}=\dfrac{11}{2}.$

Find approximate to hundredths the roots of:

39. $n^2=3n+7.$

40. $u^2-15u=60.$

41. $9A^2=1-3A.$

42. $D^2+8D+11=0.$

43. $z^2+6z=1.$

44. $12R^2=R+2.$

45. $34t=3t^2-105.$

46. $19=a+4a^2.$

47. $5M^2=2M+7.$

48. $21-2v=4v^2.$

148. Quadratics solved by a Formula. — Since any quadratic equation may be written in the general form,

$$Ax^2+Bx+C=0,$$

where x denotes the unknown number and A, B, and C are known numbers, the roots of this equation will give a *formula* by which the roots of any particular quadratic equation may be written down at once.

$Ax^2+Bx+C=0$ is solved by completing the square as follows:

Dividing by A, $x^2+\frac{B}{A}x+\frac{C}{A}=0.$

Transposing, $x^2+\frac{B}{A}x=-\frac{C}{A}.$

Completing square, $x^2+\frac{B}{A}x+\frac{B^2}{4A^2}=\frac{B^2-4AC}{4A^2}.$

Taking square root, $x+\frac{B}{2A}=\pm\frac{\sqrt{B^2-4AC}}{2A}.$

Transposing,

$$x=-\frac{B}{2A}\pm\frac{\sqrt{B^2-4AC}}{2A}$$

$$=\frac{-B\pm\sqrt{B^2-4AC}}{2A}.$$

Now, by replacing A, B, and C in the formula for the roots

$$x=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$

by the particular values which they have in any given equation, the roots of any quadratic equation may be written down at once. Thus the long process of completing the square in every equation may be avoided. Hence, *this formula should be mastered and used in all future work where the equation cannot be readily solved by factoring.*

EXAMPLE 1.—Solve $3n^2-4n=15$.

Written in the form $Ax^2+Bx+C=0$, this becomes

$$3n^2-4n-15=0.$$

Here $A=3$, $B=-4$, and $C=-15$. Substituting these values for A, B, and C in the formula,

$$n=\frac{4\pm\sqrt{16+180}}{6}$$

$$=\frac{4\pm 14}{6}$$

$$=3 \text{ or } -\tfrac{5}{3}.$$

Example 2. — Solve $4V^2+2=-3V$.

Transposing, $4V^2+3V+2=0$.

Here $A=4$, $B=3$, and $C=2$.

Substituting, $$V=\frac{-3\pm\sqrt{9-32}}{8}$$

$$=\frac{-3\pm\sqrt{-23}}{8}\text{, imaginary expressions.}$$

EXERCISES

Solve by use of the formula:

1. $y^2-3y=10$.
2. $p^2+10p+21=0$.
3. $2A^2+A=6$.
4. $3t^2+5t=2$.
5. $6K^2+K-5=0$.
6. $7u^2=50u-7$.
7. $4M^2+M+5=0$.
8. $6r_1^2+6=13r_1$.
9. $10a^2+14=39a$.
10. $8D^2=65D-8$.
11. $2n^2+3n+4=0$.
12. $5z^2=2z-7$.
13. $15q^2+3=q$.
14. $B^2+4=8B$.
15. $2v^2=7v-6$.
16. $4k^2+5=6k$.
17. $21c^2=c+4$.
18. $9y^2=6+y$.
19. $12+x=11x^2$.
20. $1-a^2-3a=0$.
21. $\frac{2}{n}=\frac{n+1}{2}$.
22. $\frac{3}{1-t}-\frac{t-1}{3}=0$.
23. $\frac{4}{v^2}=2+\frac{1}{2v}$.
24. $\frac{1}{k}+k=1\frac{1}{2}$.
25. $R+3=\frac{4}{R-1}$.
26. $s-\frac{1}{s}=3-\frac{1}{3}$.
27. $\frac{2a+3}{a-2}=\frac{a+3}{2a-1}$.
28. $\frac{E-1}{E-4}+\frac{E+9}{E+3}=0$.
29. $x+2=\frac{3x^2-4}{2x+1}$.
30. $1+V=\frac{4V}{1+V}$.
31. $b+\frac{1}{b}=1$.

32. $1-m=\frac{1+2m^2}{5+m}$. 34. $\frac{G-1}{G+1}=\frac{G+1}{G-1}-1$.

33. $\frac{6}{A+1}+\frac{2}{A}=3$. 35. $\frac{3}{x-4}=\frac{6}{x-5}-\frac{8}{x-3}$.

36. $\frac{A-2}{A+2}+\frac{A+2}{A-2}-\frac{10A-8}{4-A^2}=0$.

37. $\frac{2}{5w-10}-\frac{1-w}{w+2}=\frac{w+1}{w^2-4}$.

38. $\frac{7-3M}{4-3M}+\frac{7-M}{2M+2}=\frac{2M+3}{4M-2}$.

39. $\frac{1}{18}+\frac{y}{y+1}=\frac{y+3}{2y+8}$.

40. $\frac{1}{8}+\frac{1}{4x+4}=\frac{3}{2x^2-2}$. 43. $4W-\frac{14-W}{W+1}=14$.

41. $\frac{8+P}{8-P}+\frac{4-2P}{4+2P}=\frac{8}{3}$. 44. $\frac{(n-1)^2}{2}+\frac{n-1}{4}=9$.

42. $5s-\frac{3s-3}{s-3}=2s+\frac{3s-6}{2}$. 45. $\frac{3+T}{4+T}-\frac{T-5}{T-6}=\frac{1}{12}$.

Find to hundredths the roots of:

46. $a^2-5a=8$. 48. $12d^2=3d+19$. 50. $6Q^2+1=9Q$.

47. $4N^2+N-7=0$. 49. $9-2y=y^2$. 51. $18=3B+2B^2$.

149. Problems solved by Quadratic Equations. — It has been seen that the roots of quadratic equations may be fractions, negative numbers, surds, or even imaginary numbers. In solving a problem by means of a quadratic equation, it is necessary to consider the nature of the roots to see if both roots satisfy all requirements of the given problem. Thus, a problem may require for its solution a *positive* number. If one of the roots of the equation formed is found to be *negative*, it must be discarded, although it satisfies the equation. If the nature of a problem requires a *real* number for its solution, and the roots of the equation formed are found to be *imaginary*, they must be discarded.

EXAMPLE 1. — A train runs at a uniform rate between two points 280 miles apart. If it ran 5 miles per hour faster, it would make the distance in 1 hour less time. Find the rate of the train.

Let v equal the rate of the train.

Then $\frac{280}{v}$ = number of hours required for the run.

And $\frac{280}{v+5}$ = number of hours required at the increased speed.

Hence, $$\frac{280}{v} = \frac{280}{v+5} + 1.$$

Solving, $$v = 35 \text{ or } -40.$$

But the rate must be a positive number. Hence, the root -40 must be discarded, and the only rate possible is 35 miles per hour.

EXAMPLE 2. — Find the real number whose square increased by 32 equals 8 times the number.

Let $$n = \text{the number}.$$

Then, $$n^2 + 32 = 8n.$$

Solving, $$n = 4 \pm \sqrt{-16}.$$

Since both roots involve imaginary numbers, the problem has no solution, *i.e.* it is *impossible.*

EXERCISES

By use of one unknown number solve:

1. Find two consecutive arithmetical numbers the sum of whose squares is 313.

2. Find two consecutive arithmetical numbers whose product is 240.

3. Divide 31 into two parts the sum of whose squares is 541.

4. Find two arithmetical numbers whose sum is 50 and product 336.

5. Divide 14 into two parts such that 4 times the square of the larger shall exceed 6 times the square of the smaller by 40.

6. The denominator of a certain fraction is 5 more than the numerator. If the fraction be added to the fraction inverted, the sum will be $2\frac{25}{84}$. Find the fraction.

7. A pupil was to divide 12 by a certain number, but by mistake he subtracted the number from 12. His result was 5 too great. Find the number.

8. The side of one square exceeds that of another by 3 inches, and its area exceeds twice the area of the other by 17 square inches. Find the lengths of their sides.

9. A rectangular field is 96 ft. longer than it is wide, and it contains 298,000 sq. ft. What are its dimensions?

10. A floor can be paved with 200 square tiles of a certain size. If each tile were 1 in. shorter each way, it would require 288 tiles. Find the size of each tile.

11. In the center of a rectangular room is a rug 9 ft. by 12 ft. Around this is a border of uniform width. The area of the floor is 208 sq. ft. Find the width of the border.

12. A lawn is 40 ft. by 90 ft. Two boys agree to mow it. The first boy is to mow one half of it by cutting a strip of uniform width around it. How wide a strip must he cut?

13. A farmer has a field of wheat 80 rd. wide and 120 rd. long. How wide a strip must he cut around the field in order to have one fourth of the wheat cut?

14. In Problem 13, how wide a strip must he cut around the field in order to have one half of the wheat cut?

15. A farmer has a field 40 rd. wide and 80 rd. long to be plowed and planted in corn. He wishes to plow $4\frac{1}{2}$ acres the first day by plowing a strip of uniform width around it. How wide a strip must he plow?

16. The hypotenuse of a right triangle is 4 in. longer than one leg and 2 in. longer than the other. Find the sides of the triangle.

17. One side of a rectangle is 4 in. longer than the other, and its diagonal is 20 in. How long are the sides?

18. Two men start at the same time from the intersection of two roads, one driving south at the rate of 4 miles an hour and

the other west at the rate of 3 miles an hour. In how many hours will they be 25 miles apart?

19. Two trains are 100 miles apart on perpendicular roads, and are approaching the crossing. One train runs 10 miles an hour faster than the other. At what rates must they run if they both reach the crossing in 2 hours?

20. A train running 6 miles an hour slower than usual, due tc a storm, was 1 hour late in making a run of 252 miles. Find its speed.

21. An engineer increased the speed of his train 5 miles an hour, and made a run of 360 miles in 1 hour less than schedule time. What was the speed when running according to schedule?

22. A company of people arranged for a dinner, for which they contracted to pay $45. Five of the people were unable to attend, and as a result each of the others had to pay 30 cents more than he had expected. How many were present?

23. A farmer sold his wheat for $540. A month later the price of wheat advanced 10¢ a bushel, and he found that if he had held his wheat until that time, he could have received the same money and have had 75 bu. left. How much per bushel did he receive for the wheat?

24. A trader bought a flock of sheep for $1020. Two of them died, and he sold the rest at a profit of $3 a head. He made $225 on the transaction. How many did he buy?

25. A man bought a farm for $20,000. Later he sold all except 35 acres of it at a gain of $35 per acre over the cost, and received just what he paid for the whole farm. How many acres in the farm?

26. One pump can fill a tank in 6 minutes less time than is required for another pump to fill it. The two working together can fill it in $10\frac{2}{7}$ minutes. Find the time required for each pump alone to fill the tank.

27. One of two pumps can fill a tank in 28 minutes, and the time required for the other pump to do it is $19\frac{4}{9}$ minutes longer than is required for the two pumps together to fill it. Find the time required for the two pumps together to fill the tank.

28. With A's help B could do a piece of work in $6\frac{1}{4}$ hours less time than would be required for B to do it alone. A could do it alone in 21 hours. In how many hours could B do it alone?

29. One pipe could empty a tank in 2 hours less time than would be required for a second pipe to empty it. After the first pipe has been open 4 hours, it is closed and the second pipe is opened. The second pipe finishes emptying the tank in 5 hours more. In what time could each pipe alone empty it?

30. An aviator flew 80 miles against the wind, and back again, in 6 hours. His speed against the wind was 10 miles per hour greater than the speed of the wind. Find the speed of the wind, also what his speed would have been had there been no wind.

31. A line 16 in. long is to be divided into two parts such that the ratio of the whole line to the longer part shall be equal to the ratio of the longer part to the shorter part. Find to tenths of an inch the lengths of the parts.

NOTE.—The line in Problem 31 is said to be divided in "extreme and mean ratio." Since the time of the Greeks the problem to divide a line in extreme and mean ratio has been called the "Problem of the Golden Section."

32. Experience has shown that a book, photograph, front of a tall building, or other rectangular object, is most pleasing to the eye when its length and width are obtained by dividing their sum in extreme and mean ratio.

The page of a book is to be 5 in. wide. How long should it be made to be most pleasing to the eye?

SUGGESTION.—If n denotes the length, $\frac{n+5}{n}=\frac{n}{5}$.

33. A photograph is to be mounted on cardboard 6 in. wide. How long should the board be cut?

34. An architect is designing a building whose frontage on the street is to be 80 ft. How high should he make it for the greatest beauty?

35. An open box that shall hold 384 cu. in. is to be made from a square piece of cardboard by cutting out a 6-inch square from each corner and turning up the sides. Find the size of the piece of cardboard that must be used.

36. An open box that shall contain 324 cu. in. and that shall be twice as long as wide is to be made from a rectangular piece of tin by cutting out 2-inch squares from the corners and bending up the sides. Find the dimensions of the piece of tin required.

37. A flat disk of sheet metal 3 inches in radius is to be stamped by means of a die into a box lid 1 inch deep. The total area of the lid must equal the area of the flat disk. What will be the radius of the lid?

SUGGESTION. — If r is the radius of the lid, its total surface is composed of a circle whose area is πr^2 and a cylindrical part whose area is $2\pi r$. Hence, $\pi r^2 + 2\pi r = 9\pi$.

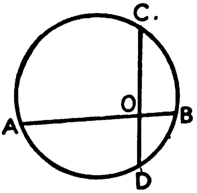
38. If the radius of the disk in Problem 37 is 2 inches and the depth of the lid $\frac{1}{2}$ inch, what will be the radius of the lid?

39. The rim of the front wheel of a carriage is 2 ft. less than the rim of the hind wheel. The front wheel makes 88 more revolutions in going a mile than the hind wheel. Find the length of the rim of each wheel.

40. The circumference of the little wheel on a sewing machine is 29 inches less than that of the big wheel which drives it. While the belt moves a distance of 1275 inches, the little wheel makes 116 more revolutions than the big one. Find the circumference of each wheel.

41. A park is 900 ft. wide and 1650 ft. long. It is desired to move out the boundary lines the same distance along one side and both ends so that the area inclosed will be just twice as great as it now is. Find how far the boundary lines must be moved.

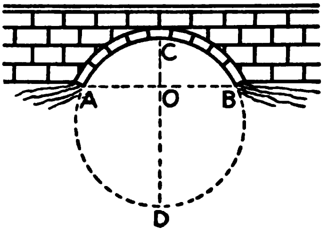
42. The longer base of a trapezoid is 14 in. longer than the other base, the altitude is equal to the shorter base, and the area is 120 sq. in. Find the lengths of the bases and the altitude.



43. If any two chords AB and CD of a circle meet at O, then $AO \times OB = CO \times OD$.

If $CO = 3$ in., $OD = 2$ in., and $AB = 8$ in., find AO and OB.

44. In Problem 43, if $CO = 9$, $OD = 7$, and $AB = 22$, find AO and OB.



45. AB, the *span* of the stone arch ACB that is to be constructed, is 40 ft. ($AO = 20$ ft. and $OB = 20$ ft.), and the diameter CD of the arch is 50 ft. Find CO, the *rise* of the arch.

46. If the span of an arch is 16 ft. and the diameter 20 ft., find the rise of the arch

47. If the span of an arch is 20 ft. and the diameter 30 ft., compute the rise.

150. Literal Equations: Formulæ. — If a literal equation or formula (see § 113) is of the second degree with respect to the unknown number, it may be solved by the methods of this chapter.

EXAMPLE. — Solve $t(t-b) = a(a+b)$ for t.

Removing parentheses, $t^2 - bt = a^2 + ab$.

Transposing, $t^2 - bt - (a^2 + ab) = 0$.

In formula, $A = 1$, $B = -b$, $C = -(a^2 + ab)$.

Substituting in formula,
$$t = \frac{b \pm \sqrt{b^2 + 4(a^2 + ab)}}{2}$$
$$= \frac{b \pm (2a + b)}{2}$$
$$= a + b \text{ or } -a.$$

EXERCISES

1. Solve $an^2 = b$ for n.

2. Solve $a^2v^2 - a^4 = 0$ for v.

3. Solve $mn = m^2 - 4$ for m.

4. Solve $2x^2 - 3r^2 = 5xr$ for x; for r.

5. Solve $W^2 + 2A^2 = 3WA$ for W; for A.

6. Solve $y^2 + 2my = n^2 + 2mn$ for y; for n.

7. Solve $w^2 + 2rw + s = 0$ for w.

8. Solve $z + \frac{1}{n} = n + \frac{1}{z}$ for n; for z.

9. Solve $P^2 + 2P = vP + 2v$ for P.

10. Solve $t^2 + at + bt + ab = 0$ for t.

11. Solve $A^2 - (m - n)A = mn$ for A.

12. Solve $\frac{1+R}{1-Rn} + \frac{1-R}{1+Rn} = 1$ for n; for R.

13. Solve $S = \frac{1}{2}n[2a + (n-1)d]$ for n.

14. Solve $S = \frac{n(n-1)}{2}$ for n.

15. If an object is let fall downward from a height s feet above the earth, the time in seconds required for it to strike the earth is computed by the formula $s = 16t^2$. Solve for t.

16. How long would it take a body to fall a distance of 1280 feet?

17. If an object is thrown downward toward the earth with an initial velocity of v feet per second, the distance s in feet that it will fall in t seconds is computed by the formula $s = vt + 16t^2$. Solve for t.

18. An object is thrown downward with a velocity of 32 feet a second from a distance of 240 feet above the earth. How long will it take for it to strike the ground?

19. The velocity of the discharge of water from a pipe is computed by the formula $4v^2 + 5v = 2 + \frac{1200\,HD}{L}$. Solve this formula for v.

SUPPLEMENTARY EXERCISES

A second method of solving a quadratic equation by completing the square besides that given in § 147 is shown below.

1. Solve $ax^2 + bx + c = 0$ for x.

Solution. — Multiply through by $4a$, or 4 times the coefficient of x^2.

Then $$4a^2x^2 + 4abx + 4ac = 0.$$

Transposing, $$4a^2x^2 + 4abx = -4ac.$$

Add b^2, the square of the coefficient of x in the original equation, to both members.

Then $$4a^2x^2 + 4abx + b^2 = b^2 - 4ac.$$

Taking square root, $$2ax + b = \pm\sqrt{b^2 - 4ac}.$$

Transposing, $$2ax = -b \pm \sqrt{b^2 - 4ac}$$

Dividing by $2a$, $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Solve by the method of Problem 1:

2. $3x^2 + 4x + 5 = 0.$
3. $7n^2 - 3n - 4 = 0.$
4. $5t^2 - 4t = 9.$
5. $12K^2 + K = 11.$
6. $9a^2 = 3a + 7.$
7. $15V^2 - 7 = 9V.$
8. $6b^2 = b + 5.$
9. $5w^2 - 17w + 6 = 0.$
10. $3M^2 = 7M + 15.$
11. $6r^2 = 13r - 6.$

CHAPTER XV

SYSTEMS INVOLVING QUADRATIC EQUATIONS

151. Systems involving Quadratic Equations. — Some problems may be expressed and solved by means of a system of two equations containing two unknown numbers in which at least one of the equations is of the second or higher degree. In this chapter we shall discuss only *systems in which one equation is linear and the other quadratic,* these being sufficient for the present needs of the student. A complete treatment of systems involving quadratic and higher equations will be found in the SECOND COURSE.

152. Elimination by Substitution. — The easiest method of eliminating one of the unknown numbers in a system consisting of one linear and one quadratic equation is by *substitution.* See § 128.

EXAMPLE. — Solve $\begin{cases} m - 2n = 5, & (1) \\ m^2 + n^2 = 10. & (2) \end{cases}$

Solving (1) for m, $m = 2n + 5.$

Replacing m by $2n + 5$ in (2), $(2n + 5)^2 + n^2 = 10.$ (3)

Solving (3), $n = -1$ or $-3.$

Replacing n by -1 in (1), $m + 2 = 5,$

$m = 3.$

Replacing n by -3 in (1), $m + 6 = 5,$

$m = -1.$

Hence there are two solutions:

$$m = 3,\ n = -1, \text{ or } m = -1,\ n = -3.$$

As shown in the above example, elimination of one of the unknown numbers in a system containing a linear and a quadratic equation always leads to a *quadratic equation* in one unknown number, which may be solved by the methods of Chapter XIV.

By substituting each of the values thus found for one of the unknown numbers in the linear equation of the given system, a corresponding value of the other unknown number is found.

Hence, a system consisting of one linear and one quadratic equation has two solutions.

EXERCISES

Solve:

1. $\begin{cases} 2a + b = 4, \\ a^2 + b^2 = 5. \end{cases}$

2. $\begin{cases} 2x + 3y = 0, \\ 3x^2 - y^2 = 23. \end{cases}$

3. $\begin{cases} m + n = 5, \\ mn = 4. \end{cases}$

4. $\begin{cases} A - B = 1, \\ AB = 2. \end{cases}$

5. $\begin{cases} u^2 + uv + v^2 = 39, \\ u - v + 3 = 0. \end{cases}$

6. $\begin{cases} V^2 = 4t, \\ V + 2t = 4. \end{cases}$

7. $\begin{cases} r_1^2 = 4r_2 + 18, \\ 3r_1 = 4r_2 + 24. \end{cases}$

8. $\begin{cases} p^2 + 3pq + q^2 = 22, \\ 2p = q. \end{cases}$

9. $\begin{cases} 2R + r = 14, \\ R^2 + 3Rr = 49. \end{cases}$

10. $\begin{cases} c + 2d = 4, \\ cd + 6 = 0. \end{cases}$

11. $\begin{cases} M + N = 2, \\ M^2 - MN + N^2 = 6. \end{cases}$

12. $\begin{cases} y - 2x = 12, \\ y^2 - x^2 - 3x + 6 = 0. \end{cases}$

13. $\begin{cases} x^2 + y^2 = 74, \\ 3x - 2y = 1. \end{cases}$

14. $\begin{cases} 2W + 2D = 5WD, \\ 2W + 2D = 5. \end{cases}$

15. $\begin{cases} 2h^2 + 3k^2 = 11 + 4hk, \\ h - 5 = 3k. \end{cases}$

16. $\begin{cases} E + F = 7, \\ E^2 - EF + F^2 = 31. \end{cases}$

17. $\begin{cases} a + b = 4, \\ b + a = ab. \end{cases}$

18. $\begin{cases} w - 3 = v, \\ w^2 + 19 + v^2 = 3wv. \end{cases}$

19. $\begin{cases} 8V + 12 = 4t, \\ 3V^2 + 2t^2 = 48 + t. \end{cases}$

20. $\begin{cases} 2x - 5 = x', \\ x + 3x' = 2xx'. \end{cases}$

21. $\begin{cases} (6 - y)(7 + z) = 80, \\ z + y = 5. \end{cases}$

22. $\begin{cases} m = 3n + 1, \\ n^2 + mn = 33. \end{cases}$

23. $\begin{cases} R^2 - RS - 6S^2 = 16, \\ R = 3S + 1. \end{cases}$

24. $\begin{cases} 3T - t - 12 = 0, \\ 9T^2 + t^2 = 72. \end{cases}$

25. $$\begin{cases} \dfrac{a}{b}+\dfrac{b}{a}=\dfrac{13}{6}, \\ a+b=5. \end{cases}$$

26. $$\begin{cases} M+N=2, \\ \dfrac{3}{N}+\dfrac{2}{M}=6. \end{cases}$$

27. $$\begin{cases} p^2+pq+q^2=97, \\ 1-\dfrac{q}{p}=\dfrac{19}{p}. \end{cases}$$

28. $$\begin{cases} G=\dfrac{3}{t}, \\ G+t=4. \end{cases}$$

29. $$\begin{cases} \dfrac{v}{u}+\dfrac{u}{v}=3\frac{1}{3}, \\ 3v-2u+12=0. \end{cases}$$

30. $$\begin{cases} 1-\dfrac{A}{B}+\dfrac{A^2}{B^2}=\dfrac{21}{B^2}, \\ A+B=9. \end{cases}$$

31. Find two numbers whose difference is 2 and the sum of whose squares is 34.

32. The hypotenuse of a right triangle is 20 feet and the sum of the legs 28 feet. Find the lengths of the legs.

33. A rectangular field is 40 rods longer than it is wide, and its area is 1152 square rods. Find the dimensions of the field.

34. A rectangular field contains 18 acres, and the length of the fence around it is 232 rods. Find the length and width of the field.

35. The base of a triangle is 5 inches longer than the altitude, and the area is 42 square inches. Find the base and altitude.

36. Of two squares the side of one is 7 feet more than the side of the other, and the area of the larger is 161 square feet more than the area of the other. Find the sides of the squares.

37. A bin is to be made to hold 144 cubic feet. It must be 3 feet deep and 3 times as long as wide. Find the dimensions of the bin.

38. Of two machines in a mill one can turn out an order of goods in 3 hours less time than the other, and if both machines are operated at once, they can turn out the goods in 2 hours. How long would it require each machine alone to turn out the goods?

39. If one machine in a mill can produce a quantity of goods in one half of the time required for a machine of smaller capacity to do it, and the two machines together can produce the goods in 4 days, how many days would it take each machine alone to produce the goods?

40. If one boiler would consume a tank of water in 2 hours less time than another, and the two together would exhaust it in 2 hours 55 minutes, how long would the tank of water supply each boiler alone?

41. In any circle whose radius is R, diameter D, and area S,

$$S = \pi R^2,$$
$$D = 2\,R.$$

By eliminating R between the two equations, get the equation between S and D.

42. In any sphere whose radius is R, diameter D, and area S,

$$S = 4\,\pi R^2,$$
$$D = 2\,R.$$

By eliminating R between these two equations, derive the equation between S and D.

43. In any right circular cylinder the radius of whose base is R, altitude H, area of cylindrical surface S, and volume V,

$$S = \pi RH,$$
$$V = \pi R^2 H.$$

Eliminate R between these two equations and get a formula between S, V, and H.

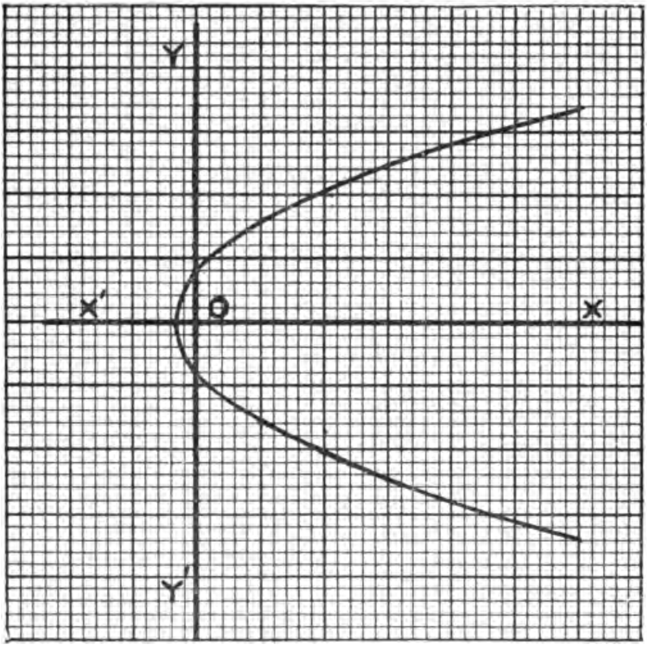
153. Graphs of Quadratic Equations. — It was shown in § 124 that the graph of a linear equation containing two unknown numbers is always a *straight line.* Similarly, it will be found that in general the graph of a quadratic equation containing two unknown numbers is some kind of *curve.*

For example, consider the graph of $y^2 = 9\,x + 16$.

By assigning values to x, and computing the corresponding approximate values of y, the following sets of values of x and y are obtained:

x	30	20	10	6	2	1	0	−1	$-1\frac{7}{9}$	Greater neg. values
y	± 16.9	± 14	± 10.3	± 8.4	± 5.8	± 5	± 4	± 2.6	0	Imaginary

By locating the points corresponding to these sets of values of x and y and joining them, as in § 124, we get the curve shown below.



Observe that the graph is perfectly symmetrical with reference to the axis XX'. By assigning greater and greater positive values to x, corresponding real values can always be found for y, which shows that the two branches of the curve continue indefinitely to the right. But if negative values are assigned to x larger than $1\frac{7}{8}$, the corresponding values of y are found to be imaginary, which shows that no part of the graph extends farther to the left than $x = -1\frac{7}{8}$.

The above curve is called a *parabola.* It is the kind of path in which astronomers have found that many comets move. Other comets and all of the planets move in closed curves called *ellipses.*

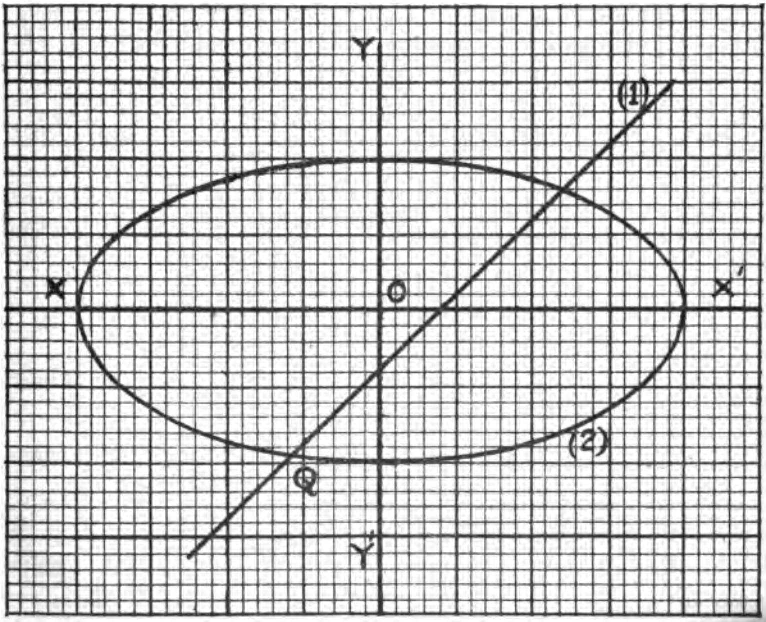
154. Systems solved Graphically. — The solution of a system consisting of one linear and one quadratic equation may be found graphically, as in § 132, by drawing the graphs of both equations upon the same axes. The sets of values of the unknown numbers corresponding to the points where the graphs meet are the solutions of the system.

Example. — Solve graphically $\begin{cases} x - y = 4, & (1) \\ x^2 + 4y^2 = 400. & (2) \end{cases}$

The graph of (1) is the straight line (1) in the figure below. By assigning values to x in (2) the following sets of values of x and y are obtained:

x	0	5	–5	10	–10	15	–15	20	–20
y	± 10	± 9.7	± 9.7	± 8.7	± 8.7	± 6.6	± 6.6	0	0

For either positive or negative values of x greater than 20, y is imaginary. And for either positive or negative values of y greater than 10, x is imaginary. Hence the graph lies entirely within the region bounded by $x = 20$ on the right, $x = -20$ on the left, $y = 10$ above, and $y = -10$ below. The graph of (2) is seen to be the ellipse (2) in the figure.



The straight line (1) meets the curve (2) at two points, P and Q. The set of values $x = 12$, $y = 8$ corresponding to P, and the set of values $x = -5.6$, $y = -9.6$ corresponding to Q, are the two solutions of the system, and will be found to satisfy both equations.

EXERCISES

Draw the graphs of:

1. $x^2 = 4y + 4$.
2. $x^2 + y^2 = 256$.
3. $4x^2 + 9y^2 = 324$.
4. $x^2 - y^2 = 16$.
5. $9x^2 - y^2 = 36$.
6. $xy + 8x - 10y = 80$.

Solve graphically:

7. $\begin{cases} x - y = 2, \\ x^2 + y^2 = 100. \end{cases}$
8. $\begin{cases} y - 2x = 2, \\ 4x^2 + y^2 = 164. \end{cases}$
9. $\begin{cases} x^2 = 2y + 1, \\ x + 2y = 4. \end{cases}$
10. $\begin{cases} 5x^2 + 2y^2 = 532, \\ 3x - y + 10 = 0. \end{cases}$
11. $\begin{cases} 2x - 5y = 3, \\ x^2 - y^2 = 171. \end{cases}$
12. $\begin{cases} 2x - 3y = 10, \\ 3x^2 - 5xy = 2y^2. \end{cases}$

13. If the two graphs of a system did not meet at all, what kind of solutions would the system have?

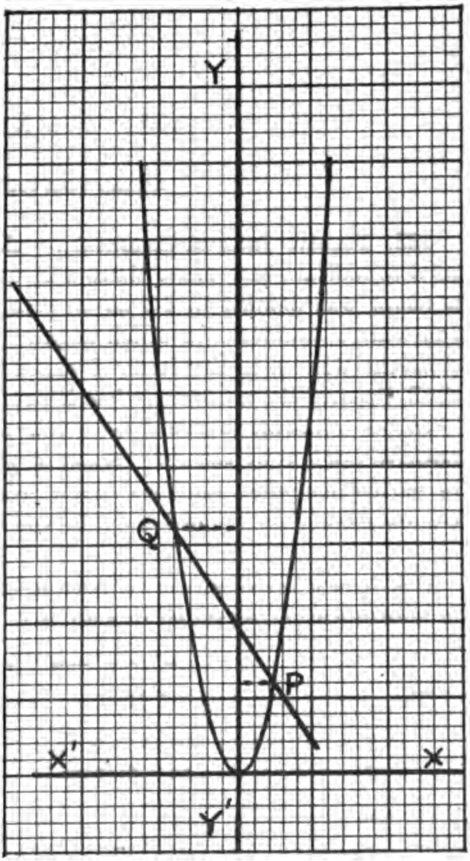
14. If the graph of the linear equation of a system just touched the graph of the quadratic, what would be the nature of the solutions?

SUPPLEMENTARY EXERCISES

The roots of a quadratic equation in one unknown number may be obtained graphically as follows:

Eliminating y from the system $\begin{cases} x^2 = y, \\ ay + bx + c = 0, \end{cases}$ gives the general quadratic equation in one unknown number $ax^2 + bx + c = 0$. Since the roots of this equation must satisfy both equations of the above system, they may be found by solving this system graphically.

1. Solve $2x^2+3x-20=0$.



Solution. — This equation corresponds to the system

$$\begin{cases} x^2 = y, & (1) \\ 2y + 3x - 20 = 0. & (2) \end{cases}$$

The graph of (1) is the parabola in the figure. The graph of (2) is the straight line meeting the parabola at P and Q. The values of x corresponding to the points P and Q are $2\frac{1}{2}$ and -4, respectively, which are the roots of the given equation.

Note. — Since the equation $x^2 = y$ will be the same for the systems formed from all quadratics, the parabola need be drawn only once and all quadratics solved by use of the one figure. To solve any quadratic in x, substitute y for x^2, and find two points of the graph of the linear equation formed. Connecting these by a ruler, observe the values of x corresponding to the points where the ruler crosses the parabola. These are the roots of the quadratic.

Draw a large perfect parabola, and by use of it and a ruler find the roots of:

2. $x^2-x-2=0$.	**6.** $x^2+x=12$.	**10.** $5x^2=1+4x$.
3. $x^2-2x=8$.	**7.** $x^2=2x+15$.	**11.** $6x^2=x+35$.
4. $x^2=3x+10$.	**8.** $2x^2-5x+2=0$.	**12.** $4x^2=21x+18$.
5. $x^2+7x+12=0$.	**9.** $2x^2+5x=3$.	**13.** $5x^2=3x+14$.

CHAPTER XVI

EXPONENTS

155. Positive Integral Exponents. — A positive whole number used to indicate how many times a given number is to be used as a factor was defined in § 5 as an **exponent.** Expressions sometimes are encountered in which negative numbers, zero, and fractions are written in the form of exponents. In this chapter we shall discover the meanings of such expressions.

In the following sections are given some laws which hold for positive integral exponents. Some of these laws have been used in earlier chapters, and are given here primarily for review. Others have been applied only in special cases.

156. Law of Exponents in Multiplication. — The law of exponents in multiplication, as stated in § 48, is expressed in symbols by

$$a^m \times a^n = a^{m+n}.$$

The general proof is as follows:

$$a^m = a \times a \times a \cdots \text{ to } m \text{ factors.}$$
$$a^n = a \times a \times a \cdots \text{ to } n \text{ factors.}$$

Hence,

$$\begin{aligned} a^m \times a^n &= (a \times a \times a \cdots \text{ to } m \text{ factors})(a \times a \times a \cdots \text{ to } n \text{ factors}) \\ &= a \times a \times a \times a \cdots \text{ to } m+n \text{ factors} \\ &= a^{m+n}, \text{ by definition of an exponent.} \end{aligned}$$

EXERCISES

Give orally the products of:

1. $a^3 \times a^4$.
2. $N^6 \times N^5$.
3. $x^{12} \times x^8$.
4. $t^{20} \times t^4$.
5. $v \times v^9$.
6. $A^{12} \times A^{12}$.

7. $r^{10} \times r^{14}$.
8. $y^{16} \times y^{4}$.
9. $P^{3} \times P^{12}$.
10. $d^{6} \times d^{17}$.
11. $a^{2} \times a^{5} \times a^{8}$.
12. $V^{4} \times V^{7} \times V^{9}$.
13. $s^{10} \times s^{4} \times s^{2}$.
14. $Q \times Q^{5} \times Q^{9}$.
15. $k^{12} \times k^{12} \times k^{12}$.
16. $x^{n} \times x^{2n} \times x^{3n}$.
✓17. $a^{n+1} \times a^{n-1} \times a^{3n}$.
18. $P^{2t-3} \times P^{3t+2}$.
19. $b^{12-a} \times b^{a+3}$.
20. $V^{n-m} \times V^{3n+2m}$.
21. $Q^{r+s} \times Q^{r-s}$.

157. Law of Exponents in Division. — The law of exponents in division was given in § 52. In symbols,

$$a^m \div a^n = a^{m-n}.$$

This follows from § 156, because the quotient a^{m-n} times the divisor a^n equals the dividend a^m.

EXERCISES

Give orally the quotients of:

1. $p^{6} \div p^{2}$.
2. $A^{12} \div A^{5}$.
3. $w^{9} \div w$.
4. $R^{16} \div R^{7}$.
5. $N^{24} \div N^{6}$.
6. $t^{20} \div t^{12}$.
7. $x^{17} \div x^{13}$.
8. $Q^{32} \div Q^{16}$.
9. $v^{22} \div v^{18}$.
10. $m^{15} \div m^{11}$.
11. $B^{28} \div B^{19}$.
12. $k^{19} \div k^{3}$.
13. $W^{5n} \div W^{n}$.
14. $R^{7a} \div R^{3a}$.
15. $x^{4n+1} \div x$.
16. $A^{2x+4} \div A^{x+1}$.
17. $y^{4t+10} \div y^{8}$.
18. $V^{3a+2b} \div V^{2a+b}$.
19. $Z^{m} \div Z^{m-2}$.
20. $G^{t+8} \div G^{t-2}$.
21. $K^{r+6} \div K^{r-2}$.

158. Power of a Power. — The law by which a power of some number is itself raised to a power is expressed by,

$$(a^n)^m = a^{mn}.$$

That is, *the mth power of the nth power of any number equals the mnth power of the number.*

For, by definition of an exponent,

$$\begin{aligned}(a^n)^m &= a^n \times a^n \times a^n \cdots \text{to } m \text{ factors}\\ &= a^{n+n+n+\cdots \text{ to } m \text{ terms}}, \text{ by § 156}\\ &= a^{mn}.\end{aligned}$$

Thus, $(a^5)^4 = a^{20}$; $(w^6)^3 = w^{18}$; $(p^4)^9 = p^{36}$.

EXERCISES

Give orally:

1. $(a^8)^4$.
2. $(x^5)^6$.
3. $(P^4)^8$.
4. $(r^9)^2$.
5. $(y^{10})^3$.
6. $(t^8)^4$.
7. $(B^{12})^7$.
8. $(k^3)^9$.
9. $(E^{10})^{10}$.
10. $(m^6)^{14}$.
11. $(A^{16})^4$.
12. $(s^{20})^5$.
13. $(a^x)^y$.
14. $(w^n)^{2m}$.
15. $(x^{2n})^{3m}$.
16. $(Q^{4x})^{3x}$.
17. $(y^{a+1})^{a-1}$.
18. $(R^{a+b})^{a-b}$.
19. $(U^{2m-n})^n$.
20. $(2^2)^4$.

159. Power of a Product. — The law by which a power of the product of two numbers is found is expressed by,

$$(ab)^n = a^n b^n.$$

That is, *the nth power of the product of two numbers equals the product of the nth powers of the numbers.*

For, by the meaning of an exponent,

$$\begin{aligned}(ab)^n &= (ab)(ab)(ab) \cdots \text{ to } n \text{ factors}\\ &= (a \times a \times a \cdots \text{ to } n \text{ factors})(b \times b \times b \cdots \text{ to } n \text{ factors})\\ &= a^n b^n.\end{aligned}$$

By similar reasoning the law can be shown to hold for any number of factors.

Thus, $(abcd)^5 = a^5b^5c^5d^5$; $(3\,xy)^4 = 3^4\,x^4y^4 = 81\,x^4y^4$.

160. Power of a Fraction. — The law by which a fraction is raised to a power is expressed by,

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

That is, *the nth power of a fraction equals the nth power of the numerator divided by the nth power of the denominator.*

For,
$$\begin{aligned}\left(\frac{a}{b}\right)^n &= \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \cdots \text{ to } n \text{ factors}\\ &= \frac{a \times a \times a \cdots \text{ to } n \text{ factors}}{b \times b \times b \cdots \text{ to } n \text{ factors}}\\ &= \frac{a^n}{b^n}.\end{aligned}$$

Thus,
$$\left(\frac{2\,ab}{3\,cd}\right)^4 = \frac{(2\,ab)^4}{(3\,cd)^4} = \frac{2^4\,a^4b^4}{3^4\,c^4d^4} = \frac{16\,a^4b^4}{81\,c^4d^4}.$$

161. Power of any Monomial. — By use of the laws of exponents in the preceding sections, any power of any monomial may be obtained.

Thus, $(2\,a^3b^5c^4)^6 = 2^6(a^3)^6(b^5)^6(c^4)^6$ by § 159

$= 64\,a^{18}b^{30}c^{24}$ by § 158.

And $\left(\dfrac{3\,m^2n^7}{5\,p^3q^5}\right)^4 = \dfrac{(3\,m^2n^7)^4}{(5\,p^3q^5)^4}$ by § 160

$= \dfrac{81\,m^8n^{28}}{625\,p^{12}q^{20}}$ by §§ 159 and 158.

EXERCISES

Give orally:

1. $(2\,a^5)^8$.
2. $(x^2y^2)^4$.
3. $(2\,N^3M^2)^5$.
4. $(a^2b^3c)^6$.
5. $(P^4t^3v^3)^8$.
6. $(w^5v^2z^7)^7$.
7. $(5\,A^2B^{10})^4$.
8. $(m^5n^4p^3)^{10}$.
9. $(3\,V^6P^2)^4$.
10. $(Q^2n^6)^{12}$.
11. $\left(\dfrac{a^4}{b^5}\right)^6$.
12. $\left(\dfrac{2\,E^6}{P^4}\right)^5$.
13. $\left(\dfrac{a^2b^2c^2}{x^3y^3z^3}\right)^8$.
14. $\left(\dfrac{a^n}{t^{2n}}\right)^m$.
15. $\left(\dfrac{x^{3a}}{y^{2b}}\right)^4$.
16. $\left(\dfrac{N^xM^4}{W^{x+y}}\right)^x$.

162. Root of a Power. — *To find any root of any power of a base divide the exponent by the index.* That is,

$$\sqrt[n]{a^m} = a^{\frac{m}{n}}.$$

For, by § 158, $\left(a^{\frac{m}{n}}\right)^n = a^{\frac{m\times n}{n}}$ or a^m.

Thus, $\sqrt[5]{x^{20}} = x^4$; $\sqrt[4]{c^{12}} = c^3$; $\sqrt[7]{m^{35}} = m^5$.

EXERCISES

Give orally:

1. $\sqrt[4]{a^{24}}$.
2. $\sqrt[5]{y^{15}}$.
3. $\sqrt[8]{N^{24}}$.
4. $\sqrt[6]{x^{18}}$.
5. $\sqrt[3]{z^9}$.
6. $\sqrt[4]{P^{12}}$.
7. $\sqrt[5]{t^{25}}$.
8. $\sqrt[9]{W^{27}}$.
9. $\sqrt[6]{n^{30}}$.
10. $\sqrt[10]{a^{40}}$.
11. $\sqrt[12]{v^{24}}$.
12. $\sqrt[8]{t^{32}}$.
13. $\sqrt[7]{R^{28}}$.
14. $\sqrt[6]{p^{36}}$.
15. $\sqrt[9]{B^{27}}$.
16. $\sqrt[7]{s^{42}}$.
17. $\sqrt[8]{W^{56}}$.
18. $\sqrt[3]{k^{36}}$.
19. $\sqrt[12]{G^{48}}$.
20. $\sqrt[4]{m^{100}}$.

163. Negative Exponents. — If the law of exponents in division be assumed to hold in case the exponent of the divisor is greater than that of the dividend, its application leads to a new kind of number symbol called a **negative exponent.**

Thus, $a^4 \div a^7 = a^{4-7} = a^{-3}$; $t^6 \div t^{11} = t^{6-11} = t^{-5}$.

The meaning of a negative exponent is shown as follows:

By § 159, $a^3 \div a^8 = a^{3-8} = a^{-5}$.

But by reduction, $a^3 \div a^8 = \dfrac{a^3}{a^8} = \dfrac{1}{a^5}$.

Hence, $a^{-5} = \dfrac{1}{a^5}$.

In general, $a^{-n} = \dfrac{1}{a^n}$.

That is, *a negative exponent indicates a power of a base that is to be used as a divisor.*

It is assumed here, and can be proved, that all of the laws that hold for positive integral exponents apply also to negative exponents.

164. Zero Exponent. — If any power of a base is divided by itself by the law in § 159, it leads to a **zero exponent.**

Thus, $a^6 \div a^6 = a^{6-6} = a^0$; $t^8 \div t^8 = t^{8-8} = t^0$.

In general, $a^n \div a^n = a^{n-n} = a^0$.

But $a^n \div a^n = 1$.

Hence, $a^0 = 1$.

That is, *any base with the exponent 0 equals 1.*

165. Fractional Exponents. — If the law in § 162 be assumed to hold in case the index of the root is not an exact divisor of the exponent, its application leads to a new kind of number symbol called a **fractional exponent.**

Thus, $\sqrt[4]{a^3} = a^{\frac{3}{4}}$; $\sqrt[3]{r^5} = r^{\frac{5}{3}}$.

In general, $a^{\frac{n}{m}} = \sqrt[m]{a^n}$.

That is, *a fractional exponent indicates a root of a power of a number, the numerator indicating the power, and the denominator indicating the root.*

It may be proved that the laws that apply to positive integral exponents apply also to fractional exponents.

EXERCISES

Simplify:

1. $n^{-2} \times n^{-3}$.
2. $x^{-5} \times x^{-8}$.
3. $w^{12} \times w^{-7}$.
4. $p^{-6} \times p^{3}$.
5. $r^{-1} \times r^{-1}$.
6. $N^{9} \div N^{-8}$.
7. $y^{-6} \div y^{-9}$.
8. $D^{-5} \div D^{3}$.
9. $V^{-12} \div V^{-8}$.
10. $k^{4} \div k^{-4}$.
11. $T \div T^{-1}$.
12. $a^{\frac{1}{2}} \times a^{\frac{1}{2}}$.
13. $x^{\frac{2}{3}} \times x^{\frac{1}{6}}$.
14. $P^{\frac{1}{2}} \times P^{\frac{3}{4}}$.
15. $t^{\frac{1}{3}} \times t^{\frac{1}{4}}$.
16. $K^{\frac{5}{9}} \times K^{\frac{3}{4}}$.
17. $n^{\frac{3}{4}} \div n^{\frac{1}{2}}$.
18. $V^{\frac{7}{8}} \div V^{\frac{1}{4}}$.
19. $R^{\frac{4}{7}} \div R^{\frac{1}{3}}$.
20. $b^{\frac{11}{12}} \div b^{\frac{2}{3}}$.
21. $L^{\frac{2}{9}} \div L^{\frac{1}{6}}$.
22. $(x^{-2})^{-3}$.
23. $(a^{4})^{-4}$.
24. $(W^{-1})^{5}$.
25. $(t^{7})^{-2}$.
26. $(m^{-5})^{-6}$.
27. $(a^{\frac{1}{2}})^{\frac{1}{2}}$.
28. $(D^{\frac{3}{4}})^{\frac{1}{3}}$.
29. $(k^{\frac{5}{9}})^{\frac{3}{4}}$.
30. $(R^{\frac{7}{12}})^{\frac{4}{5}}$.
31. $(V^{\frac{4}{9}})^{\frac{3}{8}}$.
32. $\sqrt{x^{-6}}$.
33. $\sqrt[5]{t^{-20}}$.
34. $\sqrt[6]{V^{-24}}$.
35. $\sqrt[3]{a^{\frac{6}{7}}}$.
36. $\sqrt[4]{m^{\frac{8}{9}}}$.

SUPPLEMENTARY EXERCISES

Find the value of:

1. $8^{\frac{2}{3}}$. 2. $81^{\frac{3}{4}}$. 3. $64^{\frac{5}{6}}$. 4. 2^{-1}. 5. 3^{-2}.

Simplify:

6. $n^{\frac{2}{3}} \times n^{-\frac{1}{6}}$.
7. $A^{\frac{3}{4}} \div A^{-\frac{1}{2}}$.
8. $(k^{-\frac{5}{12}})^{-\frac{4}{5}}$.
9. $\sqrt[4]{y^{-\frac{8}{9}}}$.
10. $16^{-\frac{3}{4}}$.
11. $(a^{-2} + 3a^{-1} - 4)(a^{-1} - 2)$.
12. $(x^{-1} - 1)(x^{-2} + x^{-1} + 1)$.
13. $(t^{-3} + 1) \div (t^{-1} + 1)$.
14. $(m^{-2} + 2m^{-1}n^{-1} + n^{-2}) \div (m^{-1} + n^{-1})$
15. $(N^{-3} - 1)^{2}$.

MISCELLANEOUS EXERCISES

1. Find the value of W in the formula $W=\frac{2wh}{k+1}$, when $w=1800$, $h=26$, and $k=0.75$.

2. Find the value of A in the formula.

$$A=\sqrt{s(s-a)(s-b)(s-c)},$$

when $s=140$, $a=80$, $b=90$, and $c=110$.

3. Simplify the expression $L=\sqrt{2r^2-r\sqrt{4r^2-(r\sqrt{2})^2}}$, and thus find a simple formula. Then find L when $r=12$.

4. Solve for x in the equation $\frac{d}{x+d}=\frac{r-r'}{r}$. Then find the value of x when $d=24$, $r=14$, and $r'=8$.

5. Solve $F=\frac{4\pi^2mr}{t^2}$ for each letter involved.

6. Solve $E=\frac{4n^2L^2w}{g}$ for n, L, and w.

7. In the formulæ $C=\frac{E}{R}$ and $C'=\frac{E}{R+r}$, eliminate R and express E in terms of the remaining letters.

8. Given $P=\frac{nh}{\sqrt{m^2+n^2}}$, find the value of P when $n=20$, $h=28$, and $m=16$.

9. In $a^2=b^2+c^2-2bm$, solve for m.

10. In $2x=g-\frac{n-1}{s}x$, solve for x.

11. The volume of a hollow sphere is found by the formula $V=\frac{4}{3}\pi(r^3-r'^3)$, where r is the radius of the sphere and r' the radius of the hollow. Find the weight of a hollow sphere of brass 8 inches in diameter when the diameter of the hollow is 7 inches. (A cubic inch of brass weighs 0.303 lb.)

12. The area of a ring is found by the formula $A = \pi(r^2 - r'^2)$, where r is the radius of the outer circle that bounds the ring and r' is the radius of the inner circle. When the area of a ring whose outer diameter is 30 inches is 91.1064 square inches, find the inner diameter.

Factor the following expressions:

13 $16a^2 - 12ab + 4ac$.

14. $4a^2 + 12ab + 9b^2$.

15. $9x^2 - 12xy + 4y^2$.

16. $25m^4n + 10m^2n^2 + n^3$.

17. $64r^2t^2 - 16s^2v^2$.

18. $16y^8 - 1$.

19. $m^4n - n^5$.

20. $16x^3y - 2y^4$.

21. $6m^3 + 12m^2x + 6mx^2$.

22. $ad + ay + dx + xy$.

23. $5x^3b - 5x^2b - 30xb$.

24. $r^2 - \frac{13}{4}r + \frac{3}{4}$.

25. $y^2 + 3\frac{1}{2}yz + \frac{3}{2}z^2$.

26. Reduce $\dfrac{2x^3 - 16x - 6}{3x^3 - 24x - 9}$ to its lowest terms.

27. Reduce $\dfrac{ax^m - bx^{m+1}}{a^2bx - b^3x^3}$ to its lowest terms.

28. Reduce $\dfrac{4x^3 - 2x}{2x^2 - x + 1}$ to a mixed expression.

29. Reduce $\dfrac{m^2(m^4 - n^4) + 3(m^3 - mn^2)}{m^2(m^2 - n^2)}$.

30. Change $1 + \dfrac{b^2 + c^2 - a^2}{2bc}$ to the form of a fraction.

31. Change $x^2 - 3x - \dfrac{3x(3 - x)}{x - 2}$ to a fractional form.

32. Change ab, $\dfrac{a - b}{a + b}$, and $\dfrac{a + b}{a - b}$ to fractions having a common denominator.

33. Add $2a$, $3a + \dfrac{2x}{5}$, and $a - \dfrac{8x}{9}$.

34. Add $\dfrac{a + b}{(b - c)(c - a)}$, $\dfrac{b + c}{(c - a)(a - b)}$, and $\dfrac{c + a}{(a - b)(b - c)}$.

35. Simplify $\frac{5}{2(x+1)}-\frac{1}{10(x-1)}-\frac{24}{5(2x+3)}$.

36. Multiply $\frac{ny^n}{4ax}$ by $\frac{ax}{my^{n-1}}$.

37. Multiply x^2-x+1 by $\frac{1}{x^2}+\frac{1}{x}+1$.

38. Find the product of $\frac{1-x^2}{1+y}$, $\frac{1-y^2}{x+x^2}$, and $1+\frac{x}{1-x}$.

39. Divide $x^4-\frac{1}{x^4}$ by $x^2+\frac{1}{x^2}$.

40. Simplify $\left(\frac{c-b}{c+b}-\frac{c^3-b^3}{c^3+b^3}\right)\div\left(\frac{c+b}{c-b}+\frac{c^2+b^2}{c^2-b^2}\right)$.

41. Simplify $\cfrac{1}{x+\cfrac{1}{1+\cfrac{x+1}{3-x}}}$.

42. Solve $17x-\frac{5x-4}{3}-\frac{8x+4}{5}=20x-\frac{3x+8}{2}-5$.

43. Solve $\frac{a-x}{b}-\frac{4a-x}{c}=a-b$ for x.

44. Solve $\frac{1}{ab-ax}+\frac{1}{bc-bx}=\frac{1}{ac-ax}$ for x.

Solve the following systems:

45. $\begin{cases}3x+7y=35,\\ 2x+4y=20.\end{cases}$

46. $\begin{cases}7x+3y=62,\\ 5x-2y=36.\end{cases}$

47. $\begin{cases}12x+8y=116,\\ 2x-y=3.\end{cases}$

48. $\begin{cases}6a+5b=112,\\ 8a-2b=80.\end{cases}$

49. $\begin{cases}6t+11u=115,\\ 8t-22u=-30.\end{cases}$

50. $\begin{cases}2z+3t=47,\\ 10z-12t=-63.\end{cases}$

51. Solve $\begin{cases}\frac{x}{a}-\frac{y}{b}=m,\\ \frac{x}{c}+\frac{y}{d}=n,\end{cases}$ for x and y.

Solve:

52. $\begin{cases} 8x-9y-7z=-36, \\ 12x-y-3z=36, \\ 6x-2y-z=10. \end{cases}$ 53. $\begin{cases} 7x+4y-z=78, \\ 4x-5y-3z=-21, \\ x-3y-4z=-37. \end{cases}$

54. When the cost of an article to a merchant is lowered 20 %, the merchant who keeps the same selling price makes 30 % more. Find his former per cent of gain.

55. By getting a discount of 5 % off for cash in buying, a merchant makes a net profit of 8 % more. Find his rate of gain when no discount is received.

56. A merchant has two kinds of grain, one at 60 cents per bushel, and the other at 90 cents per bushel, of which he wishes to make a mixture of 40 bushels that may be worth 80 cents per bushel. How many bushels of each must he use?

57. A merchant has three kinds of sugar. He can sell 3 lb. of the first quality, 4 lb. of the second quality, and 2 lb. of the third quality, for 60 cents; or, he can sell 4 lb. of the first quality, 1 lb. of the second quality, and 5 lb. of the third quality, for 59 cents; or, he can sell 1 lb. of the first quality, 10 lb. of the second quality, and 3 lb. of the third quality, for 90 cents. Find the price of each quality.

58. How much cream testing 36 % butter fat must be mixed with 20 gallons of milk testing 3 % butter fat to have milk testing 4.8 % butter fat?

59. Find the square root of $4a^4-16a^3+24a^2-16a+4$.

60 Find the square root of $m^2+2m-1-\frac{2}{m}+\frac{1}{m^2}$.

61. Find the square root of 33.1776.

62. Find to three decimal places the square root of 2.

Simplify the following:

63. $\sqrt{96a^2x^3}$. 64. $\sqrt{(x^2-y^2)(x+y)}$.

65. $\sqrt{\frac{ab^2}{4(a+x)}}$.

66. $\frac{x+1}{x-1}\sqrt{\frac{x-1}{x+1}}$.

67. Multiply $4a\sqrt{\frac{3a^2}{8b}}$ by $\frac{a}{2}\sqrt{\frac{3}{2b}}$.

68. Multiply $a+2\sqrt{b}$ by $a-2\sqrt{b}$.

69. Divide $\sqrt{20}+\sqrt{12}$ by $\sqrt{5}+\sqrt{3}$.

70. Solve $4x^2+16x=33$.

71. Solve $4x-\frac{14-x}{x+1}=14$.

72. Find to three decimal places the roots of

$$\frac{x}{2}+\frac{2}{x}=\frac{1}{3}+\frac{3}{x}.$$

73. A lady finds that there are just 85 square yards of floor surface to be covered in two of her square bedrooms, one of which is 3 feet longer than the other. How large is each?

74. Solve $\begin{cases} x^2-y^2=23-3xy, \\ x+2y=7. \end{cases}$

75. Solve $\begin{cases} 2x^2-5y^2=20-xy, \\ 2x-3y=1. \end{cases}$

76. A farmer has two small fields, each an exact square. It takes 200 rods of fence to inclose both. Together they contain $8\frac{1}{8}$ acres. Find the dimensions of each.

Simplify and express with positive exponents:

77. $a^{\frac{1}{2}}b^{\frac{3}{2}}c \times a^{\frac{3}{2}}b^{-\frac{1}{2}}c^{\frac{1}{2}}$.

78. $(a^9b^2c)^3 \times (a^{\frac{3}{2}}bc^{\frac{1}{2}})^4$.

79. $(x^{2n}y^m)^2 \div (x^ny^{-m})^2$.

80. $\frac{a^{-2}b^{-3}}{a^{-1}b^2} \times \frac{ab^2}{ab^{-2}}$.

81. $\frac{3^n \times 3^{n+1}}{3^{n-1} \times 3^{-2}}$.

INDEX

(Numbers refer to Sections.)

Absolute value, 26.
Addition, 14, 30, 39, 40, 103, 140.
  elimination by, 126.
Algebra, 1.
Antecedent, 114.
Arrangement of terms, 51.
Axioms, 18.

Base, 5.
Binomial, 10.
Braces, 7.
Brackets, 7.

Cancellation, 105.
Checking work, 41.
Clearing of fractions, 111.
Coefficient, 12.
  numerical, 12.
  literal, 12.
Comparison, elimination by, 127.
Complete quadratic equation, 144.
Completing the square, 147.
Complex fractions, 109.
Conditional equations, 15.
Consequent, 114.
Constant, 120.
Cubic equation, 57.

Degree of an equation, 57.
  of a term, 57.
Denominator, 97.
  lowest common, 102.
Distribution, law of, 50.
Division, 35, 52, 53, 54, 55, 108, 142.
Divisor, trial, 135.
  true, 135.

Elimination, 68.
  by addition or subtraction, 126.
  by comparison, 127.
  by substitution, 128.
Ellipse, 153.
Equations, 15.
  complete quadratic, 144.
  conditional, 15.
  cubic, 57.
  degree of, 57.
  equivalent, 131.
  fractional, 110.
  graphs of, 124, 153.
  identical, 15.
  inconsistent, 131.
  linear, 57.
  literal, 113.
  members of, 15.
  pure quadratic, 144.
  quadratic, 57.
  roots of, 17.
  simultaneous, 66.
  systems of, 67.
  to solve, 17.
Equivalent equations, 131.
Evaluation of polynomials, 11.
Exponents, 5.
  fractional, 165.
  laws of, 48, 52, 156, 157, 158, 159, 160, 162.
  negative, 163.
  positive integral, 155.
  zero, 164.
Expressions, 8.
  fractional, 100.
  integral, 100.
  literal, 8.
  mixed, 100.
  names of, 10.
  number, 8.
Extremes, 115.

Factoring, 83.
  general suggestions on, 93.
Factors. 4

Factors, grouping of, 47.
  highest common, 94.
  order of, 46.
  prime, 83.
Formula, 2.
  for solving quadratic equations, 148.
Fourth proportional, 116.
Fractional equations, 110.
  exponents, 165.
  expressions, 100.
Fractions, 97.
  clearing equations of, 111.
  complex, 109.
  denominator of, 97.
  lowest common denominator of, 102.
  numerator of, 97.
  signs of, 98.
  terms of, 97.

General numbers, 3.
Graphs, 29.
  of linear equations, 124.
  of quadratic equations, 153.
Grouping, laws of, 30, 47.
  signs of, 7.

Highest common factor, 94.

Identical equations, 15.
Identity, 15.
Imaginary numbers, 137.
Inconsistent equations, 131.
Index of root, 72.
Integral expressions, 100.

Law of distribution, 50.
  of grouping in addition, 30.
  of grouping in multiplication, 47.
  of order in addition, 30.
  of order in multiplication, 46.
Laws of exponents, 48, 52, 156, 157, 158, 159, 160, 162.
Lever, 63.
Like terms, 13.
Linear equations, 57.
  graphs of, 124.
Literal equations, 113.
  expressions, 8.
  numbers, 3.
Lowest common denominator, 102.
Lowest common multiple, 95.

Mean proportion, 116.
  proportional, 116.
Means, 115.
Members of an equation, 15.
Mixed expressions, 100.
Monomial, 10.
Multiple, 95.
  lowest common, 95.
Multiplication, 32, 49, 50, 51, 105, 106 107, 141.
  signs of, 4.

Negative exponents, 163.
  numbers, 23.
Number expression, 8.
Numbers, general, 3.
  imaginary, 137.
  literal, 3.
  negative, 23.
  opposite, 26.
  particular, 3.
  positive, 24.
  real, 137.
Numerator, 97.
Numerical coefficient, 12.

Opposite numbers, 26.
Order, laws of, 30, 46.

Parabola, 153.
Parentheses, 6.
Particular numbers, 3.
Polynomials, 10.
  evaluation of, 11.
Positive numbers, 24.
Powers, 5.
  of a fraction, 160.
  of a monomial, 161.
  of a power, 158.
  of a product, 159.
  of positive and negative numbers 34.
  roots of, 162.
Price curves, 123.
Prime factors, 83.
Proportion, 115.
  important principles in, 119.
  mean, 116.
  terms of, 115.
Proportional, fourth, 116.
  mean, 116.

Proportional, third, 116.
Pure quadratic equation, 144.

Quadratic equation, 57.
  complete, 144.
  graph of, 153.
  pure, 144.
Quadratic surd, 138.
Quality, signs of, 25.

Radical sign, 72.
Rate of motion, 61.
Ratio, 114.
Real numbers, 137.
Root, index of, 72.
  of a power, 162.
  of an equation, 17.
  square, cube, etc., 72.

Signs of a fraction, 98.
  of quality, 25.
Signs of grouping, 7.
  insertion of, 45.
  removal of, 44.
Similar terms, 13.
Simple machines, 118.
Simultaneous equations, 66.
Solution of a system, 67.
Special products and quotients, 70.
Specific gravity, 62.
Speed, 61.
Square of a binomial, 78.
  of a polynomial, 80.
  of arithmetical numbers, 79.
  perfect, 86.
Substitution, 20.
  elimination by, 128, 152.
Subtraction, 14, 31, 42, 43, 103, 140.
  elimination by, 126.
Surds, 138.
  quadratic, 138.
  simplest form of, 139.
Systems of equations, 67.

Terms, 9.
  arrangement of, 51.
  degree of, 57.
  of a fraction, 97.
  of a proportion, 115.
  similar, 13.
Third proportional, 116.
Transposition, 36.
Triangles, similar, 117.
Trinomial, 10.
True and trial divisors, 135.

Variable, 120.
Velocity, 61.
Vinculum, 7.

Zero exponent, 164.

ELEMENTARY ALGEBRA

SECOND COURSE

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PREFACE

THIS volume has been prepared as a companion to the FIRST COURSE in elementary algebra, written by the same authors. As its title suggests, the SECOND COURSE is designed for students who have had the equivalent of the work given in the FIRST COURSE. The FIRST COURSE gives one year's work, including the topics now generally taught in the first year of the average high school. The SECOND COURSE reviews the topics treated in the FIRST COURSE, amplifies the treatment in the latter by the addition of new matter where necessary, and extends the course to include all topics required for admission to college.

In the preparation of this book, the authors have been guided by the same aims and educational principles as in the preparation of the FIRST COURSE. The distinguishing features of the latter will be found also in the present volume.

The book contains a maximum of practice, abundance of drill being provided throughout. New principles and processes are adequately and psychologically developed, *only one new difficulty being presented at a time*, and these are finally mastered through plentiful application in drill exercises and problems.

A fundamental aim here, as in the other books by the same authors, has been to lead the student to see the subject, not merely as a system of exercises to be pursued for the purpose of mental discipline, although the authors are thorough believers in this feature of study, but rather as a scientific instrument for solving certain types of problems actually encountered in the world's work. To this end adequate use has been made of real applied problems of the various types encountered in practical life. These real applied problems lend intense interest, and motivate the work legitimately. Furthermore, it is through the application of the student's knowledge of algebra in the solution of these real applied problems that the knowledge is led to function. The old-time useless puzzles have been eliminated from the book.

Algebra has been correlated with arithmetic and geometry. The principles of algebra have been applied to shorten arithmetical computations. See, for example, § 3, § 5, § 11, exercises of § 15, etc. Numerical problems have been given in many lists of exercises. See, for example, exercises pp. 16, 34, 35, 39, 40, 41, 43, etc. Considerable use of percentage has been made in the solution of certain types of problems in business, etc. The facts of mensuration have been much used, and some knowledge of simple facts in geometry with which the student is familiar has been drawn upon in problems.

Graphical methods have been used in a natural way throughout the book, both in the interpretation and illustration of algebraic principles and in the solution of certain types of problems. See, for example, §§ 3, 11, 32, 51, 53, 68, 84, 85, 95, 98, 100, 109, etc.

Supplementary exercises have been given at the end of each chapter. They adapt the text to the use of schools with different kinds of courses. These exercises, as a rule, are more difficult than those in the body of the chapters. They may be used for reviews or in classes wanting a more difficult course; or they may be omitted entirely without interfering with the subsequent work.

The authors wish to acknowledge their indebtedness to all those whose timely suggestions and criticisms have helped in the preparation of this textbook.

JOHN C. STONE.
JAMES F. MILLIS.

SEPTEMBER, 1912.

CONTENTS

CHAPTER PAGE

I. THE FUNDAMENTAL OPERATIONS 1
II. FACTORS AND MULTIPLES 22
III. FRACTIONS 37
IV. LINEAR EQUATIONS IN ONE UNKNOWN NUMBER . . 49
V. SYSTEMS OF LINEAR EQUATIONS 64
VI. ROOTS, SURDS, AND IMAGINARY NUMBERS . . . 85
VII. QUADRATIC EQUATIONS 112
VIII. HIGHER EQUATIONS 131
IX. EQUATIONS INVOLVING SURDS 137
X. SYSTEMS INVOLVING QUADRATIC AND HIGHER EQUATIONS 143
XI. PROPORTION. VARIABLES 167
XII. PROGRESSIONS 190
XIII. EXPONENTS 211
XIV. THE BINOMIAL THEOREM 222
XV. LOGARITHMS 230
MISCELLANEOUS EXERCISES 245

ELEMENTARY ALGEBRA

SECOND COURSE

CHAPTER I

THE FUNDAMENTAL OPERATIONS

1. The principles and processes of algebra established in the FIRST COURSE will be employed throughout the more advanced work. Some of the most important rules that were carefully established in the FIRST COURSE will be restated in the following sections. Also, in connection with these rules, there will be introduced additional matter that was omitted from the FIRST COURSE, either because it was not needed, or because it was too difficult for a first reading.

2. Number Expressions. — In algebra and in other branches of mathematics, letters are used to represent numbers with *unknown* or *unassigned* values.

Thus, in πr^2, which expresses the area of a circle, r represents the length of the radius of any circle, which may have any numerical value, depending upon the size of the circle. Similarly, in $a - b$ the letters a and b represent numbers to which may be assigned any numerical values whatsoever.

A number represented by a letter is called a **literal number**, and any number expression in which one or more number symbols are letters is called a **literal expression**.

In any number expression the parts connected by the signs + or − are called **terms**. An expression not connected to any other expression by either of the signs + or − consists of *one term*, and is called a **monomial**. An expression consisting of *two terms* is a **binomial**. An expression consisting of *three terms* is a **trinomial**.

In general, an expression consisting of *two* or *more terms* is called a **polynomial.**

If numerical values are assigned to all of the literal numbers in any literal expression, a particular numerical value of the whole expression may be found by performing the operations indicated in it.

In finding the value of an expression containing more than one term, the value of each term must be found before the additions or subtractions indicated between the terms.

EXERCISES

1. How is the product of a numerical and a literal factor, or of two or more literal factors, indicated?

2. Express the product of 5 and n; of x and y; of 2, a, and b.

3. How is the product of two polynomials indicated?

4. Indicate the product of $t-4$ and $2t+3$.

5. What is the meaning of a^2? Of n^3? Of $(x-y)^4$?

6. In m^5, what name is given to 5? To m? To the whole expression?

7. In $8x$, what is the 8 called?

8. Name the numerical coefficients in $2A$, $12x^2$, and $6mn$.

9. Do $3a$ and a^3 mean the same? What are their values when a is 4? When $a=10$?

10. Tell how many terms in each of the following, and apply to each the appropriate name, "monomial," etc.: $4\pi R^2$; a^2-b^2; $2x-y+7z$; B^2-4AC; $a^2+2ab+b^2-c^2$; $.05V^2$; $(m+n)(m-n)$; W^2g-2Wg^2; $x^3-3x^2y+3xy^2-y^3$; $a^4+a^3+a^2+a+1$.

11. Find the value of N^4 when $N=3$; when $N=4$.

12. Find the value of $5x^2y$ when $x=2$ and $y=3$.

13. Find the value of a^2-b^2 when $a=5$ and $b=3$.

14. Find the value of $A^2-2AB+B^2$ when $A=8$ and $B=1$.

15. Find the value of $2w^3+v^3$ when $w=3$ and $v=5$.

16. Find the value of $(2x+y)(4x-3y)$ when $x=9$ and $y=4$.

17. Find the value of $(R-at)^2$ when $R=16$, $a=32$, and $t=\frac{1}{4}$.

18. The volume of a sphere whose radius is R is $\frac{4}{3}\pi R^3$. Find the volume when $R=3$ in. (Use $\pi=3.1416$.)

19. The area of a trapezoid whose altitude is h and bases b and and b', respectively, is $\frac{1}{2}h(b+b')$. Find the area when $h=12$ in., $b=7$ in., and $b'=15$ in.

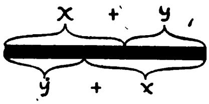
20. The cubic contents or volume of any frustum of a circular cone, such as a coffeepot, or wash tub, whose depth is H, radius at the bottom R, and radius at the top r, is $\frac{1}{3}\pi H(R^2+Rr+r^2)$. Find the cubic contents of a tub whose depth is 12 in., radius at the bottom 12 in., and radius at the top 16 in.

3. Fundamental Laws of Addition. — In the addition of two or more numbers, it is evident that the value of the sum is the same in whatever order the numbers are added. This truth is called the **Law of Order in Addition.**

In the case of two numbers the law may be expressed by

$$x+y=y+x.$$

It may be shown graphically as in the margin.

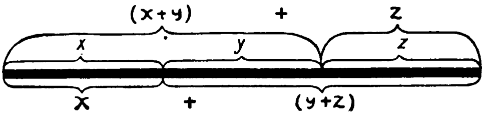


It is evident also that if three or more numbers are to be added, they may be combined or grouped in any way. In $a+b+c+d$, instead of first adding b to a, then c to the sum, etc., we may first add b to a, then d to c, then add the sums. Or we may first add c to b, then d to the sum, then this sum to a, etc. This truth is called the **Law of Grouping in Addition.**

In the case of three numbers the law may be expressed by

$$(x+y)+z=x+(y+z).$$

It may be illustrated graphically as follows:



These two laws may be used to shorten arithmetical addition.

Thus, $6+5+4+3+2$ may be written $6+4+5+3+2$, by the Law of Order, and then $(6+4)+(5+3+2)$, by the Law of Grouping. Adding 6 and 4 gives 10, adding 5, 3, and 2 gives 10, and adding these sums gives 20. In practice, the numbers are written in a column, as in the margin. The change of order is made mentally, and the numbers that make 10 are grouped.

$$\begin{array}{r}6\\5\\4\\3\\\underline{2}\\20\end{array}$$

EXERCISES

Add by grouping numbers that make 10:

1. 7, 8, 3, 2.
2. 1, 6, 9, 4.
3. 3, 5, 4, 6, 1.
4. 5, 9, 2, 1, 3.
5. 8, 6, 2, 7, 4, 3.
6. 4, 7, 7, 6, 9, 3.

Add by grouping numbers in each column:

7.	8.	9.	10.
492	347	827	3526
865	951	584	5982
628	763	276	7192
315	426	405	8304

11. Show graphically that $a+b+c+d=b+a+d+c$.

12. Show graphically that $a+b+c+d=a+(b+c+d)$.

4. Addition of Positive and Negative Numbers. — For the addition of positive and negative numbers the following rules, established in the First Course, apply:

(1) *To add two numbers with like signs, find the sum of their absolute values, and prefix the sign common to both.*

(2) *To add two numbers with unlike signs, find the difference of their absolute values, and prefix the sign of the one with the greater absolute value.*

Three or more positive or negative numbers may be added by adding two at a time, using rules (1) and (2). Or they may be added by the following rule, which follows from the Law of Order and the Law of Grouping:

(3) *To add three or more positive or negative numbers, find the sum of the positive numbers and the sum of the negative numbers separately, then find the sum of the sums.*

EXERCISES

Add:

$$\begin{array}{rrrrrrr}
\textbf{1.} & -2 & +7 & +3 & -4 & -12 & +20 \\
 & \underline{-6} & \underline{+8} & \underline{-9} & \underline{+6} & \underline{+10} & \underline{-38}
\end{array}$$

$$\begin{array}{rrrrrrrrrr}
\textbf{2.} & -5 & \textbf{3.} & +3 & \textbf{4.} & -8 & \textbf{5.} & +1 & \textbf{6.} & -16 \\
 & +7 & & -9 & & -9 & & -6 & & +20 \\
 & +2 & & -2 & & +6 & & -9 & & -4 \\
 & -8 & & +6 & & +8 & & +7 & & -8 \\
 & \underline{-4} & & \underline{7} & & \underline{-2} & & \underline{+5} & & \underline{+14}
\end{array}$$

7. $-5, +4, +6, -8, -7, +2, -1, +12$.

8. Let the student make up other drills similar to these, until he is rapid and perfectly accurate in the use of the above rules.

5. Addition of Similar Terms. — Terms which have like literal parts, as $5n$ and $-7n$, or $4V^2$ and $6V^2$, are called **like** or **similar terms**. They differ only in the numerical coefficients. Terms such as am, bm, and cm are sometimes spoken of as "similar terms in m."

Similar terms are added by the following rule:

To add two or more similar terms, add their numerical coefficients, and attach to the result the common literal part.

For convenience, terms to be added may be written in a column.

Terms that are not similar (called **dissimilar** *terms) cannot be added, but their sum may be indicated by writing them in a line, connected by their given signs.*

EXERCISES

Add:

$$\begin{array}{rrrrrrrrrr}
\textbf{1.} & +7a & \textbf{2.} & -9n & \textbf{3.} & -10W & \textbf{4.} & +8t^2 & \textbf{5.} & -3a^2b^3 \\
 & \underline{-4a} & & \underline{+2n} & & \underline{-5W} & & \underline{+4t^2} & & \underline{+5a^2b^3}
\end{array}$$

$$\begin{array}{lrlrlrlrlr} \textbf{6.} & -6\,N & 7. & +8\,x^3 & \textbf{8.} & -2\,ab & \textbf{9.} & Vt & \textbf{10.} & -9\,y^4 \\ & +4\,N & & +4\,x^3 & & +\ \ ab & & -\ \ Vt & & y^4 \\ & -7\,N & & -9\,x^3 & & +4\,ab & & -5\,Vt & & 6\,y^4 \\ & \underline{+3\,N} & & \underline{+6\,x^3} & & \underline{-7\,ab} & & \underline{9\,Vt} & & \underline{-5\,y^4} \end{array}$$

11. $-3\,k,\ -9\,k,\ +7\,k,\ -k,\ -12\,k,\ 16\,k,\ -2\,k.$

12. Let the student make up other problems similar to these, until he is proficient in the work.

Simplify by uniting similar terms:

13. $3\,a^2 - 2\,ab - b^2 + 4\,ab - b^2.$

14. $7\,x - 8 + 3\,x + 5 - 4\,x.$

15. $A^3 - B^3 + 3\,A^2B - 4\,A^3 - 4\,B^3.$

16. $y^3 - 6\,y - 5\,y^2 - 8 + 2\,y - 4\,y^3 - 2.$

Indicate the addition of:

17. $7\,a,\ -5\,b$, and $-c$.

18. $3\,w^2,\ 2\,wv$, and $-4\,v^2$.

By using the process of adding similar terms by adding their coefficients, give at sight the results of:

19. $3 \times 16 + 7 \times 16$. (Same as 10×16.)

20. $8 \times 52 + 2 \times 52.$

21. $14 \times 36 + 26 \times 36.$

22. $92 \times 48 + 8 \times 48.$

23. $36 \times 72 - 16 \times 72.$

24. $124 \times 65 - 24 \times 65.$

25. $14 \times 62 - 8 \times 62 + 24 \times 62.$

26. $81 \times 3.1416 + 19 \times 3.1416.$

27. $225 \times 3.1416 - 205 \times 3.1416.$

6. Addition of Polynomials. — Polynomials are added by the following rule:

To add polynomials, arrange the similar terms in columns, and add each column. The sums of the columns, connected by their signs, form the sum of the polynomials.

It follows from the *Law of Order* that polynomials may be rearranged, if necessary, so that the similar terms are in columns. It follows from the *Law of Grouping* that the sums of the columns, connected by their signs, form the sum of the polynomials.

EXERCISES

Add:

1. $$\begin{array}{rrr} 9a & -5b & +c \\ -2a & +3b & -4c \\ \hline -5a & -6b & -c \end{array}$$

2. $$\begin{array}{lll} a^2 & -2ab & +b^2 \\ a^2 & +2ab & +b^2 \\ \hline a^2 & & -b^2 \end{array}$$

3. $$\begin{array}{lll} x^2 & -2x & +1 \\ x^2 & +5x & -6 \\ \hline x^2 & -8x & +9 \end{array}$$

4. $$\begin{array}{llll} a^3 & & & -b^3 \\ & 3a^2b & -3ab^2 & \\ \hline a^3 & +a^2b & +ab^2 & +b^3 \end{array}$$

5. a^2-3a, $5a-6$, $2a^2+9$, and $8a-6a^2$.
6. $12-t$, $7t+3t^2$, $-4t$, and $-2t^2+16$.
7. $-2x^2+y^2$, $3xy-4y^2$, and $5x^2-xy+6y^2$.
8. a^3-1, a^2-ab+b^2, and $3-4ab+2a^2$.
9. $20-v^2$, $3v+5$, $3v^2$, and $9v-4$.
10. $\frac{1}{2}a-\frac{1}{3}b$, $\frac{1}{2}a+b$, and $a-\frac{2}{3}b$.
11. $1.5x^2-0.6x+.1$, $2.5x^2+3.4x-1$, and $5-0.8x$.
12. Simplify $a^2-3a+(5a-4a^2)$.
13. If $A=x^2-y^2$, $B=x^2+2xy+y^2$, and $C=5y^2-4xy$, find $A+B+C$.

7. Subtraction of Positive and Negative Numbers. — For the subtraction of positive and negative numbers use the following rule, established in the First Course:

To subtract, first change mentally the sign of the subtrahend, then proceed as in addition.

EXERCISES

Subtract:

1. $$\begin{array}{rrrrrr} +6 & \quad -3 & \quad +2 & \quad -4 & \quad -5 & \quad +1 \\ \underline{+8} & \quad \underline{-7} & \quad \underline{-5} & \quad \underline{+2} & \quad \underline{-9} & \quad \underline{-6} \end{array}$$

2. $$\begin{array}{rrrrrr} -12 & \quad +2 & \quad +3 & \quad -7 & \quad -4 & \quad +5 \\ \underline{+15} & \quad \underline{-10} & \quad \underline{+9} & \quad \underline{+8} & \quad \underline{-1} & \quad \underline{+9} \end{array}$$

3. $$\begin{array}{rrrrrr} -24 & +20 & -12 & +\ 4 & -\ 8 & +34 \\ \underline{-30} & \underline{-16} & \underline{-18} & \underline{-36} & \underline{-22} & \underline{+27} \end{array}$$

4. Subtract -8 from -6; $+3$ from -6; $+15$ from $+9$.

5. From $+12$ take -6, $+5$, $+8$, -7, -2, -12.

6. The temperature is $+4°$ at noon, and falls $10°$ by midnight. What is the temperature at midnight?

7. How many years is it from the year -250 to the year -140?

8. Cicero was born in the year -106 and assassinated in the year -43. At what age did he die?

9. Julius Cæsar was born in the year -100 and assassinated in the year -44. At what age did he die?

10. What is the meaning of $-6-(-5)$? The value?

11. Find the value of $8-(-4)$; $-2-(-8)$; $-6-(+4)$.

8. Subtraction of Similar Terms. — Since similar terms are subtracted by subtracting coefficients,

To subtract a term from a similar term, change mentally the sign of the subtrahend, then proceed as in addition of similar terms.

9. Subtraction of Polynomials. — One polynomial is subtracted from another when all of its terms are subtracted. Hence,

To subtract one polynomial from another, arrange similar terms in columns, and add to the minuend the subtrahend with the sign before each of its terms changed.

EXERCISES

Subtract:

1. $$\begin{array}{rrrrr} +8a & -2V & -16x & +5t^2 & -18P^3 \\ \underline{-4a} & \underline{+7V} & \underline{-12x} & \underline{+9t^2} & \underline{+11P^3} \end{array}$$

2. $$\begin{array}{rrrrr} -42ab & y^2 & -4R & 3Ww & -\pi R^2 \\ \underline{-30ab} & \underline{-y^2} & \underline{R} & \underline{-\ \ Ww} & \underline{4\pi R^2} \end{array}$$

3. From $-8n^2$ take $-9n^2$; $+5n^2$; n^2; $-n^2$; $-8n^2$.

4. $7x-(-3x)=?$ $-10v-(+4v)=?$ $-24S-(-16S)=?$

Subtract:

5. $\begin{array}{r} x^2-4x+2 \\ \underline{3x^2+7x-2} \end{array}$ 6. $\begin{array}{l} 5a \qquad -c \\ \underline{7a-4b \qquad\;\;} \end{array}$ 7. $\begin{array}{r} 4a^2 \qquad -9 \\ \underline{-4a-10} \end{array}$

8. $2m-5n$ from $-4m-n$.

9. $-6x^3+x-5$ from $2x^3-4x^2+9x$.

10. r^4-r^2+1 from r^4+2r^2+1.

11. A^3-B^3 from $A^3-3A^2B+3AB^2-B^3$.

12. $6-4x$ from $-3x^2+5-2x^3$.

13. If $A=a^2-1$, $B=a^2-4a+4$, $C=5-3a^2$, find $A-B+C$. Find $A+B-C$.

14. $3y-z-(y-6z)$ means what? Find the value.

15. $T-4-(6-5T-2T^2)=?$

16. $a^2-b^2-(a^2-2ab+b^2)=?$

10. Removal and Insertion of Signs of Grouping. — When a sign of grouping is preceded by the sign +, it indicates addition, and when preceded by the sign −, it indicates subtraction. Hence the rules:

(1) *A sign of grouping preceded by the sign + may be removed from an expression, without changing the signs of the terms inclosed.*

(2) *A sign of grouping preceded by the sign − may be removed from an expression, if the sign before each term inclosed is changed.*

NOTE. — When signs of grouping are inclosed within other signs of grouping, it is best to *remove the innermost sign first, then the innermost remaining sign, etc.*

By reversing the above rules, it follows that

Terms of a polynomial may be inclosed within a sign of grouping, when this sign of grouping is preceded by the sign +, without changing the signs of the terms; and when preceded by the sign −, by changing the signs of the terms inclosed.

EXERCISES

Remove signs of grouping and combine similar terms:

1. $a - b + (2a + b)$.
2. $x + 2y - (3y - x)$.
3. $(m - n) - (m + n)$.
4. $t - [2a - (2t - 3a)]$.
5. $[5 - (x^2 - 3x) - 4x]$.
6. $(M + 6N) - \{(7M + 3N) - (7M - 3N)\}$.
7. $\{8 - (v - 3)\} - \{8 + (v - 3)\}$.
8. $a^2 - [2ab - \{b^2 - a^2\} - 2b^2] - b^2$.
9. $-W + (3g - W) - \{2W + [g - (W - g)]\} - g$.

Leave brackets, but remove parentheses and simplify within brackets:

10. $[(a - 2b) + (3a - 4b)][(a - 2b) - (3a - 4b)]$.
11. $[P + (Q - R)][P - (Q - R)]$.
12. $[(x - a) + (5y + 2a)][(x - a) - (5y + 2a)]$.

Inclose those terms containing a and b within parentheses preceded by $+$, and all other terms within parentheses preceded by $-$:

13. $a^2 - b^2 - x^2 - 2xy - y^2$.
14. $2ab - a^2 - m^2 + n^2$.
15. $a^2 - ab + b^2 - 2xy - x^2 + y^2$.
16. $a^3 - b^3 + 3x^2y + y^3 - x^3 - 3xy^2$.
17. $5b - 2a - 3r - 4s + t$.
18. $w^2v^2 - w^4 - v^4 + a^4 - b^4$.

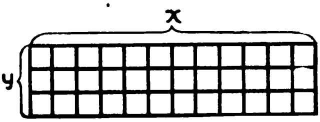
11. Fundamental Laws of Multiplication. — The product of two or more factors is the same, whatever the order in which the factors are written or taken. This truth is called the **Law of Order in Multiplication.**

The law, as applied to two factors, may be expressed by

$$xy = yx.$$

It may be illustrated graphically as follows:

The area of the rectangle is the same whether we multiply x, the number of unit squares in each row, by y, the number of rows, or multiply y, the number of unit squares in each column, by x, the number of columns.



It is evident, also, that the product of three or more factors is the same, whatever way the factors are grouped. This truth is called the **Law of Grouping in Multiplication.**

The law, as applied to three factors, may be expressed by

$$(xy)z = x(yz).$$

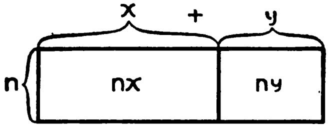
The Laws of Order and Grouping may be used to shorten arithmetical multiplication.

Thus, $2 \times 7 \times 8 \times 5$ may be written $2 \times 5 \times 7 \times 8$, by the Law of Order, and hence $(2 \times 5) \times (7 \times 8)$, by the Law of Grouping. Multiplying 2 by 5 gives 10, multiplying 7 by 8 gives 56, and multiplying these products gives 560. In practice, the change of order is made mentally, and factors whose product is easily found are grouped.

There is a third fundamental law of multiplication. It is evident that if the sum of two or more numbers is multiplied by a given number, the result obtained is the same as when each of those numbers is first multiplied by the given number and then the products added. This truth is called the **Law of Distribution.** It is expressed in symbols by

$$n\,(x + y) = nx + ny.$$

It may be represented graphically as follows:



The area of the entire rectangle is $n\,(x + y)$. But it is composed of two rectangles whose areas are nx and ny, respectively.

EXERCISES

By grouping factors to advantage, give at sight the products:

1. $5 \times 9 \times 2 \times 8$.
2. $2 \times 6 \times 8 \times 5$.
3. $4 \times 7 \times 5 \times 6$.
4. $8 \times 7 \times 5 \times 3$.
5. $50 \times 34 \times 2$.
6. $96 \times 25 \times 4$.
7. $8 \times 7 \times 12\frac{1}{2}$.
8. $2\frac{1}{2} \times 9 \times 4$.
9. $33\frac{1}{3} \times 8 \times 3 \times 2$.
10. $\frac{3}{8} \times 42 \times 16$.
11. $\frac{5}{9} \times \frac{2}{7} \times 21$.
12. $12\frac{1}{2} \times 70 \times 4 \times 5$
13. Show graphically that $n(a + b + c) = na + nb + nc$
14. Show graphically that $ab = ba$.

12. Multiplication of Positive and Negative Numbers. — The following principles apply in the multiplication of positive and negative numbers:

(1) *The product of two numbers with like signs is positive.*

(2) *The product of two numbers with unlike signs is negative.*

(3) *The absolute value of the product of two numbers is the product of the absolute values of the numbers.*

From these principles the following rule is derived for finding the product of more than two factors:

Find the product of the factors regardless of signs, and prefix + or − according as the number of negative factors is even or odd.

EXERCISES

Multiply:

1.
$$\begin{array}{cccccc} +6 & -3 & +4 & -7 & -3 & +5 \\ \underline{+2} & \underline{-5} & \underline{-8} & \underline{+2} & \underline{-9} & \underline{-7} \end{array}$$

2. $(-2)(-4)$, $(-7)(+8)$, $(+5)(-6)$, $(+4)(+9)$.
3. $(-7)(+6)$, $(-10)(-4)$, $(+12)(+3)$, $(-9)(+15)$.
4. $(-2)(-3)(-1)$, $(-4)(+2)(+2)$, $(+6)(-2)(-5)$.
5. $(-1)(+7)(-9)$, $(+4)(-4)(-4)$, $(-5)(-5)(-5)$.
6. $(-1)(-1)(-1)(-1)$, $(+2)(-2)(-3)(-5)(+6)(-1)$.
7. $(-2)^2$, $(-4)^2$, $(+3)^2$, $(-1)^3$, $(+2)^3$.

8. $(-1)^5, (-2)^2(-3)^2, (-5)^2(-2)^2, (-4)(-3)^2(-1)$.

9. Find the value of a^3 when $a = -2$; of abc when $a = -5$, $b = -1$, $c = +4$; of $a^2 + b^2$ when $a = -6$, and $b = +4$.

10. Find the value of $(x+y)(x-y)$ when $x = -4$ and $y = -9$.

3. Law of Exponents in Multiplication. — *In multiplying two powers of the same base, the exponent of the product is obtained by adding the exponents of the factors.*

In general symbols,

$$a^m \times a^n = a^{m+n}.$$

This law follows immediately from the meaning of an exponent. Thus,

a^m means $aaa \cdots$ to m factors,

and a^n means $aaa \cdots$ to n factors.

Hence, $a^m \times a^n = aaaa \cdots$ to $m + n$ factors $= a^{m+n}$.

14. Multiplication of Monomials and Polynomials. — From the laws in the preceding sections we have the following rule:

To multiply monomials, find the product of the numerical coefficients using the laws of signs, then annex to the result the products of the like literal factors, using the law of exponents.

It follows from the Law of Distribution that

To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial, and add the partial products obtained.

It follows also from the Law of Distribution that

To multiply one polynomial by another, multiply the multiplicand by each term of the multiplier, and add all of the partial products obtained.

In multiplying one polynomial by another, the similar terms among the partial products obtained should be arranged in columns to be added.

For convenience, the terms of each of two polynomials to be multiplied should be arranged according to either the ascending or the descending powers of some letter, before multiplying.

EXERCISES

Find the products of:

1. $a^2 \times a^5$; $a^6 \times a^4$; $x^7 \times x^3$; $n^{12} \times n^4$; $P^8 \times P^8$.
2. $y \times y^2 \times y^3$; $t^4 \times t \times t^7$; $D^6 \times D^5 \times D$; $m^2 \times m^{10} \times m^6$.
3. $5a^3 \times 3a^2$; $6V^4 \times 5V^2$; $9z \times 2z^5$; $b^2 \times 7b^2 \times 4b^6$.
4. $(-3a^5)(-4a^3)$; $(+8x)(-5x^4)$; $(-2N^4)(+6N)$.
5. $(+6a^3b^2)(-4ab^4)$; $(-9m^6n)(-2mn^6)$; $(+12x^4y^4)(+4x^2y^6)$.

Find the value of:

6. $\begin{array}{r}a-b\\ a\\ \hline\end{array}$ $\qquad\begin{array}{r}2m+3n\\ mn\\ \hline\end{array}$ $\qquad\begin{array}{r}x^2-3y^2\\ -4xy\\ \hline\end{array}$

7. $12n^2(3n-5)$; $8A^4(3A^2-4A+1)$; $x^2(x^2-xy+y^2)$.
8. $5t^4(-3t^2+7t-2)$; $-3R^2(\pi R^2-2\pi RT+\pi T^2)$.
9. $(x+3)(x-7)$; $(w^2-5)(w^2-3)$; $(N-6)(N^2-4N+4)$.
10. $(a^2-1)(a^2+1)$; $(y^3+1)(y^3-1)$; $(t^4-1)(t^4-1)$.
11. $(a^2-a+1)(a^2+a+1)$.
12. $(3-s+4s^2)(2s^2+7s-5)$.
13. $(3x^2-8)(5-4x^3+2x-x^2)$.
14. $(D^2-4+4D)(3-4D^2)$.
15. $(n-1)(n^3+n^2+n+1+n^4)$.
16. $(p+1)(p+2)(p+3)$.
17. $(A+2)(A-2)(A^2+4)$.
18. $(3y-5)(9y^2+25)(5+3y)$.
19. $(x^3+v^3)(v^6+x^6)(v^3-x^3)$.

Remove signs of grouping, and combine similar terms:

20. $a(2a-1)+2a(a+1)$.
21. $5(6x-3)-8(2x+4)$.
22. $9x(y-2x)-5y(2x-7y)$.
23. $(t+5)(t-1)+(t+1)(t-5)$.
24. $(n^2-4)(n^2+3n)+(n-5)(n^3-1)$.

25. $(4Q-1)(Q-4)-(2Q+3)(3Q-2)$.

26. $(x^2-y^2)(x^2+y^2)-(x^2+y^2)(x^2+y^2)$.

27. $(t^3)^2=?\quad(v^4)^5=?\quad(-x^2)^4=?$

28. $(-5x^2y^3)^3=?\quad(6w^4v^7)^2=?\quad(-2H^6t^5)^3=?$

29. $(x+3)^2=?\quad(a-b)^2=?\quad(2a-3b)^2=?$

30. $(3m+5n)^2=?\quad(a+b+c+d)^2=?$

15. Special Products. — The products of certain **types** of expressions may be obtained by special rules found by actual multiplication.

By actual multiplication, $(2x^4y^3)^4=16x^{16}y^{12}$; $(2m^2n^5)^5=32m^{10}n^{25}$; and in general,

$$(ax^my^n)^p=a^px^{mp}y^{np}.$$

Hence, to find any power of a monomial, raise the numerical coefficient to that power and multiply each exponent in the monomial by the exponent of the power.

The above law also implies the form $((a^m)^n)^p=a^{mnp}$.

Also, by actual multiplication,

$$(a+b)(a-b)=a^2-b^2.$$

Hence, to find the product of the sum and the difference of the same two terms, take the difference between their squares.

$$(x+a)(x+b)=x^2+(a+b)x+ab.$$

Hence, to find the product of two binomials having a common term, take the square of the common term, plus the algebraic sum of the unlike terms multiplied by the common term, plus the algebraic product of the unlike terms.

$$(a+b)^2=a^2+2ab+b^2 \text{ and } (a-b)^2=a^2-2ab+b^2.$$

Hence, to square a binomial, take the square of the first term, plus (or minus) twice the product of the terms, plus the square of the second term.

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc.$$

Hence, the square of any polynomial equals the sum of the squares of all of its terms, plus twice the algebraic product of each term into each of the terms following it.

EXERCISES

Give at sight the value of:

1. $(2^2)^3$.
2. $(10^2)^2$.
3. $(5^2)^2$.
4. $(30^2)^2$.
5. $(20^2)^3$.
6. $(2a^2b)^3$.
7. $(3a^4b^2)^2$.
8. $(10a^3b^5)^3$.
9. $(20xy^6)^2$.
10. $(30x^2y^5)^3$.
11. $(a^2b^3)^n$.
12. $(xy^2z^3)^m$.
13. $(2a^2b)^n$.
14. $((a^nb^m)^c)^d$.
15. $((2a^3b)^2)^3$.

Give at sight the products:

16. $(n+4)(n-4)$.
17. $(x-10)(x+10)$.
18. $(R+1)(R-1)$.
19. $(w-6)(w+6)$.
20. $(3-t)(3+t)$.
21. $(2a-3b)(2a+3b)$.
22. $(6r+s)(6r-s)$.
23. $(7m-2n)(2n+7m)$.
24. $(x+5)(x+2)$.
25. $T-4)(T-7)$.
26. $(V+9)(V-4)$.
27. $(y-7)(y-6)$.
28. $(2a+3)(2a+5)$.
29. $(5x-7)(5x+3)$.
30. $(12P+9)(12P-5)$.
31. $(2+8a)(7+8a)$.
32. $(x+y)^2$.
33. $(2n+5)^2$.
34. $(R-7)^2$.
35. $(5d-12)^2$.
36. $(3x+4y)^2$.
37. $(15m-2n)^2$.
38. $(1-4t^2)^2$.
39. $(9+2V)^2$.
40. $(x+y-z)^2$.
41. $(4m-3n+p)^2$.
42. $(1-2x+x^2)^2$.
43. $(4a+2b-5c)^2$.
44. $(a^2-b^2+ab)^2$.
45. $(x+y-z-w)^2$.
46. $(16t+9)(9-4t)$.
47. $(3K-7L)(7L+3K)$.
48. $(xy+13)(xy-11)$.
49. $(8M^2-1)(3+8M^2)$.
50. $(16vt-s^2)(16vt+s^2)$.
51. $(14A^3-1)(1+14A^3)$.

Give at sight the products of the following, using the principles of this section:

52. 41×41; 72×68; 97×103; 65×65; 81×84.

53. 109×101; 92×92; 64×56; 35×45; 79×79.

16. Division. — Division is the inverse of multiplication. To divide a number M (dividend) by a number N (divisor) is to find a number (quotient) which multiplied by N will give M. In general,

Divisor × Quotient = Dividend.

Hence, the principles and rules of division, deduced from the corresponding principles and rules of multiplication, are as follows:

(1) *If the dividend and divisor have like signs, the quotient is positive.*

(2) *If the dividend and divisor have unlike signs, the quotient is negative.*

(3) *In dividing powers of the same base, the exponent of the quotient is obtained by subtracting the exponent of the divisor from that of the dividend. That is,*

$$n^a \div n^b = n^{a-b}.$$

(4) *In dividing monomials, divide the coefficient of the dividend by that of the divisor, using the above laws of signs, and annex to the result the quotients of the literal factors obtained by the above law of exponents.*

(5) *To divide a polynomial by a monomial, divide each term of the dividend by the divisor, and add the partial quotients obtained.*

EXERCISES

Divide as indicated:

1. $\frac{-16}{-8}$; $\frac{+24}{+4}$; $\frac{-12}{+6}$; $\frac{+32}{-4}$; $\frac{-21}{-7}$; $\frac{+63}{-9}$.

2. $(-24) \div (-4)$; $(-42) \div (+6)$; $(+72) \div (-9)$.

3. $a^7 \div a^2$; $t^{12} \div t^7$; $V^{22} \div V^{16}$; $R^{18} \div R^6$; $n^{42} \div n^{34}$.

4. $16\,x^6 \div 2\,x^4$; $28\,P^9 \div 7\,P^5$; $96\,y^{13} \div 6\,y$; $84\,T^6 \div 12\,T^6$.

5. $(-26\,a^{14}) \div (-2\,a^9)$; $(+32\,V^{16}) \div (-8\,V^{12})$.

6. $(+100\,x^3y^7) \div (-20\,xy^4)$; $(-a^{10}b^2) \div (-ab^2)$.

7. $4\,x^2\overline{)16\,x^4 - 8\,x^2}$; $-5\,m^2n\overline{)15\,m^4n - 10\,m^3n^2 + 20\,m^2n^3}$.

8. $(4\,a^2b^4 - 12\,ab^5 - 6\,a^3b^3) \div (-2\,ab^3)$.

9. $(21\,r^4s^2 + 28\,rs^5 - 35\,r^3s^3) \div (-7\,rs^2)$.

10. $(-56\,W^5 - 49\,W^4V + 63\,W^3V^2 - 21\,W^2V^3) \div (-7\,W^2)$.

11. $(42\,x^4y^4 - 36\,x^2y^6 - 48\,x^6y^2) \div (+6\,x^2y^2)$.

12. $(\frac{1}{2}\,M^3N - \frac{1}{4}\,M^2N^2 + \frac{1}{2}\,MN^3) \div (\frac{1}{2}\,MN)$.

17. Division of One Polynomial by Another. — The method of dividing one polynomial by another, as shown in the First Course, is as follows:

(1) *Arrange both the dividend and divisor according to the descending or ascending powers of some letter.*

(2) *Divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient.*

(3) *Multiply the whole divisor by this term of the quotient, and subtract the result from the dividend.*

(4) *Treat the remainder as a new dividend (having the terms arranged as before), and repeat the process, continuing until either the remainder zero, or a true remainder, is found.*

In arranging the terms for division, if terms containing powers of the letter of arrangement between the highest and lowest powers of that letter in the dividend are missing, spaces should be left for them in the written work, as shown in the illustration below.

$$
\begin{array}{r|lll}
 & a^2 & -\,ab & +\,b^2 \\
\hline
a + b & a^3 & & +\,b^3 \\
 & \underline{a^3 + a^2b} & & \\
 & -\,a^2b & & +\,b^3 \\
 & \underline{-\,a^2b} & \underline{-\,ab^2} & \\
 & & ab^2 & +\,b^3 \\
 & & \underline{ab^2} & \underline{+\,b^3}
\end{array}
$$

EXERCISES

Divide:

1. $n^2 + 7n + 10$ by $n + 2$.
2. $x^2 - 12x + 32$ by $x - 8$.
3. $54 + 3a - a^2$ by $9 - a$.
4. $4t - 60 + t^2$ by $t + 10$.
5. $y^2 - 16$ by $y - 4$.
6. $9 - v^2$ by $3 + v$.
7. $1 - 25N^2$ by $5N + 1$.
8. $R - 72 + R^2$ by $R - 8$.
9. $a^3 - 1$ by $a - 1$.
10. $27 + m^3$ by $m + 3$.
11. $1 - x^4$ by $1 - x$.
12. $s^3 + t^3$ by $s^2 - st + t^2$.
13. $A^4 + A^2B^2 + B^4$ by $A^2 + AB + B^2$.
14. $8b^3 + 125 - 30ab + a^3$ by $a + 5 + 2b$.

18. Synthetic Division. — An abbreviated method of division, called **synthetic division,** may be used to advantage when the dividend in a polynomial is some letter, say x, and the divisor of the form $x - a$.

Thus, the usual process of division of $x^3 - 9x^2 + 27x - 40$ by $x - 5$, bringing down only such terms in each remainder as are immediately needed, is as follows:

$$
\begin{array}{rl}
 & \underline{x^2 - 4x + 7} \\
x - 5) & x^3 - 9x^2 + 27x - 40 \\
 & \underline{x^3 - 5x^2} \\
 & \quad -4x^2 + 27x \\
 & \quad \underline{-4x^2 + 20x} \\
 & \qquad\qquad\quad 7x - 40 \\
 & \qquad\qquad\quad \underline{7x - 35} \\
 & \qquad\qquad\qquad\quad -5, \text{ rem.}
\end{array}
$$

Observe that the coefficients of the terms in the quotient are the coefficients of the first term of the dividend and of each of the remainders, taken in order.

Since the first term of each partial product cancels the term just above it in the subtraction, the former need not have been written.

Hence, if the dividend is written in descending powers of x, the first term of the divisor need not be written at all.

If the sign of each term of the partial products had been changed, this term might have been added to the term above it, instead of subtracted from it. The same result may be obtained by changing the sign of each term of the divisor, multiplying, and adding the partial products to the terms above them.

Hence, omitting all work that is unnecessary, writing the coefficients only, and raising all of the remainders into one line, the division may be written as follows:

$$\begin{array}{r} 5)1-9+27-40 \\ \underline{+5-20+35} \\ 1-4+\ 7-\ 5 \end{array}$$

where 1, -4, and $+7$ are the coefficients of the quotient, and -5 the remainder.

Thus we see that in **synthetic division**:

The first term of the divisor is omitted, the sign of the second term of the divisor changed, and only coefficients of the dividend, partial products, remainders, and quotient, written.

The first coefficient of the dividend is brought down for the first coefficient of the quotient.

This number is multiplied by the number in the divisor and added to the next term of the dividend for the second coefficient of the quotient, and so on to the last term. The last sum is the true remainder.

Example. — Divide $3x^3-7x^2+13$ by $x+2$.

$$\begin{array}{r} -2)3-\ 7+\ 0+13 \\ \underline{-\ 6+26-52} \\ 3-13+26-39 \end{array}$$

The 2 of the divisor is placed at the left, with its sign changed. Since the first power of x is missing from the dividend, the coefficient 0 must be supplied. The first coefficient 3 is brought down for the first term of the quotient. Then $(-2)(3)=-6$; $-7-6=-13$; $(-2)(-13)=+26$; $0+26=+26$; $(-2)(+26)=-52$; $+13-52=-39$. Hence the quotient is $3x^2-13x+26$, and remainder -39.

If any power of x in the dividend is missing, its place must be supplied by a zero.

EXERCISES

Divide by the synthetic method:

1. x^3+3x^2-7x-3 by $x-2$.
2. $2x^4-9x^2+3x-5$ by $x-4$.
3. a^4-3a^2+7a-1 by $a-2$.
4. $3n^3+7n^2-3n+15$ by $n+3$.
5. $y^4-7y^3+2y-12$ by $y+2$.
6. t^3+2t^2+96 by $t+4$.
7. V^5+3V^2+V-7 by $V-3$.
8. $2r^3+r^2-7$ by $r+3$.
9. A^4+5A^3+A-8 by $A+4$.
10. n^4+3n^2-9 by $n+6$.

SUPPLEMENTARY EXERCISES

NOTE. — These exercises and the supplementary exercises at the ends of other chapters may be used by all classes for reviews and for more difficult exercises than those contained in the body of the text. They may be omitted, however, without interfering with the continuity of the work.

1. Express $BBB \cdots$ to n factors by use of an exponent.

2. Express t^a without an exponent.

By adding the literal coefficients of the like literal parts, find the sums of:

3. an, bn, and cn.

SOLUTION. — Indicating the addition of a, b, and c, the sum is $(a+b+c)n$.

4. ax, $-bx$, and cx.

5. mr^2, nr^2, and $-pr^2$.

6. $2ay$, $-3by$, and cy.

7. $4ax^2y$, $9nx^2y$, and $-5bx^2y$.

8. $ax+by+cz$, $bx+cy+az$, and $cx+ay+bz$.

9. From $5m-2n$ subtract $n-4p$.

10. From $Ax-By$ subtract $Cx+Dy$.

11. From $2ax+4cz$ subtract $5by-az$.

12. Simplify $-2-[a+\{5-(3a-1)+a\}-1]-a$.

13. Remove parentheses, leaving brackets, and simplify
$$[(5x-y)+(2y-4z)][(5x-y)-(2y-4z)].$$

Multiply:

14. x^n, x^{4n}, and x^{3n}.

15. $a^{n+1}b$, ab^{n-1}, and a^2b^2.

16. R^{a-4}, $2R^{2a+1}$, and $-3R^{5-a}$.

17. A^x+B^x and A^x-B^x.

18. a^n-b^n, a^n+b^n, $a^{2n}+b^{2n}$.

19. $T^{2a}+t^{2b}$, $T^{2a}-t^{2b}$, $T^{4a}+t^{4b}$.

20. $x^n \cdot x^n \cdot x^n \cdots$ to 8 factors.

21. $x^n \cdot x^n \cdot x^n \cdots$ to m factors.

22. $(x^ay^b)(x^ay^b) \cdots$ to n factors.

23. $(2a^nb)(2a^nb) \cdots$ to c factors.

Divide:

24. M^{4a} by M^a; B^{3x-1} by B^{x-3}.

25. $-48x^{3m}y^n$ by $-4x^ny^{3m}$.

26. $32W^{a+2}V^{a-2}$ by $4W^6V^{a-4}$.

27. $N^b-3N^{b+1}+2N^{b+2}$ by N^b.

28. $x^{2n}-y^{2n}$ by x^n-y^n.

29. $A^{4n}-B^{4n}$ by A^n-B^n.

30. Simplify $-n[a-n(3a-n)-n^2]+an$.

CHAPTER II

FACTORS AND MULTIPLES

19. Factoring. — Those expressions whose product is a given expression are called the **factors** of the given expression. The process of finding the factors of a given expression is called **factoring**. If a factor of an expression cannot itself be factored, it is called a **prime factor**. In general, to factor an expression means to find its prime factors.

Ability to factor an expression depends upon recognition of the **special type form** of the expression. The methods of factoring expressions of certain *special type forms* that were discussed in the FIRST COURSE are restated in §§ 20–23.

20. Monomial Factors. — In factoring any expression, if it has a monomial factor, this factor should be found first, and the expression written as the product of the monomial and a polynomial factor. The polynomial factor should then be factored if possible. To find the monomial and polynomial factors of an expression,

Find, by inspection, the monomial of highest power that will divide each term of the expression. Divide the expression by this monomial. The divisor and quotient are the monomial and polynomial factors, respectively, of the given expression.

EXAMPLE. — Factor $3m^4n - 9m^3n^2 + 3m^2n^3$.

Dividing each term by $3m^2n$,

$$3m^4n - 9m^3n^2 + 3m^2n^3 = 3m^2n(m^2 - 3mn + n^2).$$

EXERCISES

Factor:

1. $6a^4 - 18a^3$.

2. $15m^3n + 12m^2n^2 + 9mn^3$.

3. $4x^6 - 14x^4y^2 + 10x^3y^3$.

4. $2\pi R^2 + 4\pi RH$.

5. $7t^4 - 28t^3v - 14t^2v^2$.

6. $-A^3 - A^2 - A$.

7. $-a^2 - 2ab - b^2$.

8. $x^4y^3z^6 + x^5y^4z^4 + x^3y^5z^5$.

9. $4\pi R^2 + 4\pi r^2 + 4\pi Rr$.

10. $8V_1^2 - 16V_1V_2 + 8V_1V_3$.

11. $3m^4n - 6m^3n^2 + 3m^2n^3$.

12. $x^n + ax^n$.

13. $a^nb^n - a^n$.

14. $t^2v^{n+2} + t^{n+2}v^2$.

15. $P^{n+1} + P^nQ$.

16. $5a^ny^{n-1} + 10a^{n-1}y^n$.

21. Factoring Binomials. — To factor a binomial, see if it is of either of the following special forms, and determine the factors accordingly.

I. A BINOMIAL THE DIFFERENCE OF TWO SQUARES

By multiplication,

$$a^2 - b^2 = (a + b)(a - b).$$

Hence, the factors of the difference of the squares of two numbers are the sum and the difference of the numbers.

EXAMPLE. — $64V^2 - 81T^2 = (8V)^2 - (9T)^2$
$= (8V + 9T)(8V - 9T)$.

II. A BINOMIAL THE DIFFERENCE OF TWO CUBES

By multiplication,

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

Hence, one factor of the difference of the cubes of two numbers is the difference of the numbers, and the other is the sum of their squares and their product.

EXAMPLE. — $8R^3 - 27D^3 = (2R)^3 - (3D)^3$
$= (2R - 3D)(4R^2 + 6RD + 9D^2)$.

III. A BINOMIAL THE SUM OF TWO CUBES

By multiplication,

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

Hence, one factor of the sum of the cubes of two numbers is the sum of the numbers, and the other is the sum of their squares minus their product.

EXAMPLE. — $64\,n^{15} + 1 = (4\,n^5)^3 + 1$
$= (4\,n^5 + 1)(16\,n^{10} - 4\,n^5 + 1)$.

A binomial may be one of the above forms, and then when factored, one or both of its factors may be some one of these forms. In such a case the factors should be factored.

EXAMPLE. — $x^6 - 1 = (x^3)^2 - 1$
$= (x^3 + 1)(x^3 - 1)$
$= (x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1)$.

EXERCISES

Factor:

1. $9\,A^2 - 16$.
2. $4\,t^2 - 25$.
3. $1 - 49\,n^2$.
4. $25\,x^2 - 64\,y^2$.
5. $81\,V^2 - 1$.
6. $4\,P^2 - 9\,Q^2$.
7. $25\,R^2 - 16$.
8. $36\,v^2 - 49\,w^2$.
9. $a^3 - 8\,b^3$.
10. $M^3 - 27$.
11. $1 - 64\,k^3$.
12. $x^3 - 125\,y^3$.
13. $216 - T^3$.
14. $m^3 + 27\,n^3$.
15. $A^3 + 1$.
16. $8\,r^3 + 1$.
17. $1 + 64\,P^3$.
18. $125\,p^3 + 8\,q^3$.
19. $8\,W^3 - 1$.
20. $27 + s^3$.
21. $3\,a^2 - 12\,b^2$.
22. $5 - 40\,N^3$.
23. $48\,x^3 + 6\,y^3$.
24. $x^4 - 1$.
25. $y^6 - z^6$.
26. $A^8 - B^8$.
27. $1 - t^6$.
28. $m^6 + n^6$.
29. $x^{12} - y^{12}$.
30. $am^4 - an^4$.
31. $p^6q^3 + p^3q^6$.
32. $\pi R^3 - \pi RH^2$.
33. $8\,v^4t - 64\,vt^4$.
34. $(a + b)^2 - c^2$.
35. $(m - n)^2 - x^2$.
36. $p^2 - (q - r)^2$.
37. $(v + t)^2 - (w + u)^2$.
38. $(M - N)^2 - (P - Q)^2$.
39. $(2\,A + B)^2 - (C - 3\,D)^2$.

22. Factoring Trinomials. — To factor a trinomial, see if it is of either of the following special forms, and determine the factors accordingly..

I. TRINOMIALS THAT ARE PERFECT SQUARES

By multiplication,

$$(a + b)^2 = a^2 + 2\,ab + b^2 \text{ and } (a - b)^2 = a^2 - 2\,ab + b^2.$$

Evidently, a trinomial is the square of a binomial when two of its terms are squares and the other term twice the product of their square roots.

To factor such a trinomial express it as the square of a binomial. It follows from the above identities that:

To find the binomial of which a given trinomial is a square, take as the terms the square roots of the two square terms of the trinomial, and connect them by the sign of the other term.

EXAMPLE. — $25M^2 - 30MN + 9N^2 = (5M - 3N)^2$.

II. TRINOMIALS OF THE FORM $x^2 + ax + b$

Since, by § 15, the product of two binomials having a common term x must be of the form $x^2 + ax + b$, in which a is the algebraic sum of the unlike terms of the binomials and b their product, it follows that:

To factor a trinomial of the form $x^2 + ax + b$, *take as factors two binomials with a common term* x *and having for their other terms two numbers whose algebraic sum is* a *and algebraic product* b.

EXAMPLE. — Factor $R^2 - 4R - 12$.

For the second terms of the factors we must find two numbers whose sum is -4 and whose product is -12. These are evidently -6 and $+2$. Hence,

$$R^2 - 4R - 12 = (R-6)(R+2).$$

III. TRINOMIALS OF THE FORM $ax^2 + bx + c$

If a trinomial of the form $ax^2 + bx + c$ has binomial factors, they are of the forms $mx + p$ and $nx + q$. Of the various methods of factoring such a trinomial, there is none better than the following:

Find by trial two binomials whose product is the given binomial.

EXAMPLE. — Factor $4A^2 - 3A - 27$.

The product of the first terms of the factors must be $4A^2$. Hence, they must be either A and $4A$, or $2A$ and $2A$. The product of the last terms must be -27. Hence, they must have unlike signs, and must be either 1 and

-27, -1 and 27, 3 and -9, or -3 and 9. Hence, the possible trial sets of factors are:

$$\begin{array}{ccccc} A+1 & A-1 & A+3 & A-3 & 2A+1 \\ \underline{4A-27} & \underline{4A+27} & \underline{4A-9} & \underline{4A+9} & \underline{2A-27\text{ etc.}} \end{array}$$

Only the fourth of these gives the right sum of cross products, $-3A$. Hence,

$$4A^2-3A-27=(A-3)(4A+9).$$

EXERCISES

Factor:

1. $x^2-10x+25.$
2. $4a^2+4a+1.$
3. $9t^2-24tv+16v^2.$
4. $1+14P+49P^2.$
5. $R^2-12Rr+36r^2.$
6. $n^2+7n+10.$
7. $x^2-11x+18.$
8. $A^2-A-56.$
9. $w^2+2wv-35v^2.$
10. $T^2-7TU-30U^2.$
11. $8a^2-10a+3.$
12. $2V^2+7V-30.$
13. $15M^2-2M-8.$
14. $12R^2+25Rr+12r^2.$
15. $10x^2+29xy-21y.^2$
16. $64p^2+48pq+9q^2.$
17. $3W-27+4W^2.$
18. $L^2+5Lv-14v^2.$
19. $mn^2-3mn-54m.$
20. $5nx^2+10nxy^2+5ny^4.$
21. $4aF^2+8aF-60a.$
22. $4k^2n-2kmn-20m^2n.$
23. $-B^3-2B^2-B.$
24. $6h^2m^2+17h^2m-14h^2$
25. $5r_1^2-12r_1r_2+4r_2^2.$
26. $36-30D-6D^2.$
27. $tT^2-30tTv+225tv^2$
28. $-4-24d^2-36d^4.$
29. $4\pi R^2+28\pi-22\pi R.$
30. $45V_1^2+12V_2^2+69V_1V_2.$
31. $(x+b)^2-5(x+b)+6.$
32. $(m-n)^2+2(m-n)(w+v)+(w+v)^2.$
33. $4(A+B)^2+3C(A+B)-27C^2.$

23. Factoring Polynomials by Grouping Terms. — By grouping terms, some polynomials may be written in one of the special forms given in §§ 20, 21, and 22, and factored accordingly.

I. POLYNOMIALS WHOSE TERMS MAY BE GROUPED TO GIVE A COMMON FACTOR

Group the terms of the polynomial to be factored so that a factor may be divided out of each group, and that the expressions left in parentheses are the same in all of the groups. Then divide by the expression in parentheses, writing the divisor as one factor and the quotient as the other.

EXAMPLE 1. — Factor $2v + v^2 - 2w - vw$.

Grouping the first and second terms, and also the third and fourth terms, and dividing out the monomial factor of each group,

$$\begin{aligned} 2v + v^2 - 2w - vw &= v(2+v) - w(2+v) \\ &= (2+v)(v-w). \end{aligned}$$

EXAMPLE 2. — Factor $a^2 - b^2 - ac + bc$.

Grouping the first two terms, and also the last two, and factoring the groups,

$$\begin{aligned} a^2 - b^2 - ac + bc &= (a+b)(a-b) - c(a-b) \\ &= (a-b)(a+b-c). \end{aligned}$$

II. POLYNOMIALS WRITTEN AS THE DIFFERENCE OF TWO SQUARES

Group those terms that form the square of a binomial, and express the polynomial in the form $a^2 - b^2$.

EXAMPLE. — Factor $4a^2 - b^2 + 6bc - 9c^2$.

Grouping the last three terms in parentheses preceded by the sign −,

$$\begin{aligned} 4a^2 - b^2 + 6bc - 9c^2 &= 4a^2 - (b^2 - 6bc + 9c^2) \\ &= 4a^2 - (b-3c)^2 \\ &= (2a + [b-3c])(2a - [b-3c]) \\ &= (2a+b-3c)(2a-b+3c). \end{aligned}$$

EXERCISES

Factor:

1. $a^2 - 2ab + b^2 - c^2$.
2. $x^2 - y^2 - 2y - 1$.
3. $an - bn - am + bm$.
4. $y^2 + xy - yz - xz$.
5. $xP - yQ - xQ + yP$.
6. $25A^2 + 2BC - B^2 - C^2$.
7. $m^2 + n^2 - 2mn - v^2 - t^2 - 2vt$.
8. $B^2 - 4AC - A^2 - 4C^2$.

9. $sP - 5s + 5t - tP.$
10. $na - nb + a - b.$
11. $V^2 - V - Vv + v.$
12. $R^2 - r^2 - r + R.$
13. $1 + 4y + 4y^2 - z^2.$
14. $1 - 6b + 4mn + 9b^2 - m^2 - 4n^2.$
15. $H^2 - 16t^2 + Hv - 4vt.$
16. $2a - 2b + ad - bd + 2c + cd.$
17. $am^2 + am + bm^2 + bm + a + b.$
18. $x^2 - y^2 + z^2 - w^2 - 2(xz - yw).$

24. Trinomials of the Form $a^4 + na^2b^2 + b^4$. — Some trinomials of the special form $a^4 + na^2b^2 + b^4$, which cannot be factored by the methods of §22, may be written in the form $a^2 - b^2$, by the addition and subtraction of a term and by grouping terms as in § 23.

Add such a positive term to the middle term as will make the trinomial a perfect square, and indicate the subtraction of the term added.

EXAMPLE. — Factor $4x^4 - 16x^2y^2 + 9y^4$.

By adding and indicating the subtraction of $4x^2y^2$,

$$\begin{aligned}4x^4 - 16x^2y^2 + 9y^4 &= 4x^4 - 12x^2y^2 + 9y^4 - 4x^2y^2\\ &= (2x^2 - 3y^2)^2 - (2xy)^2\\ &= (2x^2 - 3y^2 + 2xy)(2x^2 - 3y^2 - 2xy).\end{aligned}$$

EXERCISES

1. $n^4 + n^2 + 1.$
2. $1 + 3a^2 + 4a^4.$
3. $x^8 + x^4 + 1.$
4. $t^4 + 2t^2v^2 + 9v^4.$
5. $V^4 - 3V^2 + 9.$
6. $4a^4 + 16a^2b^2 + 25b^4.$
7. $4M^4 + 4M^2N^2 + 25N^4.$
8. $R^4 - 15R^2r^2 + 9r^4.$
9. $36A^4 - 76A^2T^2 + 25T^4.$
10. $49y^8 + 19y^4z^2 + 4z^4.$
11. $4W^8 - 16W^4w^6 + 25w^{12}.$
12. $64a^4b^4 + 12a^2b^2 + 1.$
13. $m^8 - 22m^4x^2y^4 + 9x^4y^8.$
14. $8\pi R^4 - 90\pi R^2r^2 + 50\pi r^4.$
15. $4v^{12}t - 4v^6t^3 + 64t^5$
16. $75a^5 + 48a^3b^4c^2 + 12ab^8c^4.$

25. Factoring by the Remainder Theorem. — A polynomial that cannot be factored by any of the methods in the preceding sections may sometimes be factored by use of the following principle, called the **Remainder Theorem.**

If a polynomial in x be divided by $x - a$, the remainder is the same as the dividend when x is replaced by a.

Thus, dividing $x^3 - x^2 + 3x + 2$ by $x - 2$ gives a remainder 12. Now, if x is replaced by 2, the dividend becomes $8 - 4 + 6 + 2$, or 12. Hence, the remainder is the same as the dividend when x is made 2.

$$\begin{array}{r|l}
 & x^2 + x + 5 \\ \hline
x - 2 & x^3 - x^2 + 3x + 2 \\
 & \underline{x^3 - 2x^2} \\
 & \quad x^2 + 3x \\
 & \quad \underline{x^2 - 2x} \\
 & \qquad 5x + 2 \\
 & \qquad \underline{5x - 10} \\
 & \qquad\quad 12, \text{ rem.}
\end{array}$$

To show that this principle is true in general, let any polynomial in x be represented by the general expression:

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \cdots + Mx + N.$$

Let the quotient and remainder, when this is divided by $x - a$, be represented by Q and R respectively. Then since

$$\textit{Dividend} = \textit{Divisor} \times \textit{Quotient} + \textit{Remainder},$$

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \cdots + Mx + N = (x - a)Q + R.$$

Since x may be any number, this equation must be true when $x = a$. But when $x = a$, it becomes

$$Aa^n + Ba^{n-1} + Ca^{n-2} + \cdots + Ma + N = (a - a)Q + R.$$

Hence, since the term $(a - a)Q$ is zero,

$$Aa^n + Ba^{n-1} + Ca^{n-2} + \cdots + Ma + N = R;$$

which establishes the general principle.

Now, since $x - a$ is a *factor* of the polynomial when the remainder is zero, the following **factor theorem** may be stated:

If a polynomial in x becomes zero when x is replaced by a number a, then $x - a$ is a factor of the polynomial.

EXAMPLE. — Factor $x^3 - 5x^2 + 10x - 12$.

If x is replaced by 3, the polynomial becomes $27 - 45 + 30 - 12$, or 0. Hence, $x - 3$ is a factor.

$$\begin{array}{r|rrrr}
3 & 1 & -5 & +10 & -12 \\
 & & +3 & -6 & +12 \\ \hline
 & 1 & -2 & +4 &
\end{array}$$

Dividing the polynomial by $x - 3$ gives the quotient $x^2 - 2x + 4$ for the other factor.

Now $x^2 - 2x + 4$ is of the form $x^2 + ax + b$, but cannot be factored.

Hence, $x^3 - 5x^2 + 10x - 12 = (x - 3)(x^2 - 2x + 4)$.

Observe that, in the above example, since the product of the last terms of the factors must be -12, the only values that should be substituted for x in the given expression are divisors of -12. In general, when an expression to be factored is written in descending powers of the letter of arrangement, only divisors of the last term should be used for the letter of arrangement in discovering the factors.

When a binomial factor has been discovered by the Remainder Theorem, the quickest way to divide it out is by *synthetic division*, as shown in the above example.

If a polynomial in x has more than one factor of the form $x - a$, these may *all* be found by trial before any division is performed, and the polynomial be divided by their product to obtain the remaining factor. In some problems the division may not be necessary at all.

Example. — Factor $x^3 - 5x^2y - 8xy^2 + 12y^3$.

The only values that may be substituted for x are y, $-y$, $2y$, $-2y$, $3y$, $-3y$, $4y$, $-4y$, $6y$, $-6y$, $12y$, or $-12y$. By trial it is found that when x is replaced by y, $-2y$, or $6y$, the given expression becomes zero. Hence, three factors are $x - y$, $x + 2y$, and $x - 6y$. It is evident that there can be only these three factors. Hence,

$$x^3 - 5x^2y - 8xy^2 + 12y^3 = (x - y)(x + 2y)(x - 6y).$$

EXERCISES

Factor by the Remainder Theorem:

1. $2x^2 - x - 1$.
2. $3x^2 + 5x + 2$.
3. $2y^2 - 5y + 2$.
4. $4n^2 + 7n - 2$.
5. $2t^2 - 9t + 9$.
6. $3A^2 + 14A + 15$.
7. $3V^2 - 13V - 10$.
8. $x^4 - 3x^2 - 4$.
9. $a^3 - 21a - 20$.
10. $y^3 - 6y^2 + 11y - 6$.
11. $b^3 + b^2 - 3b - 2$.
12. $P^3 + 7P^2 + 7P - 15$.
13. $n^4 + n^3 - 2n - 2$.
14. $2v^3 + 3v^2 - 50v + 24$.
15. $3T^3 - 2T^2 + T - 2$.
16. $a^3 + 3a^2 + 3a + 2$.
17. $R^3 + 2R^2 - 4R - 3$.
18. $s^4 + 7s^3 + 13s^2 + 7s + 12$.
19. $z^6 - z^4 + z^2 - 1$.
20. $a^3 + 2a^2b - ab^2 - 2b^3$.
21. $2x^2 - 3xy + y^2$.

22. $3x^2 + 2xy - y^2$.

23. $m^3 + 2m^2y - my^2 - 2y^3$.

24. $s^3 - 2s^2y - 9sy^2 + 18y^3$.

25. $t^3 + 4t^2d + td^2 - 6d^3$.

26. $A^3 - 3A^2B + 3AB^2 - B^3$.

27. $N^3 - 7a^2N - 6a^3$.

28. Show that $a - b$ is a factor of $a^{20} - b^{20}$.

29. Show that $a + b$ is a factor of $a^{15} + b^{15}$.

26. Binomials of the Forms $a^n - b^n$ and $a^n + b^n$.—The factors of $a^n - b^n$ when n is 2, and of both $a^n - b^n$ and $a^n + b^n$ when n is 3, have been found in § 21. The factors of $a^n - b^n$ and $a^n + b^n$ for all values of n may be found by the Remainder Theorem.

In $a^n - b^n$, let $a = b$. Then $a^n - b^n$ becomes $b^n - b^n$ or 0.

(1) *Hence, $\mathbf{a} - \mathbf{b}$ is a factor of $\mathbf{a} - \mathbf{b}^n$ for all values of $\mathbf{n}$.*

In $a^n - b^n$, let $a = -b$. Then $a^n - b^n$ becomes $(-b)^n - b^n$, which is 0 when n is even, but not 0 when n is odd.

(2) *Hence, $\mathbf{a} + \mathbf{b}$ is a factor of $\mathbf{a}^n - \mathbf{b}^n$ only when $\mathbf{n}$ is even.*

In $a^n + b^n$, let $a = b$. Then $a^n + b^n$ becomes $b^n + b^n$, which is not 0 for any value of n.

(3) *Hence, $\mathbf{a} - \mathbf{b}$ is not a factor of $\mathbf{a}^n + \mathbf{b}^n$ for any value of $\mathbf{n}$.*

In $a^n + b^n$, let $a = -b$. Then $a^n + b^n$ becomes $(-b)^n + b^n$, which is 0 when n is odd, but not 0 when n is even.

(4) *Hence, $\mathbf{a} + \mathbf{b}$ is a factor of $\mathbf{a}^n + \mathbf{b}^n$ only when $\mathbf{n}$ is odd.*

The polynomial factors of $a^n - b^n$ and $a^n + b^n$ for special values of n are found by division. Thus, by division,

$$a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4);$$
$$a^6 - b^6 = (a - b)(a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5);$$
$$a^7 + b^7 = (a + b)(a^6 - a^5b + a^4b^2 - a^3b^3 + a^2b^4 - ab^5 + b^6);$$
$$a^8 - b^8 = (a + b)(a^7 - a^6b + a^5b^2 - a^4b^3 + a^3b^4 - a^2b^5 + ab^6 - b^7); \text{ etc.}$$

In general,

When the binomial factor is $\mathbf{a} - \mathbf{b}$, the signs of all terms of the polynomial factor are +; and when the binomial factor is $\mathbf{a} + \mathbf{b}$, the signs of the successive terms of the polynomial factor are alternately + and −.

In every polynomial factor the exponents of a *diminish by* 1 *and the exponents of* b *increase by* 1, *as we proceed term by term from left to right.*

Example. — Factor $x^{10} - 32y^5$.

Indicating the powers of the factors in the terms, and simplifying afterward,

$$\begin{aligned} x^{10} - 32y^5 &= (x^2)^5 - (2y)^5 \\ &= (x^2 - 2y)[(x^2)^4 + (x^2)^3(2y) + (x^2)^2(2y)^2 + (x^2)(2y)^3 + (2y)^4] \\ &= (x^2 - 2y)(x^8 + 2x^6y + 4x^4y^2 + 8x^2y^3 + 16y^4). \end{aligned}$$

EXERCISES

Factor:

1. $a^7 - b^7$.
2. $x^7 + 1$.
3. $a^5 - 32b^5$.
4. $1 - t^5$.
5. $1 - n^7$.
6. $a^{11} - b^{11}$.
7. $a^{11} + b^{11}$.
8. $R^{13} + r^{13}$.
9. $R^{13} - r^{13}$.
10. $32x^5 + 243y^5$.
11. $2 - 256B^7$.
12. $m^{10} + n^5$.
13. $x^6 + y^6$.

Suggestion. — Write it as the sum of odd powers, thus: $(x^2)^3 + (y^2)^3$.

14. $1 + a^{10}$.
15. $x^{12} + y^{12}$.
16. $n^{14} + 1$.
17. $v^{18} + w^{18}$.
18. $5 + 5P^{10}$.
19. $4 + 4N^6$.
20. $a^{15} + b^{15}$.
21. $1 + y^{25}$.
22. $a^6 - 1$.

Suggestion. — Write it as the difference of two squares, thus, $(a^3)^2 - 1$.

23. $x^{12} - y^{12}$.
24. $m^{10} - n^{10}$.
25. $1 - t^{16}$.
26. $256 - a^6$.
27. $a^{14} - 1$.
28. $p^8 - q^8$.
29. $w^{12} - v^6$.
30. $1 - P^9$.

27. General Directions for Factoring. — In attempting to factor any given expression,

(1) *First see if there is a monomial factor, and if there is, factor the expression into its monomial factor and the corresponding polynomial factor.*

(2) *Then, to factor the polynomial factor, see if it is of either one of the special type forms discussed in §§ 21, 22, 23, 24, and 26; and if so, use the method applicable to that special form.*

(3) *If unsuccessful, try the Remainder Theorem.*

(4) *Factor each polynomial factor obtained, if possible, until the expression is resolved into its prime factors.*

MISCELLANEOUS EXERCISES

Find the prime factors of:

1. $2N^2 - N - 1$.
2. $ax^4 - 64a^5$.
3. $3abn^2 + 2abn - ab$.
4. $4P^2 + 32PQ + 39Q^2$.
5. $bt^2 - 6b^2t + 5t^3$.
6. $2V^2 - 3VW + W^2$.
7. $x^4 + 3x^2 - 4$.
8. $xy + z^2 - xz - yz$.
9. $n^3 + 3n^2 + 6n + 18$.
10. $4(a - b)^2 - (y + v)^2$.
11. $9R^2 - 27Rr + 20r^2$.
12. $B^4 + B^2C^2 + C^4$.
13. $x^4 - 18x^2y^2 + y^4$.
14. $T^6 + T^3 - 42$.
15. $h^4 + 2h^2k^2 - 35k^4$.
16. $y^2 + 3y^3 - y^4 - 3y$.
17. $1 + bt - (x^2 + bx)t^2$.
18. $(x^2 + y^2 - z^2)^2 - 4x^2y^2$.
19. $V^4 - 2Vv^3 - v^4 + 2V^3v$.
20. $ay^2 + 7ay - 30a$.
21. $ac + cd - ab - bd$.
22. $p^4 - (p - 6)^2$.
23. $72t^2 + 41t - 45$.
24. $a^2b^2 - a^2 - b^2 + 1$.
25. $2z^2 + 5zt - 12t^2$.
26. $6a^3N - a^2N^2 - 9a^4$.
27. $1 - 16m^8$.
28. $a^2 - b^2 - 3a + 3b$.
29. $bcn + acn^2 + bn^3 + an^4$.
30. $a^4 - 2(b^2 + c^2)a^2 + (b^2 + c^2)^2$.
31. $x^2 - (4a^2 + b^2)a^2x + 4a^6b^2$.
32. $n^2 + (x - y)n - 2x(x + y)$.
33. $4w^4 + 3w^2v^2 + 9v^4$.
34. $A^2B - BC^2 + A^2C - C^3$.
35. $K^3 - K^2 - 4K - 6$.
36. $n^3 - 7n + n^2 - 3$.
37. $D^3 + 15 - 17D + D^2$.
38. $432 + 2F^3$.
39. $1 - 2x + x^2 + 2ab - b^2 - a^2$.
40. $P^4 + P^2 + 1$.
41. $ax^2 + (ab + 1)x + b$.
42. $2t^2 - (2a + b)t + ab$.
43. $2H^2 + (4a + b)H + 2ab$.
44. $az^2 + (b - a)z - b$.
45. $x^4y^4 + 9ax^2y^2 + 14a^2$.
46. $220 - 2V - 2V^2$.
47. $20c^3 - 60c^2 + 45c$.
48. $1 + 32Q^5n^{10}$.
49. $1 - d^9$.
50. $y^3 - 11y^2 + 31y - 21$.
51. $L^3 - 6L^2 + 11L - 6$.
52. $y^2 - 9z^2 + 25x^2 - 10xy$.

53. $3(a-b)^2-14(a-b)+8$.

54. $7(a+b)^2-11\,c(a+b)-6\,c^2$.

55. $x^4-x^3-17\,x^2+5\,x+60$.

56. a^3-a^2A-a+A.

57. $2\,m^3b+3\,m^2b-8\,mb-12\,b$.

58. $a^{10}+ab^9$.

59. $v^{16}-t^{16}$.

60. $P^4-P^2+12\,PQ-36\,Q^2$.

61. $1-n+n^6-n^7$.

62. $289\,g^7-83\,g^5+g^3$.

63. x^3+x+y^3+y.

64. $M-N+M^3-N^3$.

65. $1-a^{21}$.

66. $x^2+4\,x+4-4\,a^2+4\,ay-y^2$.

28. Highest Common Factor. — Since the *highest common factor* of two or more expressions must contain all factors common to the expressions, and no others,

To find the highest common factor of two or more expressions, find the prime factors of each. Then form the product of all prime factors that are common to the expressions, using each factor the least number of times that it occurs in any one of the expressions.

EXERCISES

Find the highest common factor of:

1. 24, 42, 54, and 78.

2. 36, 84, 120, and 108.

3. 216, 360, 504, and 576.

4. $12\,a^3b^4$, $18\,a^4b^3$, and $24\,a^3b^3c$.

5. $16\,m^2na^4$, $42\,ma^3$, and $18\,m^3a^2b$.

6. x^2-1, $x^2-3\,x+2$, and x^2+x-2.

7. V^3-1, $V^2-2\,V+1$, and $V^2-3\,V+2$.

8. a^5+b^5, a^3+b^3, and a^2-b^2.

9. x^5-xy^2, and $x^3+x^2y+xy+y^2$.

10. $7\,N^3-2\,N^2-5$ and $7\,N^3+12\,N^2+10\,N+5$.

11. $p^2+3\,pq+2\,q^2$, $p^2+5\,pq+4\,q^2$, and $p^2-6\,pq-7\,q^2$.

12. $1-x^6$, $1-2\,x^3+x^6$, and $1+x+x^2$.

13. $R^2-7\,R+10$, and $4\,R^3-25\,R^2+20\,R+25$.

14. $6\,v^2t+4\,vt^2-2\,t^3$, and $8\,v^3+4\,v^2t-4\,vt^2$.

15. $4\,P^2+20\,P$, $P^3+4\,P^2-5\,P$, and $4\,aP^2+20\,aP$.

16. m^2+m-6, $m^3-2\,m^2-m+2$, $m^3+3\,m^2-6\,m-8$.

29. Lowest Common Multiple. — The lowest common multiple of two or more expressions must contain all factors of the expressions, and no others. Hence,

To find the lowest common multiple of two or more expressions, find the prime factors of each. Then form the product of all of the different kinds of factors, using each kind of factor the greatest number of times that it occurs in any one of the expressions.

EXERCISES

Find the lowest common multiple of:

1. 18, 24, 42, and 54.
2. 30, 45, 75, and 120.
3. 32, 56, 63, and 112.
4. $36\,a^{10}b$, $9\,a^2b^5$, and $16\,a^3b^2$.
5. $25\,\pi R^2(R-r)(2\,R+r)^2$, and $45\,\pi Rr(2\,R+r)^3$.
6. x^2-1, and $x^2-2\,x+1$.
7. x^3-y^3, x^3-xy^2, and $2\,x^2y$.
8. N^4-1, $N^2+3\,N+2$, and N^2+N-2.
9. m^2-n^2, m^3-n^3, and m^4-n^4.
10. $1-V^2$, $1-V^4$, and $1-V^8$.
11. $a^3+2\,a^2-3\,a$, $2\,a^3+6\,a^2+2\,a+6$, and $3\,a^3-3\,a^2$.
12. P^4-Q^4, $P^4+2\,P^2Q^2+Q^4$, and $P^4-2\,P^2Q^2+Q^4$.
13. w^2-v^2, $w^3+w^2v-wv^2-v^3$, and $w^3-w^2v-wv^2+v^3$.
14. $y^4-13\,y^2+36$, and $y^4+y^3-7\,y^2+y+6$.
15. $27\,T^3-8$, $9\,T^2-4$, and $9\,T^2-12\,T+4$.
16. $6\,M^3-7\,M^2N-2\,MN^2$, and $3\,M^2+MN-4\,N^2$.

SUPPLEMENTARY EXERCISES

Factor:

1. $1-x-x^2+x^3$.
2. $4(a-b)^3-a+b$.
3. $4\,c^3-c^2-8\,c+2$.
4. $x^{16}-y^{16}$.
5. $2^{5x}-32$.
6. x^3y^3-512.

7. $a^{3n} - a^{2n}b^n$.

8. $x^{2n} - x^n - 2$.

9. $t^{4n} + 3\,t^{2n}v^n - 28\,v^{2n}$.

10. $p^{2n} - q^{2n}$.

11. $A^{4n} - B^{4n}$.

12. $x^{2n+1} - xy^n$.

13. $V^{3t} - 1$.

14. $nk^{5n} - n^{6n}$.

15. $a^{2n} - b^{2n} - a^nc^n + b^nc^n$.

16. $6\,b^{2n} + b^nc^n - 15\,c^{2n}$.

17. $x^{2n-4} - y^{2n+4}$.

18. $\pi R^{2n} + (\pi r + 1)R^n + r$.

19. $V^{4t} + V^{2t} + 1$.

20. $N^{6a} + 1$.

21. $16^n + 2 \cdot 2^{2n} + 1$.

22. $a^4 - 11\,a^2b^2 + b^4$.

23. $4^n + 3 \cdot 2^n + 2$.

24. $a^{15} + b^{15}$.

25. $9^n - 4 \cdot 3^n - 5$.

26. $4^a - 16^b$.

27. $8^a - 27^b$.

28. $2 \cdot 3^2 - 9\,a - 2\,a^2$.

Find the highest common factor of:

29. $D^{2m} - 1$, and $D^{3m} - 1$.

30. $a^{5n} - a^nb^{2n}$, and $a^{3n} + a^{2n}b^n + a^nb^n + b^{2n}$.

31. $2\,V^{2t} + V^t - 6$, and $6\,V^{2t} - 7\,V^t - 3$.

32. $x^3 + 3\,x^2 - 7\,x - 21$, and $x^3 + 3\,x^2 - 3\,x - 9$.

33. $4\,ax - 6\,x - 2\,ay + 3\,y$, and $4\,a^2 - 10\,a + 6$.

34. $2\,a^2 - 15\,a + 25$, and $a^3 - 5\,a^2 - a + 5$.

Find the lowest common multiple of:

35. $x^3 - y^3$, $x^3 + y^3$, $x^6 - y^6$, and $x^{12} - y^{12}$.

36. $x^2 - 4$, $x^2 - 7\,x + 10$, and $x^3 - 5\,x^2 + 4x - 20$.

37. $9\,by^3 - 3\,by^2 - 42\,by$, and $6\,b^2 - 26\,b + 28$.

38. $a^{2n} - b^{2n}$, $a^{3n} - b^{3n}$, and $a^{4n} - b^{4n}$.

39. $4\,P^{2x} - 4\,P^x + 1$, $4\,P^{2x} - 1$, and $4\,P^{2x} + 4\,P^x + 1$.

40. $a^{n+1}x - a^{n+1}y - b^{n-1}x + b^{n-1}y$, and $x^2 - 2\,xy + y^2$.

CHAPTER III

FRACTIONS

30. Fractions. — A **fraction** is an indicated quotient of two number expressions. The dividend is called the **numerator** and the divisor the **denominator** of the fraction. The numerator and denominator are called the **terms** of the fraction.

31. The Signs of a Fraction. — In an algebraic fraction three signs are involved: (1) the sign before the fraction; (2) the sign of the numerator; and (3) the sign of the denominator.

Since a fraction is an indicated quotient, by the *laws of signs in division,* $\frac{+a}{+b}$ and $\frac{-a}{-b}$ are both positive and are equal, and $\frac{-a}{+b}$ and $\frac{+a}{-b}$ are both negative and are equal. Hence,

$$\frac{+a}{+b}=\frac{-a}{-b}=-\frac{-a}{+b}=-\frac{+a}{-b}.$$

It follows that to preserve the value of a fraction:

(1) *If the signs of both terms are changed, the sign before the fraction must be left unchanged.*

(2) *If the sign of only one term is changed, the sign before the fraction must be changed.*

If the numerator is a polynomial, its sign is changed by changing the sign of each of its terms.

If the numerator or denominator is the product of two or more factors, its sign is changed by changing the signs of an odd number of factors, and it is left unchanged by changing the signs of an even number of factors.

Thus, in $\frac{1-x}{(x-3)(x-4)}$ the denominator $(x-3)(x-4)=x^2-7x+12$. If the sign of $x-3$ is changed, by changing the signs of its terms, we get

$(3-x)(x-4)=-x^2+7x-12$. If the signs of both factors are changed, we get $(3-x)(4-x)=x^2-7x+12$.

EXERCISES

Write with positive numerators and denominators:

1. $\frac{-4}{7}$.
2. $\frac{6}{-9}$.
3. $\frac{-2}{-5}$.
4. $\frac{+8}{-9}$.
5. $\frac{-2x^2}{-3y^2}$.
6. $\frac{4ab}{-5c^3}$.
7. $\frac{-1}{t^2}$.
8. $\frac{-\pi r}{-R}$.

Write with denominators $m-n$:

9. $\frac{1}{n-m}$.

10 $\frac{-mn}{n-m}$.

11. $\frac{a-b}{n-m}$.

Write all of the fractions with the same denominator:

12. $\frac{3}{a-b}-\frac{4}{b-a}$.

13. $\frac{x}{x-1}+\frac{1}{1-x}-\frac{x+1}{1-x}$.

14. $\frac{1+2m}{m^2-1}-\frac{3m}{1-m^2}+\frac{2-m}{1-m^2}$.

15. $\frac{6\pi r}{r^2-s^2}-\frac{r^2}{s^2-r^2}+\frac{r^2+s^2}{r^2-s^2}$.

Write with the factors of denominators in descending powers of a, the first term of each factor being positive:

16. $\frac{1}{(a-b)(c-a)}$.

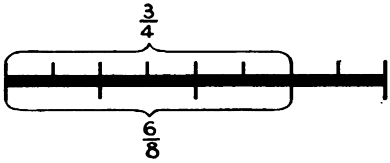
17. $\frac{a^3}{(a-b)(c-a)(d-a)}$.

18. $\frac{a+b+c+d}{(b-a)(c-a)(d-a)}$.

32. Reduction of a Fraction to an Equivalent Fraction. — *Multiplying or dividing both terms of a fraction by the same expression does not change the value of the fraction. That is,*

$$\frac{a}{b}=\frac{na}{nb} \text{ and } \frac{a}{b}=\frac{a\div n}{b\div n}.$$

This principle may be illustrated graphically as follows:



If each of the 3 fourths in $\frac{3}{4}$ is halved, they make 6 eighths. That is, $\frac{3}{4}=\frac{6}{8}=\frac{2\times 3}{2\times 4}$. Starting with $\frac{6}{8}$, if each of the 6 eighths in $\frac{6}{8}$ is doubled, they make 3 fourths. That is, $\frac{6}{8}=\frac{3}{4}=\frac{6\div 2}{8\div 2}$.

Multiplying or dividing both terms of a fraction by the same expression is called **reducing** the fraction.

33. Reduction of Fractions to Lowest Terms. — *To reduce a fraction to its lowest terms, factor the numerator and denominator, and divide both terms by their common factors.*

The division may be indicated by *cancellation* of the common factors of the terms.

Thus, $$\frac{a^3-ab^2}{a^4-ab^3}=\frac{\cancel{a}\cancel{(a-b)}(a+b)}{\cancel{a}\cancel{(a-b)}(a^2+ab+b^2)}=\frac{a+b}{a^2+ab+b^2}.$$

EXERCISES

Reduce to lowest terms:

1. $\frac{18}{24}$.
2. $\frac{15}{45}$.
3. $\frac{42}{49}$.
4. $\frac{64}{72}$.
5. $\frac{250}{350}$.
6. $\frac{45}{630}$.
7. $\frac{xy^2}{yz^2}$.
8. $\frac{4A}{8A^3}$.
9. $\frac{6a^3b^2}{9a^2b^3}$.
10. $\frac{2\pi R}{4\pi R^3}$.
11. $\frac{x^2-1}{x^3-1}$.
12. $\frac{a^3+b^3}{a^3+a^2b}$.
13. $\frac{m^4-n^4}{m^4+2m^2n^2+n^4}$.
14. $\frac{t^2-5t-14}{t^2+8t+12}$.
15. $\frac{10p^2-13pq-3q^2}{8p^2-2pq-15q^2}$.
16. $\frac{1+N^3}{N^3-4N^2-5N}$.
17. $\frac{w^3v-wv^3}{w^3v^2+w^2v^3}$.
18. $\frac{F^5+1}{(F+1)^5}$.
19. $\frac{A^2+AB+B^2}{A^4+A^2B^2+B^4}$.
20. $\frac{a^2+an+ax+nx}{a^2+an-ay-ny}$.

34. Reduction of Fractions to a Common Denominator. — Two or more fractions may be reduced to equivalent fractions that have the same denominator, called their **common denominator**. The lowest common denominator of two or more fractions is the lowest common multiple of their denominators.

To reduce two or more fractions to equivalent fractions with a common denominator, factor all denominators, and form their lowest common multiple for the common denominator. Multiply both terms of each fraction by the factor or factors of the common denominator that its denominator does not already contain.

EXERCISES

Reduce to equivalent fractions with the lowest common denominator:

1. $\frac{5}{6}, \frac{3}{4}, \frac{1}{2}$. 3. $\frac{2}{7}, \frac{1}{2}, \frac{2}{3}$. 5. $\frac{2}{15}, \frac{8}{25}, \frac{7}{45}$. 7. $\frac{1}{2}, \frac{16}{21}, \frac{25}{28}$.
2. $\frac{1}{3}, \frac{1}{4}, \frac{3}{8}$. 4. $\frac{5}{12}, \frac{3}{16}, \frac{7}{24}$. 6. $\frac{1}{8}, \frac{5}{36}, \frac{20}{27}$. 8. $\frac{9}{16}, \frac{2}{9}, \frac{4}{15}$.

9. $\dfrac{4}{3a^2}, \dfrac{1}{5a}, \dfrac{7}{12a^3}$. 10. $\dfrac{a}{b^2}, \dfrac{b}{c^2}, \dfrac{c}{a^2}$. 11. $\dfrac{m}{np}, \dfrac{p}{mn}, \dfrac{n}{mp}$.

12. $\dfrac{1}{1-x}, \dfrac{1}{1+x}, \dfrac{x}{1-x^2}$. 15. $\dfrac{t}{t^2-9}, \dfrac{t-1}{t^2-5t+6}$.

13. $\dfrac{a}{a+b}, \dfrac{b}{a-b}, \dfrac{ab}{a^2+b^2}$. 16. $\dfrac{R-3}{R^2-9R+14}, \dfrac{-3}{R^2-4}$.

14. $\dfrac{M}{N}, \dfrac{M^2}{MN-N^2}, \dfrac{MN}{M^2-N^2}$. 17. $\dfrac{m+n}{m^3-n^3}, \dfrac{2n}{m^2n-mn^2}$.

18. $\dfrac{3}{x^2+3x+2}, \dfrac{5}{2x^2+5x+2}, \dfrac{7}{2x^2+3x+1}$.

19. $\dfrac{1}{(a-b)(a-c)}, \dfrac{1}{(b-c)(b-a)}, \dfrac{1}{(c-a)(c-b)}$.

20. $\dfrac{y+z}{(y-x)(z-x)}, \dfrac{z+x}{(y-z)(y-x)}, \dfrac{x+y}{(z-x)(z-y)}$.

35. Addition and Subtraction of Fractions. — *To add or subtract fractions, reduce them to their lowest common denominator, then find the sum or difference of their numerators, and place the result over the common denominator.*

Before beginning to factor the denominators, it is best to rewrite the fractions, if necessary, so that the denominators are all arranged in descending or all in ascending powers of some letter of arrangement, with the first term of each denominator positive.

Example. — Simplify $\frac{n}{1+n}-\frac{n}{1-n}+\frac{n^2}{n^2-1}$.

Rewriting the first two fractions, with denominators in descending powers of n, making changes of sign as in § 31, the expression becomes

$$\begin{aligned}\frac{n}{n+1}+\frac{n}{n-1}+\frac{n^2}{n^2-1}&=\frac{n}{n+1}+\frac{n}{n-1}+\frac{n^2}{(n+1)(n-1)}\\&=\frac{n^2-n}{(n+1)(n-1)}+\frac{n^2+n}{(n+1)(n-1)}+\frac{n^2}{(n+1)(n-1)}\\&=\frac{3n^2}{(n+1)(n-1)}\text{ or }\frac{3n^2}{n^2-1}.\end{aligned}$$

36. Reduction of Mixed Expressions to Fractions. — An expression that contains no fractions with literal denominators is an **integral expression**; one that consists of one or more fractions is a **fractional expression**; and one consisting of an integral part and a fractional part is a **mixed expression.**

A mixed expression may be changed to a fraction by the process of § 35, the integral part being written in the form of a fraction with the denominator 1.

Example. — Reduce $m-n+\frac{2mn}{m-n}$ to a fraction.

$$\begin{aligned}m-n+\frac{2mn}{m-n}&=\frac{m-n}{1}+\frac{2mn}{m-n}\\&=\frac{m^2-2mn+n^2}{m-n}+\frac{2mn}{m-n}\\&=\frac{m^2+n^2}{m-n}.\end{aligned}$$

EXERCISES

Simplify by adding or subtracting, as indicated:

1. $\frac{4}{5}+\frac{3}{4}$. 2. $\frac{7}{16}-\frac{1}{12}$. 3. $\frac{5}{12}+\frac{2}{7}-\frac{1}{8}$. 4. $\frac{15}{16}-\frac{2}{5}+\frac{8}{15}$.

5. $\frac{3}{a^2}+\frac{1}{2a}$. 7. $\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}$. 9. $\frac{t}{1-t^2}-\frac{1}{t-1}$.

6. $\frac{1}{x}+\frac{x-1}{x^2}-\frac{3}{2x}$. 8. $\frac{N+1}{N-1}-\frac{N-1}{N+1}$. 10. $\frac{5}{s^2-s}+\frac{2}{s^2+s}$.

11. $\dfrac{r-2}{r^2+10r+16}+\dfrac{r+1}{r^2+9r+14}$. 13. $\dfrac{V-3}{V+3}+\dfrac{V+3}{V-3}+\dfrac{V^2}{9-V^2}$.

12. $\dfrac{2\pi R}{\pi R+\pi H}-\dfrac{2\pi R}{\pi R-\pi H}$. 14. $\dfrac{a^2}{a^3-b^3}-\dfrac{1}{b-a}$.

15. $\dfrac{1}{ab-ac-b^2+bc}+\dfrac{1}{bc-ab-c^2+ac}$.

16. $\dfrac{1}{(a-b)(b-c)}+\dfrac{1}{(b-a)(a-c)}-\dfrac{1}{(c-a)(c-b)}$.

17. $\dfrac{M^2}{(M-N)(M-P)}-\dfrac{N^2}{(P-N)(N-M)}+\dfrac{P^2}{(P-M)(N-M)}$.

18. $\dfrac{1}{R+r}-\dfrac{1}{R-r}+\dfrac{1}{R+3r}-\dfrac{1}{R-3r}$.

SUGGESTION. — First combine the first two fractions, then the last two, then add the results. This will save much multiplication.

19. $\dfrac{1}{n-4}+\dfrac{1}{n-5}-\dfrac{1}{n+4}-\dfrac{1}{n+5}$.

20. $\dfrac{2}{a-b}-\dfrac{4}{c+d}-\dfrac{2}{a+b}+\dfrac{4}{c-d}$.

21. $\dfrac{1}{x^2-9}-\dfrac{1}{x^2+9}+\dfrac{9}{x^2-1}-\dfrac{9}{x^2+1}$.

Reduce to a fraction:

22. $t-\dfrac{1}{t}$. 24. $\dfrac{a^2}{a-b}-a$. 26. $m+n-\dfrac{2mn}{m+n}$.

23. $\dfrac{V}{V+3}+2$. 25. $1-R+\dfrac{R^2}{1+R}$. 27. $y+2+\dfrac{3-y^2}{y-4}$.

28. $1+\dfrac{M}{N}+\dfrac{M^2}{N^2}$. 30. $\dfrac{a}{a-b}-\dfrac{b}{a-b}-1$.

29. $x^2+xy+y^2+\dfrac{y^3}{x-y}$. 31. $D^2-\dfrac{2D^2-D^4}{D^2-D+1}+D+1$.

37. Multiplication of Fractions. — As in arithmetical fractions, *The product of two or more fractions is the fraction whose numerator is the product of their numerators and whose denominator is the product of their denominators.*

In the multiplication of fractions, if a factor is common to a numerator and any denominator, the product may be reduced to lower terms by canceling this factor from the terms of the product. In such a case, work may be saved by factoring the *terms* of the fractions and canceling any factor common to any numerator and any denominator, before performing the multiplications.

Thus,
$$\begin{aligned}\frac{x^2-1}{x^2+5x+6}\times\frac{x^2+6x+8}{x^2+6x-7}&=\frac{\cancel{(x-1)}(x+1)}{(x+3)\cancel{(x+2)}}\times\frac{\cancel{(x+2)}(x+4)}{\cancel{(x-1)}(x+7)}\\&=\frac{(x+1)(x+4)}{(x+3)(x+7)}\text{ or }\frac{x^2+5x+4}{x^2+10x+21}.\end{aligned}$$

Integral, mixed, or fractional expressions to be multiplied should first be reduced to fractions.

Thus,
$$\begin{aligned}(m^2-n^2)\left(1-\frac{m}{m+n}\right)&=\frac{\cancel{(m+n)}(m-n)}{1}\times\frac{n}{\cancel{m+n}}\\&=mn-n^2.\end{aligned}$$

EXERCISES

Find the product of:

1. $\frac{3}{4}\times\frac{2}{5}$.
2. $\frac{2}{3}\times\frac{6}{7}$.
3. $6\times\frac{2}{3}$.
4. $10\times\frac{4}{5}$.
5. $\frac{1}{2}\times\frac{4}{9}\times\frac{3}{8}$.
6. $1\frac{5}{12}\times\frac{10}{16}\times\frac{9}{15}$.
7. $9\frac{1}{2}\times\frac{5}{12}\times\frac{2}{5}$.
8. $21\times\frac{25}{36}\times\frac{4}{35}$.
9. $\frac{a^2b}{cd}\times\frac{ac^2}{b^2d}\times\frac{d^3}{a^2c}$.
10. $\frac{x^2}{y^2}\times\frac{y^3z}{x^3}\times\frac{x}{yz}$.
11. $\frac{m^4}{n^2p}\times\frac{mn^2}{p^2}\times\frac{np^3}{m^2}$.
12. $\frac{x^4-1}{x^2+x+1}\times\frac{x^3-1}{x^2-1}$.
13. $\frac{v-t}{v+2t}\times\frac{v^2-2t^2}{v^3-t^3}$.
14. $\frac{n^2-14n-15}{n^2-4n-45}\times\frac{n^2-6n-27}{n^2-12n-45}$.
15. $\frac{\pi R^2+\pi RH}{R-H}\times\frac{H}{\pi R^2-\pi H^2}$.
16. $\frac{b^3-1}{b+1}\times\frac{b^2-1}{b^2+b+1}$.
17. $\left(1-\frac{27T^3}{V^3}\right)\left(\frac{1}{V-3T}\right)$.
18. $(A^2+A+1)\left(\frac{1}{A^3-1}\right)$.
19. $\left(\frac{a^2+ab+b^2}{a^3-b^3}\right)(a^2-b^2)$.
20. $\left(\frac{1}{N^2}-1\right)\left(1+\frac{1}{N^2}\right)$.
21. $\left(1+\frac{1}{F}+\frac{1}{F^2}\right)\left(\frac{1}{F^3-1}\right)$.

22. $\left(1-\frac{b^2}{a^2}\right)\left(\frac{1}{a-b}\right)$.

23. $\left(\frac{mn}{m^2+n^2}\right)\left(1-\frac{m^4}{n^4}\right)$.

24. $\left(\frac{h^2+hk}{h^2+k^2}\right)\left(\frac{h}{h-k}-\frac{k}{h+k}\right)$.

25. $\left(\frac{1}{P}-1\right)\left(\frac{P}{1+P}-\frac{1-P}{P}\right)$.

Multiply by the lowest common denominator:

26. $\frac{1}{n-1}+\frac{1}{n+1}$.

27. $\frac{A}{A^2-B^2}-\frac{1}{A+B}$.

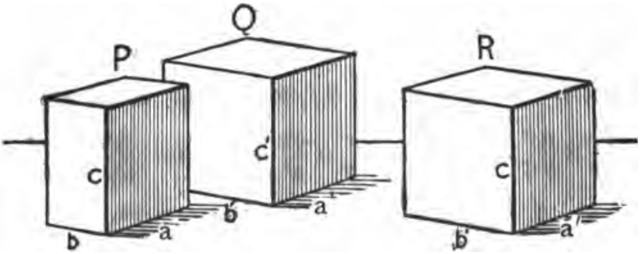
28. $\frac{1}{x-y}-\frac{xy+2y^2}{x^3-y^3}$.

29. $\frac{2K^2}{K^4-1}+\frac{1}{K^2+1}-\frac{1}{K^2-1}$.

30. $\frac{2v+1}{v^3-1}-\frac{3}{v^2-1}$.

31. $4-\frac{7M-2}{M^2-M}$.

32. P, Q, and R are the volumes of the three rectangular solids shown in the figure. It is known from geometry that $\frac{P}{R}=\frac{ab}{a'b'}$ and that $\frac{R}{Q}=\frac{c}{c'}$.



Show by multiplication that $\frac{P}{Q}=\frac{abc}{a'b'c'}$.

38. Division of Fractions. — *To find the quotient of two fractions multiply the dividend by the divisor inverted.*

That is, $$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}.$$

For, the product of the quotient $\frac{a}{b}\times\frac{d}{c}$ and the divisor $\frac{c}{d}$ equals the dividend $\frac{a}{b}$, by cancellation.

If the dividend or divisor is an integral, mixed, or fractional expression, it must be changed to a fraction, and the division performed by the above rule.

EXAMPLE. — $\left(1-\frac{x^2}{y^2}\right)\div\frac{x+y}{xy}=\frac{y^2-x^2}{y^2}\times\frac{xy}{x+y}=\frac{xy-x^2}{y}.$

EXERCISES

Perform the indicated divisions:

1. $\frac{3}{4}\div\frac{6}{7}$. 3. $12\div\frac{3}{5}$. 5. $21\div\frac{7}{9}$. 7. $\frac{16}{27}\div 3\frac{1}{3}$.
2. $\frac{15}{16}\div\frac{9}{10}$. 4. $\frac{3}{8}\div 4$. 6. $3\frac{2}{3}\div\frac{5}{6}$. 8. $1\frac{5}{6}\div 6\frac{7}{8}$.

9. $\frac{4m^2n}{5pq^2}\div\frac{6n^2}{q^2}$.

10. $\frac{4\pi R^2}{9D}\div\frac{2\pi R}{3D}$.

11. $\frac{x^2-1}{x}\div\frac{x+1}{x^2}$.

12. $\frac{9-V^2}{2+V}\div\frac{6-2V}{4-V^2}$.

13. $\frac{a^3+b^3}{2ab}\div\frac{a^2+ab}{2b}$.

14. $\frac{t^2-1}{t}\div(1+t)$.

15. $\frac{m^3-n^3}{6m^2+mn-n^2}\div\frac{4m^2+mn-5n^2}{6m^2-5mn-4n^2}$.

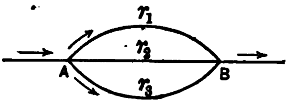
16. $\left(1-\frac{P^3}{Q^3}\right)\div\frac{P-Q}{PQ}$.

17. $(a^2+ab+b^2)\div\frac{a^4-ab^3}{b}$.

18. $\left(x+1+\frac{3}{x-1}\right)\div\left(x+2-\frac{1}{x-1}\right)$.

19. $\left(\frac{M^2+N^2}{N}-M\right)\div\left(\frac{1}{N}-\frac{1}{M}\right)$.

20. $\left(\frac{k^2}{t^2}-\frac{2k}{t}+1\right)\div\left(1-\frac{k^3}{t^3}\right)$.



21. If a conductor of an electrical current is divided between the points A and B into three branches with resistances r_1, r_2, r_3, respectively, the total resistance of the conductor between A and B is $1\div\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)$. Simplify this expression.

39. Complex Fractions. — Fractions one or both of whose terms contain fractions are called **complex fractions.** A complex fraction may be simplified by reducing it to a simple fraction or an integral expression. Since a complex fraction is an indicated quotient,

To simplify a complex fraction, reduce both terms to simple fractions, then perform the indicated division.

A complex fraction of the special form indicated in the example below is called a **continued fraction.** To simplify such a fraction, begin with the last or lowest complex fraction, and simplify step by step.

Example. — Simplify $\dfrac{1}{x+\dfrac{1}{x-\dfrac{1}{x}}}$.

Simplifying $\dfrac{1}{x-\dfrac{1}{x}}$, $\quad \dfrac{1}{x+\dfrac{1}{x-\dfrac{1}{x}}}=\dfrac{1}{x+\dfrac{x}{x^2-1}}=\dfrac{x^2-1}{x^3}$.

EXERCISES

Simplify:

1. $\dfrac{\frac{7}{8}}{\frac{9}{16}}$.

2. $\dfrac{8}{\frac{5}{6}}$.

3. $\dfrac{3\frac{1}{3}}{2\frac{1}{2}}$.

4. $\dfrac{\dfrac{M}{N}}{\dfrac{M^3}{N^3}}$.

5. $\dfrac{\dfrac{x^3}{y^2}}{x}$.

6. $\dfrac{1-x}{\dfrac{1}{x^2}-1}$.

7. $\dfrac{\dfrac{1}{A+B}}{\dfrac{1}{A-B}}$.

8. $\dfrac{1-\dfrac{m}{n}}{1+\dfrac{m}{n}}$.

9 $\dfrac{\dfrac{b}{c}-\dfrac{c}{b}}{1+\dfrac{c}{b}}$.

10. $\dfrac{\dfrac{a}{a+b}-1}{a+b}$.

11. $\dfrac{2t+\dfrac{t^2+s^2}{s}}{\dfrac{t}{s}+1}$.

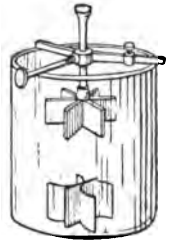
12 $\dfrac{\dfrac{a^6+b^6}{a^6-b^6}}{\dfrac{a^4-a^2b^2+b^4}{a^4+a^2b^2+b^4}}$.

13. $\dfrac{\dfrac{m}{n}+1+\dfrac{n}{m}}{\dfrac{m^2}{n^2}+1+\dfrac{n^2}{m^2}}$.

14. $\dfrac{1}{x+\dfrac{1}{1+\dfrac{x+1}{1-x}}}$.

15. $\dfrac{1}{a-\dfrac{a^2-1}{a+\dfrac{1}{a-1}}}$.

16. $\dfrac{2}{2+\dfrac{2}{2+\dfrac{2}{2+\dfrac{2}{x}}}}$.



17. If n equal electric cells are connected by wires in one way (parallel), so as to form a battery, the current strength produced by the battery is $\dfrac{E}{R+\dfrac{r}{n}}$. Simplify this fraction.

18. It has been shown that π, the ratio of the circumference to the diameter of a circle, is equal to the continued fraction in the margin. Using only so much of the fraction as is here written, find the value of π approximated to hundredths.

$$\frac{4}{1+\dfrac{1}{2+\dfrac{9}{2+\dfrac{25}{2+\dfrac{49}{2+\dfrac{81}{2+\text{etc.}}}}}}}$$

SUPPLEMENTARY EXERCISES

Simplify:

1. $\dfrac{n^2-100}{n^2-n-90} \div \dfrac{n^2+10n}{n^3+9n^2} \div \dfrac{2n^2t+3nt}{4n+6}$.

2. $\left(1-V+\dfrac{2V^2}{1+V}\right) \div \left(\dfrac{1}{1+V}-\dfrac{1}{1-V}\right) \times \left(\dfrac{3V^2}{V^4-1}\right)$.

3. $\left(\dfrac{A}{A-B}-\dfrac{B}{A+B}\right)\left(\dfrac{1}{A^2}-\dfrac{1}{B^2}\right) \div \left(\dfrac{1}{A^2}+\dfrac{1}{B^2}\right)$.

4. $\left(\dfrac{x^2-y^2-z^2-2yz}{x^2-xy-xz}\right) \div \left(1-\dfrac{2z}{x+y+z}\right)$.

5. $\left(\dfrac{5a^2}{b}+b\right)\left(\dfrac{b^2}{5a^2}-1\right) \div \dfrac{b}{a}+\dfrac{a+b}{b}$.

6. $\left(\dfrac{m}{n}-\dfrac{n}{m}\right) \div \left(\dfrac{n}{m}-\dfrac{m}{n}\right)+\dfrac{m^2n^2-n^4}{m+n} \times \dfrac{1}{mn^2-n^3}$.

7. $\left(\frac{a+b}{b+c}-\frac{a+b}{a+c}\right)\left(\frac{a}{a+b}+\frac{c}{a-b}-\frac{a^2-2ab-c^2}{a^2-b^2}\right).$

8. $\left(\frac{2a-2n}{a+n}-\frac{a+n}{a-n}\right)\left(\frac{1}{a}+\frac{1}{n}\right)-\left(\frac{1}{a}-\frac{1}{n}\right)\div\left(\frac{1}{a}+\frac{1}{n}\right).$

9. $\left(\frac{w^2+v^2}{w^2-v^2}-\frac{w-v}{w+v}\right)\left(\frac{w+v}{2wv}\right)+\frac{w^2-1-v^2}{w-v}.$

10. $\frac{a^3-x^3}{a^3+x^3}\times\frac{a^2-x^2}{a^2+x^2}\times\frac{a-x}{a+x}\times\frac{a^2-ax+x^2}{a^2+ax+x^2}\times\frac{a^2+2ax+x^2}{a^2-2ax+x^2}.$

11. $\frac{x+1}{y}\times\frac{y^2+2y+1}{x^2-1}\times\frac{(x-1)y}{(y+1)^2}.$

12. $\frac{a+b}{m}\div\frac{b^2-a^2}{m^2}\times\left(\frac{1}{x}\div\frac{m}{a-b}\right).$

13. $\left(1+\frac{1}{a}\right)\div\left(a-\frac{1}{a}\right)\times\left(1-\frac{1}{a}\right).$

14. $\dfrac{a-\dfrac{a-x}{1+ax}}{1+\dfrac{a(a-x)}{1+ax}}.$

15. $\dfrac{\dfrac{m^2+mn+n^2}{m^3+n^3}}{\dfrac{m^3-n^3}{m^2-mn+n^2}}.$

16. $\dfrac{\dfrac{a^6+b^6}{a^3+b^3}}{\dfrac{a^6-b^6}{a^3-b^3}}.$

17. $\dfrac{\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}.$

18. $\dfrac{\dfrac{AB-B^2}{A^2-AB}+\dfrac{A^2-B^2}{AB}}{\dfrac{A(A+B)}{B(A-B)}}\div\dfrac{1}{\dfrac{(A+B)^2}{A^2-B^2}}.$

CHAPTER IV

LINEAR EQUATIONS IN ONE UNKNOWN NUMBER

40. The Equation. — An **equation** is the statement that two expressions are equal in value.

Thus, $a^2 - b^2 = (a+b)(a-b)$, and $2N + 4 = 5N + 1$ are equations.

The two expressions connected by the sign $=$ are called the **members** of the equation.

If an equation is true for all particular values of the literal numbers involved, it is called an **identical equation**, or simply an **identity**. In the work of the preceding chapters all of the processes really consisted of establishing identities.

Thus, $a^2 - b^2 = (a+b)(a-b)$ is true when $a = 2$ and $b = 1$, when $a = 5$ and $b = 2$, and for any values whatsoever that may be assigned to a and b, and hence is an identity. This identity was established in *factoring*.

If an equation is not true for all particular values of the literal numbers involved, it is called a **conditional equation**, or simply an **equation**.

Thus, $5t - 4 = 2t + 8$ is true only when t is 4.

It is by use of conditional equations that problems are solved. The unknown numbers, whose values are sought, are represented by letters, and the relations between the unknown numbers and the known numbers of the problems are expressed by means of equations. An equation is **solved** when the values of the unknown number are found that reduce the equation to an **identity**.

41. Integral and Fractional Equations. — If no unknown number of an equation be involved in the denominator of a fraction, the equation is **integral**. If one or more unknown numbers of an equation be involved in the denominators of one or more fractions, the equation is **fractional**.

Thus, $6A^2 - 4 = 7A$ is *integral*, and $\frac{1}{x} - \frac{1}{y} = 8$ is *fractional*.

42. The Degree of an Equation. — The **degree of an integral equation** is the greatest sum of the exponents of its unknown numbers in any one term. If an equation has only one unknown number, its degree is the largest exponent of the unknown number in the equation.

Thus, $V + 5 = 9 - 2V$ is of the *first degree.*

$x^2 + 7x - 2 = 0$ is of the *second degree.*

$x^3 - xy^2 = 16x + 2$ is of the *third degree.*

$m^2n^2 + 2mn = 9$ is of the *fourth degree.*

An equation of the first degree is called a **linear equation.**

One of the second degree is called a **quadratic equation.**

One of the third degree is called a **cubic equation.**

43. Solving an Equation in One Unknown Number. — If a conditional equation contains only one unknown number, the values of the unknown number for which the equation is true are called the **roots** of the equation, and are said to **satisfy** the equation.

The process of finding the values of the roots of an equation is called **solving the equation.**

As shown in the FIRST COURSE, equations may be solved by applying one or more of the following principles, called **axioms**:

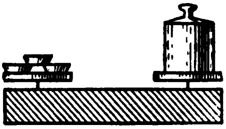
I. *If equal numbers are added to equal numbers, the sums are equal.*

II. *If equal numbers are subtracted from equal numbers, the remainders are equal.*

III. *If equal numbers are multiplied by equal numbers, the products are equal.*

IV. *If equal numbers are divided by equal numbers (not zero), the quotients are equal.*

The use of these axioms in solving an equation may be compared with changing the weights on the two pans of a balance. Just as adding equal weights to both pans, removing equal weights from both pans, etc., when the instrument is balanced will keep it balanced, so adding equal expressions to, or subtracting equal expressions from, both members of an equation, etc., will preserve the equality.



EXAMPLE. — Solve $6E - 5 = 4E + 1$.

Adding 5 to each member, $6E = 4E + 6$ Ax. I.

Subtracting $4E$ from each member, $2E = 6$. Ax. II.

Dividing each member by 2, $E = 3$. Ax. IV.

EXERCISES

Solve by use of axioms:

1. $n + 5 = 8$.
2. $t - 3 = 7$.
3. $4V = 6 + 3V$.
4. $3x + 2 = x + 4$.
5. $7P - 1 = 2P - 11$.
6. $20k - 5 = 5k + 25$.
7. $7N + 13 = 12N - 35 + N$.
8. $\frac{a}{6} = 4$.
9. $\frac{x}{2} + \frac{x}{3} + x = 2$.
10. $\frac{2H}{5} - \frac{3H}{4} = 1 + \frac{H}{15}$.
11. $12 - \frac{4m}{5} + \frac{m}{12} = 2m$.

44. Equivalent Equations. — Two equations that have the same roots are called **equivalent equations.**

Thus, $2t - (5t + 5) = 7$, and $4t + 1 = t - 11$ are equivalent, because each is satisfied when $t = -4$, and for no other values of t.

It is evident that the process of solving an equation by use of the axioms in § 43 consists of deriving from the given equation an equation equivalent to it, then deriving from the new equation a third one equivalent to the second, etc., until finally an equation equivalent to the given equation is obtained that shows the value of the unknown number directly.

In general, if the same number expression is added to or subtracted from both members of an equation, an equivalent equation is derived. An equivalent equation is derived also if both members of an equation are multiplied or divided by any expression that is free of the unknown number. But, if both members are multiplied or divided by an expression containing the unknown number, the equation derived may not be equivalent to the given equation.

Example 1.—Let $n + 3 = 5$. (1)

Multiplying by $n - 6$, $n^2 - 3n - 18 = 5n - 30$. (2)

Now, (1) is satisfied only when $n = 2$. But (2) is satisfied when $n = 2$ or when $n = 6$. Hence (2) is not equivalent to (1).

Example 2.—Let $t^2 - 5t + 4 = 2t - 2$. (1)

Dividing by $t - 1$, $t - 4 = 2$. (2)

Now, (1) is satisfied when $t = 1$ or $t = 6$. But (2) is satisfied only when $t = 6$. Hence (2) is not equivalent to (1).

The above examples illustrate the following principles:

(1) *If both members of an equation are multiplied by an expression containing the unknown number, one or more extraneous roots may be introduced.*

(2) *If both members of an equation are divided by an expression containing the unknown number, one or more roots may be lost.*

Hence, in solving an equation, if both members are multiplied by an expression containing the unknown number, all roots obtained should be tested to see if they satisfy the original equation, and any root that does not should be discarded. And, to avoid losing roots, when the members of an equation are divided by an expression containing the unknown number, the roots obtained by equating the divisor to zero should be kept.

45. Transposition.—In the equation

$$a - b = c - d,$$

by adding b, $a = c - d + b$;

or adding d, $a - b + d = c$;

or subtracting a, $-b = c - d - a$;

or subtracting c, $a - b - c = -d$.

It is evident from the above equations that the effect of adding a term to, or subtracting a term from, each member of an equation is to make it disappear from its original position and to reappear in the other member of the equation with its sign changed.

The mechanical process of changing a term from either member of an equation to the other and changing its sign, which is equivalent to adding it to or subtracting it from both members, is called **transposition.** Hence,

A term of an equation may be transposed from either member to the other member by changing its sign.

46. Solution of Linear Equations in One Unknown Number. — To solve any linear equation in one unknown number:

(1) *If either member contains signs of grouping, remove them.*

(2) *Transpose all unknown terms to the left member and all known terms to the right member, changing the sign of each term transposed.*

(3) *Combine similar terms in each member.*

(4) *Divide each member by the resulting coefficient of the unknown number.*

(5) *Check the work by replacing the letter representing the unknown number in the original equation by the root found, and seeing if the two members have the same value.*

EXERCISES

Solve:

1. $5n - 4 = 3n + 8.$
2. $9V = 1 - V + 9.$
3. $4(x - 1) = 3(x + 1) - 4.$
4. $7(3 - 4A) = 5(4 - 3A).$
5. $5(R - 1) + 3(R + 1) = 0.$
6. $6t - (3t + 2) = 1.$
7. $N - (6 - 2N) = 9(N - 1).$
8. $2(12 + y) - (y - 3) = 0.$
9. $(P + 2)^2 - P^2 = P - 5.$
10. $5 + (a - 3)^2 = (a + 1)^2.$
11. $7(r + 2) - 6(r + 1) = 5r - 8(r + 3).$
12. $(x + 3)(x - 2) = (x - 5)(x + 1) + 29.$
13. $(K - 4)(K - 5) = (K + 9)(K - 8) + 2.$
14. $(b - 6)^2 = b(b - 22) + 136.$
15. $2y(y - 7) = (y + 8)(2y - 7).$
16. $(1 - 4B)(3B + 7) = 1 + (4B + 3)(5 - 3B).$

47. Fractional Equations. — To solve a fractional equation it is necessary first to clear it of fractions.

To clear an equation of fractions, multiply each member by the lowest common denominator of all fractions in the equation.

Caution. — Test all roots by substituting in the original equations to see that they were not introduced by clearing of fractions. (See (1), § 44, and (5), § 46.)

EXERCISES

Solve:

1. $\frac{1}{2a}+\frac{5}{8}=\frac{3}{a}$

2. $\frac{1}{x}+\frac{2}{x}=3-\frac{3}{x}.$

3. $\frac{m+4}{3m}=\frac{3m-8}{4m}.$

4. $\frac{4}{2t+6}=\frac{3}{4t-8}.$

5. $\frac{b}{b-1}-\frac{3}{b+1}=1.$

6. $\frac{V}{V+1}+2=\frac{3V}{V+2}.$

7. $\frac{2R+1}{2R-1}-\frac{2R-1}{1+2R}=\frac{8}{4R^2-1}.$

8. $\frac{t+1}{t-1}+\frac{t^2-3}{1-t^2}=\frac{4}{1+t}.$

9. $\frac{7}{a^2-3a-10}-\frac{2}{a^2+a-2}=\frac{12}{a^2-6a+5}.$

10. $\frac{k^2}{k^2+2k-15}=\frac{2k+10}{4k-12}+\frac{3k-1}{6k+30}.$

11. $\frac{1}{x-1}-\frac{1}{x-2}=\frac{1}{x-3}-\frac{1}{x-4}.$

Suggestion. — Perform the subtraction indicated in each member, before clearing of fractions.

12. $\frac{n-2}{n-8}-\frac{n-4}{n-10}=\frac{n-6}{n-12}-\frac{n-8}{n-14}.$

13. $\dfrac{W+7}{W+9}+\dfrac{W+15}{W+17}=\dfrac{W+13}{W+15}+\dfrac{W+9}{W+11}.$

14. $\dfrac{\dfrac{x}{x-1}-1}{\dfrac{x}{x-1}+1}=\dfrac{\dfrac{x}{x+1}-1}{\dfrac{x}{x+1}+1}.$

48 Literal Equations: Formulæ. — If an equation contains other letters than the one representing the unknown number, it is a **literal equation.** A literal equation may be solved by the same process as the equations in the preceding sections. The root, or value of the unknown number, will be an expression containing the other letters involved in the equation.

Many practical formulæ are solved as literal equations.

Example. — The formula $C=\dfrac{nE}{R+nr}$ is used in electrical work. Solve it for n.

Clearing of fractions, $CR+nCr=nE.$

Transposing, $nCr-nE=-CR.$

Combining terms (factoring), $(Cr-E)n=-CR.$

Dividing by $Cr-E$, $n=\dfrac{-CR}{Cr-E}$ or $\dfrac{CR}{E-Cr}$

EXERCISES

1. Solve $5mx+n^2=m^2$ for x.

2. Solve $P(n+1)+P=n$ for n.

3. Solve $V^2+1+d(1-V)=V(V+d)$ for V.

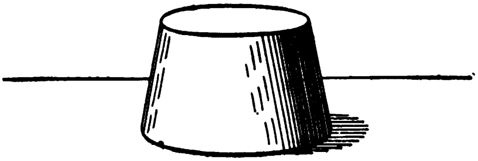
4. Solve $n=\dfrac{nx+y}{a}$ for x.

5. Solve $\dfrac{1}{A}+\dfrac{1}{B}=\dfrac{1}{C}+1$ for A.

6. Solve $\dfrac{at}{a^2-b^2}-\dfrac{3}{a-b}=\dfrac{t}{4a+4b}$ for t.

7. The area of a trapezoid is expressed by $A=\frac{1}{2}h(b+b')$. Solve for b.

8. The area of the conical surface of a frustum of a right circular cone is computed by the formula $S = \pi L(R + r)$. Solve for R; for r; for L.



9. The area of the entire surface of the solid in Ex. 8 is computed by the formula $S = \pi L(R + r) + \pi(R^2 + r^2)$. Solve for L.

10. The volume of the solid in Ex. 8 is computed by the formula $V = \frac{1}{3}\pi H(R^2 + r^2 + Rr)$. Solve for H.

11. A formula for the use of lenses is $\frac{1}{D} = \frac{1}{F - d} - \frac{1}{f}$. Solve for F; for f; for D; for d.

12. A formula used in electrical work is

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} + \frac{1}{r_5}.$$

Solve for R; for r_1, etc.

13. In the formula $H = KND(D - 1)(R + 2)$, for computing the horse power of gasoline engines, solve for K; for N; for R.

14. In the formula $P = H\left(\frac{7.6}{460 + T} - \frac{7.9}{460 + t}\right)$, for computing the draft in a chimney, solve for H; for T; for t.

15. In the formula $\frac{VP}{273 + t} = \frac{V'P'}{273 + t'}$, which expresses the relation between the pressure, volume, and temperature of gases, solve for V; for V'; for P; for P'; for t; for t'.

49. Problems. — Some problems may be solved by use of linear equations in one unknown number. The procedure is as follows:

(1) *Let some letter represent one of the unknown numbers in the problem.*

(2) *Express in terms of this letter and the known numbers given or implied in the problem every quantity involved in the problem.*

(3) *Express by an equation the relation between these quantities that is either directly stated or implied in the problem.*

(4) *Solve the equation, and from the root found determine the values of all numbers sought.*

Example. — Two grades of coffee costing the dealer 20 cents and 28 cents per pound, respectively, are to be mixed so that the blend shall be worth 25 cents per pound. How many pounds of each grade must be taken to make 50 pounds of the blend?

Let n = number of pounds of 20-cent grade.
Then $50 - n$ = number of pounds of 28-cent grade.
$20\,n$ cents = cost of n pounds of 20-cent grade.
$28(50 - n)$ cents = cost of $50 - n$ pounds of 28-cent grade.
Hence $20\,n + 28(50 - n)$ cents = cost of 50 pounds of blend.
But 50 pounds of blend cost 50×25 cents, or 1250 cents.
Hence, $20\,n + 28(50 - n) = 1250.$
Solving, $n = 18\tfrac{3}{4}.$
And $50 - n = 31\tfrac{1}{4}.$

EXERCISES

1. Find two consecutive even numbers whose product exceeds the square of the smaller by 48.

2. How many pounds each of two grades of coffee costing 20 cents and 28 cents, respectively, a pound, must a dealer use to make 100 pounds of a blend worth 25 cents a pound?

3. If one grade of coffee costs 24 cents a pound and another grade costs 30 cents a pound, how many pounds of each grade must be taken to make 9 pounds of a blend worth 28 cents a pound?

4. How many pounds each of tea worth 50 cents a pound and tea worth 80 cents a pound must be mixed to make 10 pounds of mixture worth 60 cents a pound?

5. A money drawer contains $2.40, consisting of nickels and dimes. There are 33 coins in all. How many coins of each kind are there?

6. A man made two investments together amounting to \$10,000. On the first investment he made annually 6 %, and on the other 5 %. His total income annually from both investments was \$535. Find the amount of each investment.

7. \$25,000 is invested in two places, part at 4 % and the rest at $5\frac{1}{2}$ %. The annual income from the two investments is \$1135. How much in each investment?

8. I invested \$12,000, part in 4 % bonds and the rest in a business enterprise that paid me 8 % annually. My annual income was \$800. Find the amount of each investment.

9. How many gallons of cream containing 26 % butter fat and milk containing 4 % butter fat must be mixed to make 60 gallons of milk testing 6 % butter fat?

10. How many gallons of cream containing 28 % butter fat and milk containing 4.5 % butter fat must be mixed to make 20 gallons of cream containing 25 % butter fat?

11. A certain solution contains 15 % alcohol. How many gallons of other ingredients must be added to 50 gallons of it so that the new mixture shall contain only 10 % alcohol?

12. Ordinary commercial alcohol is 95 % pure. How much water must be added to 20 quarts of it so that the diluted alcohol is only 80 % pure?

13. In an alloy of copper and tin weighing 24 pounds, 35 % is copper. How many pounds of copper must be added to this so that the new alloy is 40 % copper?

14. Tin appears to lose $\frac{2}{15}$ of its weight in air when immersed in water, and zinc appears to lose $\frac{1}{10}$ of its weight in air when immersed in water. If I find that a mass of alloy of these two metals weighing 50 pounds in air appears to weigh only 44 pounds in water, how many pounds of each metal in the alloy?

15. A man is 36 years old and his son 6 years old. In how many more years will the man be only twice as old as his son?

16. At what advance must a merchant mark an article costing him $12 in order that he may then sell it at a reduction of 25 % from the marked price, and yet make a profit of 20 % on it?

17. A milliner advertises a lot of hats that cost her $2.50 each for special sale at a reduction of 40 % from the regular marked price. At what advance must she first mark the hats in order that she may give this reduction and still make a profit of 20 %?

18. A teamster has one horse weighing 1200 pounds and another weighing 1500 pounds. If their draft power is proportional to their weight, how must he divide the 48-inch doubletree in order to justly distribute the load?

Suggestion. — The draft power of one multiplied by the number of units from the end of doubletree to the point of division equals the draft power of the other multiplied by the distance to the same point.

19. A messenger going 4 miles an hour has been gone 3 hours when it is found that the message is wrong. A second messenger, riding at an average of 6 miles an hour, is sent to overtake him. In how many hours will the second messenger overtake the first, and at what point?

20. An automobile was driven 125 miles in 4 hours 20 minutes. Part of the distance was in the country at a speed of 30 miles per hour, and the rest in the city at a speed of 15 miles per hour. Find how many hours of the drive were in the country.

21. A bullet with a velocity of 1650 feet per second is heard by the marksman to strike an object $2\frac{1}{2}$ seconds after it is fired. Sound travels approximately 1100 feet per second. How far away was the object that was hit?

22. Find the time between 2 and 3 o'clock when the hands of a clock are together.

Suggestion. — Represent by some letter the number of minute spaces on the clock face from the 12 o'clock point around to the point where the hands are together.

23. Find the time between 3 and 4 o'clock when the hands of a clock are opposite each other.

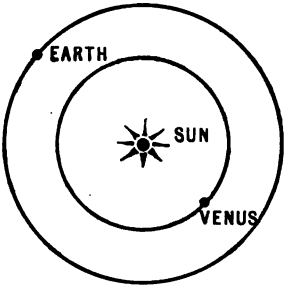
24. Find the time between 8 and 9 o'clock when the hands of a clock are opposite each other.

25. Find the time between 7 and 8 o'clock when the minute hand is 15 minute spaces ahead of the hour hand.

26. How long is it from the time that the hands of a clock are opposite each other until they are opposite each other again?

SUGGESTION. — Let t = number of minutes required. Show that $\frac{1}{60}t - \frac{1}{720}t = 1$.

27. If a planet and the earth are on opposite sides of the sun, and in a straight line with it, the planet is said to be *in opposition.* Venus makes a circuit around the sun in 225 days, and the earth makes a circuit in 365 days. If Venus is in opposition, how long will it be until it is in opposition again?



SUGGESTION. — Let n = the number of days required.

In n days the earth will make $\frac{n}{365}$ revolutions and Venus $\frac{n}{225}$ revolutions about the sun.

But to be again in opposition Venus must make one more revolution than the earth.

Hence, $$\frac{n}{225} - \frac{n}{365} = 1.$$

28. Mars makes a circuit about the sun in 687 days. Mars was in opposition November 11, 1911. Find the date upon which it was again in opposition.

29. Mercury makes a circuit about the sun in 88 days. How long is it from the time that it is in opposition until it is in opposition again?

30. Jupiter makes a circuit about the sun in 4333 days. How long is it from the time that it is in opposition until it is in opposition again? It was in opposition February 28, 1909. Find the next date of opposition.

31. In a factory a large machine can turn out a number of articles in 6 days, and a smaller sized machine can turn them out in 9 days. If both machines are operated at once, how long will it take them to turn out the articles?

SUGGESTION.—If $n =$ the number of days required, show that

$$\frac{1}{n}=\frac{1}{6}+\frac{1}{9}.$$

32. There are two large machines and three small ones in a factory. To manufacture a given order of goods would take either one of the large machines alone 20 days, and either one of the small machines alone 35 days. If all five machines are operated, how long does it take to fill the order?

33. A water tank can be filled by one pipe in 5 hours and emptied by another in 8 hours. If both pipes are open, how long will it take to fill the tank?

34. In a city waterworks plant are two steam pumps. One pump would fill the reservoir in 15 hours, and the other would fill it in 18 hours. If both pumps are operated at once, how long does it require to fill the reservoir?

35. An aviator flew to a point 75 miles away, and back in 5 hours, 30 minutes. His rate returning was 30 miles per hour. What was his rate going?

36. An aviator made 30 miles per hour against the wind on a trip of 40 miles. After he had gone 25 miles, the wind ceased. The entire time required for the trip was 1 hour 10 minutes. What was the velocity of the wind?

37. A boatman rows 5 miles in $\frac{3}{4}$ hour with the current and it takes him $1\frac{1}{4}$ hours to return the same distance against the current. Find the rate of the current.

38. Find a number such that if its pth part be added to it, the sum will be equal to the qth part of it plus r.

SUPPLEMENTARY EXERCISES

Solve:

1. $\frac{2n+1}{2n-1}-\frac{2n-1}{1+2n}=\frac{8}{4n^2-1}$.

2. $\frac{2A+1}{2A-16}-\frac{2A-1}{12+2A}-\frac{17+9A}{A^2-2A-48}=0$.

3. $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-3}{x-4}-\frac{x-4}{x-5}$.

4. $\frac{x-4}{x-5}-\frac{x-5}{x-6}=\frac{x-7}{x-8}-\frac{x-8}{x-9}$.

5. $\frac{t-8}{t-6}+\frac{t+1}{t-1}-\frac{t}{t-2}=\frac{t-9}{t-7}$.

6 Solve $\frac{ax^2}{b-cx}+a+\frac{ax}{c}=0$ for x.

7. Solve $\frac{3}{an-ab}-\frac{b}{b^2-bn}-\frac{9+3a}{2ab}=0$ for n

8. The formula $R=\frac{\frac{3pV}{2T}+\frac{pV}{T}+E}{\frac{3pV}{2T}+E}$ expresses a law of science relating to matter, energy, and heat. Solve it for V; for p; for E; for T.

9. A train leaves a station and runs at a speed of 35 miles an hour. Two hours later a second train leaves the same station and follows the first train at a speed of 45 miles an hour. In how many hours will the second train pass the first?

10. A number is composed of two digits whose sum is 10. This number exceeds by 54 the number obtained by interchanging the digits. Find the number.

11. An express train traveling 42 miles per hour started from the same depot 50 minutes after a freight train, which it overtook in 2 hours, 5 minutes. Find the velocity of the freight train.

12. A messenger was sent to carry a despatch. Five hours after his departure a second messenger was sent, who ought to overtake the first one in 8 hours. To do this he was obliged to travel $2\frac{1}{2}$ miles per hour faster. How many miles per hour did the first travel?

13. Two trains move in the same direction. The first starts from A at 2 o'clock in the morning, and travels 32 miles per hour. The second starts from B at 8:30 o'clock in the morning, and travels 42 miles per hour. The distance between A and B is 80 miles. When and where does the second overtake the first?

14. A decimal, such as 0.343434 ⋯, in which certain figures repeat themselves indefinitely, is called a **repeating decimal.** Every repeating decimal is equivalent to some common fraction. Thus 0.6666 ⋯ is equal to $\frac{2}{3}$.

Find the common fraction to which 0.272727 ⋯ is equivalent.

Solution. — The figures 27 repeat. Hence we separate the number into two parts, thus: $0.272727 \cdots = 0.27 + 0.002727 \cdots$. It is clear that the second term is just $\frac{1}{100}$ of the whole decimal. Hence, if v represents the value of the fraction sought,

$$v = 0.27 + \frac{v}{100}$$

Solving $$v = \frac{3}{11}.$$

15. Find the common fraction to which 0.5135135 ⋯ is equivalent.

Suggestion. — $v = 0.513 + \frac{v}{1000}$.

Find the common fraction to which each of the following is equivalent:

16. 0.0727272 ⋯.
17. 0.121212 ⋯.
18. 0.060606 ⋯.
19. 0.123123 ⋯.
20. 0.818181 ⋯.
21. 0.432432 ⋯.
22. 0.522522 ⋯.
23. 0.61716171 ⋯.
24. 1.66666 ⋯.

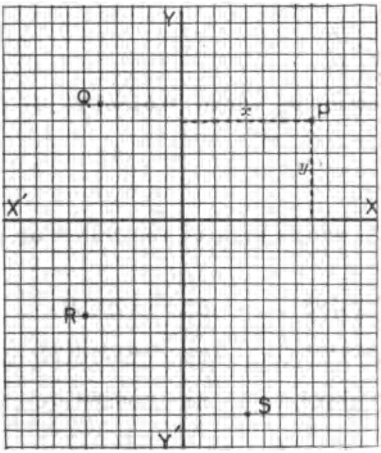
CHAPTER V

SYSTEMS OF LINEAR EQUATIONS

50. Equations in Two Unknown Numbers. — A linear equation containing two unknown numbers is satisfied by an unlimited number of *sets of values* of the unknown numbers. For any particular value that may be assigned to either one of the unknown numbers, one particular value of the other can be found for which the equation is true.

Thus, in $2x - 3y = 24$, by assigning values to y and solving for the corresponding values of x, we find that when $y = 0$, $x = 12$; when $y = 1$, $x = 13\frac{1}{2}$; when $y = 2$, $x = 15$; when $y = 4$, $x = 18$; when $y = 6$, $x = 21$; when $y = -1$, $x = 10\frac{1}{2}$; when $y = -2$, $x = 9$; when $y = -4$, $x = 6$; etc. The number of sets of values may be continued indefinitely.

51. Graph of a Linear Equation in Two Unknown Numbers. — The relation between two unknown numbers which is expressed by an equation may be shown also by means of a **graph.** The construction of the **graph** of a linear equation in two unknown numbers was discussed in the First Course, § 124.



Just as a point on the earth's surface is located by knowing its directions and distances from the prime meridian and the equator (longitude and latitude), so if two lines XX' and YY' are drawn at right angles on squared paper, the position of any point P on the paper is determined by its directions and distances from these lines. The line XX' is called the **x-axis,** YY' the **y-axis,** and their point of intersection the **origin.** The distance x (measured parallel to XX') of P from the y-axis and the distance y (measured parallel to YY') of P from the x-axis are called the **coördinates** of P. Distances measured to the right of YY' are *positive,* and to the left *negative;* and distances measured upward from XX' are *positive,* and downward *negative.*

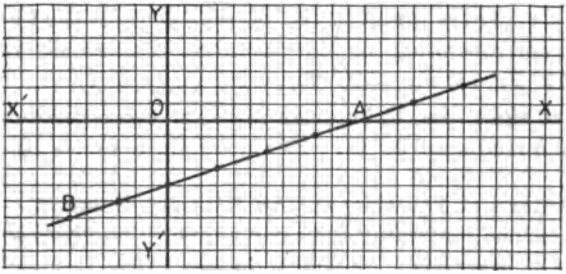
Thus, the coördinates of P are $x = +8$ and $y = +6$; of Q, $x = -5$ and $y = +7$; of R, $x = -6$ and $y = -6$; and of S, $x = +4$ and $y = -12$.

To draw the graph of a linear equation in x and y, a number of sets of values of x and y that satisfy the equation are found as in § 50. Axes are drawn, and points are located, the coördinates of each of which correspond to one of the sets of values found for x and y. A smooth line is drawn connecting these points. This line is the **graph of the equation.**

EXAMPLE. — Draw the graph of $x - 3y = 12$.

Assigning values to y and solving for x, the following sets of values of x and y are found:

x	12	15	18	9	6	8	0	−3	−6
y	0	1	2	−1	−2	−3	−4	−5	−6



Locating the points whose coördinates correspond to these sets of values of x and y, and connecting them, gives the straight line AB shown in the fig

It may be shown that,

The graph of every linear equation in two unknown numbers is a straight line.

Hence, to draw the graph of such an equation it is sufficient to locate only *two points* of it, and draw the straight line through them with a ruler. The two points should be taken a good distance apart for accuracy in drawing.

EXERCI ES

1. By assigning values to y and computing the corresponding values of x, find ten sets of values of x and y that satisfy $3x + 2y = 24$. Locate the ten points whose coördinates correspond to these values. Join the points.

By locating two points of each, and drawing the straight line through them with a ruler, construct the graphs of the following:

2. $x - 2y = 8$.
3. $4x + y = 12$.
4. $5x - y = 20$.
5. $2x + 5y = 15$.
6. $8x - 3y = 32$.
7. $x + 6y = 18$.
8. $10x + y = 30$.
9. $4x - 7y = 20$.
10. $3x + 24 = y$.

11. If C is the temperature as indicated on a Centigrade thermometer, and F the temperature as indicated on a Fahrenheit thermometer, then $C = \frac{5}{9}(F - 32)$. Draw the graph of this equation.

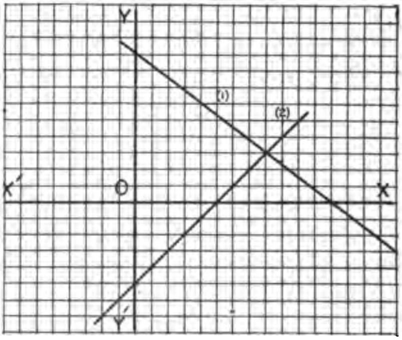
12. By using the graph in Ex. 11, give at sight the Centigrade readings corresponding to the following Fahrenheit readings: 40° F.; 12° F.; 18° F.; − 10° F.; − 20° F.; − 5° F.

Give at sight the Fahrenheit readings corresponding to the Centigrade readings: 34° C.; 15° C.; 40° C.; 8° C.; − 10° C.

52. Systems of Equations in Two Unknown Numbers. — Two or more equations containing the same unknown numbers are called **a system of equations.** As shown in the FIRST COURSE (§ 131), in a system of two linear equations in two unknown numbers there is, in general, one and only one set of values of the unknown numbers that satisfy both equations. Such a set of values is called the **solution of the system.**

Thus, in the system $\begin{cases} x + 2y = 8, \\ 3x - y = 10, \end{cases}$ the values $x = 4$ and $y = 2$ satisfy both equations. Hence, the solution of the system is $x = 4$, $y = 2$.

53. Systems solved Graphically. — The solution of a system of two linear equations in two unknown numbers may be found by drawing the graphs of the equations, and observing the coördinates of the point where the graphs meet. Since this point is on both graphs, its coördinates must satisfy both equations.



EXAMPLE. — Solve

$$\begin{cases} 3x + 4y = 36, & (1) \\ x - y = 5. & (2) \end{cases}$$

The graphs of equations (1) and (2) are lines (1) and (2), respectively. They meet at a point whose coördinates are $x = 8$, $y = 3$. Hence, the solution is $x = 8$, $y = 3$.

EXERCISES

Solve graphically:

1. $\begin{cases} x - y = 4, \\ x + y = 8. \end{cases}$

2. $\begin{cases} x + 4y = 10, \\ 3x - y = 12. \end{cases}$

3. $\begin{cases} x + y = 5, \\ 2x + 3y = 12. \end{cases}$

4. $\begin{cases} 5x + 4y = 22, \\ 3x + y = 9. \end{cases}$

5. $\begin{cases} 7x - 2y = 31, \\ 4x + 3y + 3 = 0. \end{cases}$

6. $\begin{cases} 3x + 2y = 12, \\ 4x - 3y + 1 = 0. \end{cases}$

7. $\begin{cases} 2x + 5y = 28, \\ 5x - 8y = 29. \end{cases}$

8. $\begin{cases} 7x + y = 42, \\ 3x - y = 8. \end{cases}$

9. $\begin{cases} x + y = 30, \\ 3x - 2y = 25. \end{cases}$

10. By drawing the graphs of the equations, show that

$$\begin{cases} x - 2y = 8, \\ 2x - 4y = 40, \end{cases}$$

has no solution. What kind of lines are the graphs?

54. Systems solved by Elimination. — A system of equations may be solved by a process called **elimination.** This process, applied to a system of two linear equations in two unknown numbers, consists of combining the members of the two equations so as to derive a new equation which contains only one unknown number. The new equation is then solved. By substituting the value of the unknown number found in either of the original equations, and solving, the value of the other unknown number is obtained.

The three most common methods of elimination, as shown in the FIRST COURSE, are as follows:

ELIMINATION BY ADDITION OR SUBTRACTION

To eliminate one of the unknown numbers, multiply the members of each equation, if necessary, by such a number as will make the numerical coefficients of that unknown number the same in both of the resulting equations. Add or subtract (according to signs) the corresponding members of the resulting equations.

EXAMPLE. — Solve $\begin{cases} 2M + 5N = 15, & (1) \\ 3M - 4N = 11. & (2) \end{cases}$

Let us eliminate N.

Multiplying (1) by 4, $8M + 20N = 60.$ (3)

Multiplying (2) by 5, $15M - 20N = 55.$ (4)

Adding (3) and (4), $23M = 115.$

Solving, $M = 5.$

Substituting 5 for M in (1), $10 + 5N = 15.$

Solving, $N = 1.$

ELIMINATION BY COMPARISON

Solve each of the equations for the value of either one of the unknown numbers, expressed in terms of the other, and write one of the values equal to the other. Solve the resulting equation.

EXAMPLE. — Solve $\begin{cases} 6v - 5t = 1, & (1) \\ 9v + 10t = 12, & (2) \end{cases}$

Let us eliminate v.

Solving (1) for v, $v = \frac{1 + 5t}{6}.$ (3)

Solving (2) for v, $v = \frac{12 - 10t}{9}$. (4)

From (3) and (4), $\frac{1 + 5t}{6} = \frac{12 - 10t}{9}$.

Solving, $t = \frac{3}{5}$.

Substituting $\frac{3}{5}$ for t in (1), $6v - 3 = 1$.

Solving, $v = \frac{2}{3}$.

ELIMINATION BY SUBSTITUTION

Solve one of the equations for the value of either unknown number, expressed in terms of the other, and substitute this value in place of that number in the other equation. Solve the resulting equation.

Example.—Solve $\begin{cases} 4a - 5b = 26, & (1) \\ 3a - 6b = 15. & (2) \end{cases}$

Let us eliminate a.

Solving (1) for a, $a = \frac{26 + 5b}{4}$. (3)

Substituting the value of a in (3) for a in (2),

$$\frac{3(26 + 5b)}{4} - 6b = 15. \quad (4)$$

Solving (4), $b = 2$.

Substituting 2 for b in (1), $4a - 10 = 26$.

Solving, $a = 9$.

EXERCISES

Eliminate by addition or subtraction, and solve:

1. $\begin{cases} 7m + n = 42, \\ 3m - n = 8. \end{cases}$

3. $\begin{cases} 2x + 5y = 15, \\ 3x - 4y = 11. \end{cases}$

5. $\begin{cases} 4R - 3r = 1, \\ 3R - 4r = 6. \end{cases}$

2. $\begin{cases} 2A - B = 5, \\ 5A - 2B = 14. \end{cases}$

4. $\begin{cases} 5v - 2t = 35, \\ t + 4v = 25. \end{cases}$

6. $\begin{cases} 8V = t + 34, \\ V + 8t = 53. \end{cases}$

Eliminate by comparison and solve:

7. $\begin{cases} 2p - q = 9, \\ 5p - 3q = 14. \end{cases}$

9. $\begin{cases} 3x - y = 1, \\ 2x + 5y = 41. \end{cases}$

11. $\begin{cases} 2E - F = 5, \\ E + 2F = 25. \end{cases}$

8. $\begin{cases} 3M + 2N = 2, \\ M - N = 9. \end{cases}$

10. $\begin{cases} 9a - 5b = 13, \\ 5a + b = 11 \end{cases}$

12. $\begin{cases} 10m + 3n = 174, \\ 3m + 10n = 125. \end{cases}$

Eliminate by substitution and solve:

13. $\begin{cases} a+b=30, \\ 3a-2b=25. \end{cases}$

14. $\begin{cases} 4w=5v+26, \\ 3w-6v=15. \end{cases}$

15. $\begin{cases} 3r+7r'=16, \\ 2r+5r'=13. \end{cases}$

16. $\begin{cases} 7F+4C=1, \\ 9F+4C=3. \end{cases}$

17. $\begin{cases} 8A+2B=90, \\ 3A-5B=51. \end{cases}$

18. $\begin{cases} 5y+11z=102, \\ y-3z+16=0. \end{cases}$

Eliminate by any method and solve:

19. $\begin{cases} 7D-9T=13, \\ 5D+2T=10. \end{cases}$

20. $\begin{cases} 18k-20s=1, \\ 15k+16s=9. \end{cases}$

21. $\begin{cases} 8x+6y=10, \\ 5x+2y=1. \end{cases}$

22. $\begin{cases} 7a-4b=12, \\ 8a=5b. \end{cases}$

23. $\begin{cases} \frac{1}{3}a-\frac{1}{4}b=2, \\ \frac{1}{4}a-\frac{1}{2}b=7. \end{cases}$

24. $\begin{cases} \frac{1}{2}m+\frac{1}{3}n=13, \\ \frac{1}{5}m+\frac{1}{8}n=5. \end{cases}$

Solve for x and y:

25. $\begin{cases} ax-by=a^2-b^2, \\ bx-ay=a^2-b^2. \end{cases}$

26. $\begin{cases} ax+by=a^2+b^2, \\ bx+ay=2ab. \end{cases}$

27. $\begin{cases} x-ay=a+b, \\ ax-ay=a^2+b. \end{cases}$

28. $\begin{cases} x+ny=m, \\ 2x-my=n. \end{cases}$

29. $\begin{cases} x+p^2y=0, \\ x-q^2y=1. \end{cases}$

30. $\begin{cases} ax+by=c, \\ dx+ey=f. \end{cases}$

Eliminate before clearing of fractions, and solve:

31. $\begin{cases} \frac{2}{x}-\frac{1}{y}=9, \\ \frac{5}{x}-\frac{3}{y}=14. \end{cases}$

32. $\begin{cases} \frac{4}{a}-\frac{3}{b}=1, \\ \frac{3}{a}-\frac{4}{b}=6. \end{cases}$

33. $\begin{cases} \frac{5}{A}-\frac{2}{B}=1, \\ \frac{8}{A}+11=\frac{5}{B}. \end{cases}$

34. $\begin{cases} \frac{5}{m}+\frac{2}{n}=1, \\ \frac{13}{m}+\frac{8}{n}=11. \end{cases}$

35. $\begin{cases} \frac{3}{v}+\frac{2}{t}=2, \\ \frac{1}{v}-\frac{1}{t}=9. \end{cases}$

36. $\begin{cases} \frac{4}{M}+\frac{1}{N}=34, \\ \frac{4}{N}+\frac{1}{M}=16. \end{cases}$

55. Systems in Three or More Unknown Numbers. — The process of elimination described in § 54 may be applied in the solution of a system of three equations in three unknown numbers, a system of four equations in four unknown numbers, etc. In any case, the equations of the system are combined so as to derive a new system in which the number of equations is one less than the number in the given system, and in which one of the unknown numbers does not appear. By repeating the process, a new system is obtained in which the number of equations and the number of unknown numbers is one less than before, etc. Finally a single equation is obtained which contains only one unknown number.

Example. — Solve
$$\begin{cases} 2a + b - c + 2d = 11, & (1) \\ 2a - b + 2c + d = 2, & (2) \\ a + b + c + 3d = 5, & (3) \\ 5a - 2b + c - d = 10. & (4) \end{cases}$$

Eliminating d between (1) and (2) gives

$$2a - 3b + 5c = -7. \qquad (5)$$

Eliminating d between (1) and (3) gives

$$4a + b - 5c = 23. \qquad (6)$$

Eliminating d between (1) and (4) gives

$$12a - 3b + c = 31. \qquad (7)$$

Now (5), (6), and (7) form a new system free of d.

Eliminating c between (5) and (6) gives

$$6a - 2b = 16. \qquad (8)$$

Eliminating c between (5) and (7) gives

$$58a - 12b = 162. \qquad (9)$$

Now (8) and (9) form a new system free of c and d.

Eliminating b between (8) and (9) gives

$$22a = 66. \qquad (10)$$

Solving (10), $a = 3.$

Substituting 3 for a in (8), $b = 1.$

Substituting 3 for a and 1 for b in (5), $c = -2.$

Substituting 3 for a, 1 for b, and -2 for c in (1), $d = 1.$

EXERCISES

Solve:

1. $\begin{cases} a+b+c=6, \\ 3a-b+2c=7, \\ 2a+3b-c=5. \end{cases}$

2. $\begin{cases} x+y+z=9, \\ 27x+9y+3z=93, \\ 8x+4y+2z=36. \end{cases}$

3. $\begin{cases} 2M-3N-7P=19, \\ 5M-8N+11P=24, \\ 7M-11N+4P=43. \end{cases}$

4. $\begin{cases} 7r+s+3t=54, \\ 4r-5s+6t=12, \\ r+15s-9t=58. \end{cases}$

5. $\begin{cases} 3A+2C=6-2B, \\ A-5B+6C=2, \\ 6A-8C=12. \end{cases}$

6. $\begin{cases} x+y=1, \\ y+z=9, \\ x+z+6=0. \end{cases}$

7. $\begin{cases} 2a+b-c=7, \\ b-a=1, \\ c-b=1. \end{cases}$

8. $\begin{cases} x+y+z+w=10, \\ x-y-z+w=0, \\ 2x+y+3z-w=9, \\ 3x-y+z-2w+4=0. \end{cases}$

9. $\begin{cases} 2p-q-r+s=13, \\ p+q-r-s=-1, \\ p-q+r-s+5=0, \\ 3p+2q-r+2s=17. \end{cases}$

10. $\begin{cases} 4E-G+K=0, \\ 7G+2K+H=0, \\ 4K+H+8E=0, \\ 5G+16E-H=4. \end{cases}$

11. $\begin{cases} \frac{2}{a}+\frac{5}{b}-\frac{7}{c}=9, \\ \frac{5}{a}-\frac{1}{b}+\frac{3}{c}=16, \\ \frac{7}{a}+\frac{6}{b}+\frac{1}{c}=34. \end{cases}$

12. $\begin{cases} \frac{1}{x}+\frac{2}{y}=26, \\ \frac{3}{x}+\frac{4}{z}=56, \\ \frac{5}{y}+\frac{6}{z}=65. \end{cases}$

Solve for x, y, and z:

13. $\begin{cases} x+y+z=4a, \\ x-y+2z=4a+2b, \\ 2x+y-z=a+b. \end{cases}$

14. $\begin{cases} ax+by+cz=p, \\ dx+ey+fz=q, \\ gx+hy+iz=r. \end{cases}$

56. Number of Equations and Unknown Numbers in a System. — If a system of linear equations contains just as many equations as there are unknown numbers, in general *one* and *only one solu-*

tion of the system can be found. Such a system is called a **determinate system.** The systems solved in § 55 were *determinate systems.*

If there are more unknown numbers than equations in a system, in general the number of solutions of the system is *indefinitely great.* Hence, such a system is called an **indeterminate system.**

Thus, $\begin{cases} x+y+z=6, \\ 2x-y+z=3, \end{cases}$ is *indeterminate.* Some of its many solutions are $x=1$, $y=2$, $z=3$; $x=3$, $y=3$, $z=0$; $x=-1$, $y=1$, $z=6$; etc.

If there are more equations than unknown numbers in a system, in general no set of values of the unknown numbers exists which will satisfy all equations of the system. Hence, such a system is called an **impossible system.**

Thus, in $\begin{cases} m-n=5, \\ m+2n=14, \\ 2m-3n=4, \end{cases}$ the only values of m and n that satisfy the first two equations are $m=8$ and $n=3$. But these values do not satisfy the third equation.

There are exceptions to some of the above principles in cases in which special relations exist between the equations of a system.

Thus, the system $\begin{cases} 2x-y=3, \\ 4x=6+2y \end{cases}$ is *not determinate.* Any values of x and y that satisfy either of the equations satisfy the other also. The equations are not independent. The second may be derived from the first by multiplying by 2 and transposing a term.

Two equations in which all values of the unknown numbers that satisfy one satisfy the other also are called **equivalent equations.**

The system $\begin{cases} 2x+3y=14, \\ 3x-2y=8, \\ 5x+y=22, \end{cases}$ is *not impossible,* for each equation is satisfied when $x=4$ and $y=2$. The equations are not independent, for the third is derived from the other two by addition.

The system $\begin{cases} x+5y=3, \\ 2x+10y=15, \end{cases}$ has *no solution.* If the first is multiplied by 2 and subtracted from the second, it gives $0=9$, which is absurd. Hence, the system is *not determinate.*

Two equations which are not satisfied by any set of values of the unknown numbers are called **inconsistent equations.**

EXERCISES

1. Show that the equations $x+y=7$, $2x-y=8$, $x-2y=1$ do not form an impossible system. What is the solution?

2. Draw the graphs of the three equations in Ex. 1 upon the same set of axes. What do you observe about the points where they meet?

3. Show that $3m+n-2=0$ and $3n=6-9m$ are equivalent. How may the second be obtained from the first?

4. Draw the graphs of the two equations in Ex. 3. What do you observe about them?

5. Show that the system $x-2y=7$ and $3x=36+6y$ has no solution.

6. Draw the graphs of the equations in Ex. 5. What do you observe about them?

57. Problems solved by Systems of Linear Equations. — If a problem contains more than one unknown number, each of these may be represented by some letter, and from relations given in the problem, as many equations containing these letters may be formed as there are unknown numbers. By solving the system of equations thus obtained, the values of all unknown numbers in the problem are found directly.

EXERCISES

1. Of two partners, A and B, A has twice as much invested in a business as B. If the profits from the business are \$3600, how much should each receive?

2. A man invested part of \$10,000 at 5% and the rest at 8%. His annual income from both investments was \$608. Find the amount of each investment.

3. A man invested \$24,000, part in municipal 4% bonds, and the rest in railroad stock that yielded him 7%. His annual

income from the entire investment was \$1230. Find the amount invested in bonds and the amount invested in stocks.

4. A dealer wishes to mix coffee costing him 20 cents a pound with coffee costing 30 cents a pound so as to make a blend that will cost him 24 cents a pound. How many pounds of each grade must he use to make 50 pounds of the blend?

5. A tea merchant has one grade of tea costing 40 cents a pound and another grade costing 65 cents a pound. How many pounds of each grade must he use to make 20 pounds of a mixture costing him 50 cents a pound?

6. How much cream containing 22 % butter fat and milk containing 4 % butter fat must be mixed to make 40 gallons of cream containing 20 % butter fat?

7. How much milk testing $3\frac{1}{2}$ % butter fat and cream testing 24 % butter fat must be mixed to make 100 gallons of milk testing 4 % butter fat?

8. A certain mixture contains 30 % alcohol. Commercial alcohol is 95 % pure. How many gallons each of the mixture and commercial alcohol must be taken to make 60 quarts of a mixture which contains 45 % alcohol?

9. A chemist has in his laboratory the same acid in two strengths. Six parts of one mixed with 4 parts of the other gives a mixture 86 % pure, and 4 parts of the first mixed with 6 parts of the second gives a mixture 84 % pure. What is the per cent of purity of each acid?

10. When weighed in water, gold appears to lose $\frac{4}{77}$ of its weight in air. Silver appears to lose $\frac{2}{21}$ of its weight in air when weighed in water. An alloy of gold and silver that weighs 18 ounces in air appears to weigh only $16\frac{16}{21}$ ounces in water. How much gold and how much silver in the alloy?

11. Hieron, king of Syracuse, had a crown that weighed 20 pounds. The workman who made it claimed that it was of pure gold, but King Hieron suspected that it was an alloy of gold and silver. His friend Archimedes, the greatest mathematician of

antiquity, weighed it in water and found that it appeared to weigh only $18\frac{3}{4}$ pounds. How much gold and how much silver did it contain? (Use the data of Problem 10.)

12. When weighed in water tin appears to lose $\frac{2}{15}$ of its weight in air, and zinc $\frac{1}{10}$ of its weight in air. An alloy of tin and zinc that weighed 45 pounds in air appeared to weigh only $39\frac{1}{2}$ pounds in water. Find the amount of each metal in the alloy.

13. The relation between the readings on the Centigrade and the Fahrenheit thermometers is expressed by $C=\frac{5}{9}(F-32)$, where C and F are the temperatures indicated on the Centigrade and Fahrenheit thermometers, respectively. Alcohol boils at a temperature that is 96° lower on the Centigrade than on the Fahrenheit thermometer scale. Find the boiling points of alcohol on both the Centigrade and Fahrenheit thermometers.

14. The temperature of boiling mercury is 312° higher when read upon the Fahrenheit thermometer than when read upon the Centigrade. Find the boiling point of mercury on each scale.

15. Two men lift a weight of 216 pounds which is suspended from a 6-foot pole carried between them. If the weight is $2\frac{1}{2}$ feet from one man and $3\frac{1}{2}$ feet from the other, how much does each man lift?

NOTE. — The equations for Problems 15 and 16 depend upon the principle that the amount lifted by one multiplied by the distance to the weight equals the amount lifted by the other multiplied by the distance to the weight.

16. Two boys carry a basket of groceries suspended on a stick between them. If the basket weighs 60 pounds, and is placed 24 inches from one end and 30 inches from the other end of the stick, how much does each boy carry if one grasps each end of the stick?

17. An aviator flew to a point 30 miles away in 40 minutes, with the wind. It took 2 hours for him to make the return trip against the wind. Find the speed of the wind and what his speed would have been if there had been no wind.

18. A freight train is 715 feet long, and a passenger train 275 feet long. They pass each other on parallel tracks. When going in opposite directions, it takes them just 9 seconds to pass; but when going in the same direction, it takes them 45 seconds to pass. Find the speed of each train.

19. A man said to his son: "I am now 4 times as old as you; but in 20 years I will be only twice as old as you." Find the age of each.

20. A pumping station contains two steam pumps. If both pumps are driven at once, they will fill the reservoir in $8\frac{8}{9}$ hours. If one is driven 10 hours and the other 8 hours, the reservoir will be filled. In what time can each pump alone fill the reservoir?

21. Two machines operated together can turn out a piece of work in 13 hours 20 minutes. After both have been operated for 6 hours, one machine is stopped, and the other finishes the work in $16\frac{1}{2}$ hours more. In how many hours could each machine alone turn out the work?

22. The perimeter (sum of the sides) of a triangle is 36 inches. One side is 3 inches longer than another, and the third side is 3 inches longer than the longer of the other two. Find the length of each side.

23. The sum of the two sides of a triangle which meet at one vertex is 28 inches, at another vertex 34 inches, and at the third vertex 38 inches. Find the length of each side.

24. Three circles are drawn tangent to (touching) each other, and with their centers at the points P, Q, and R, respectively. The distance between P and Q is 12 inches, between P and R 16 inches, and between Q and R 20 inches. Find the radii of the circles.

NOTE.—The lines PQ, QR, and PR pass through the points at which the circles touch.

25. The sides of a triangle are all tangent to a circle. The sides of the triangle are 28 inches, 36 inches, and 40 inches respectively. Find the points of the sides at which they touch the circle.

Note. — Use the fact from geometry that the distances from any vertex of the triangle to the two points where the sides from that vertex touch the circle are equal.

26. Three pumps together can fill a reservoir in $3\frac{3}{23}$ hours, the two larger can fill it in $4\frac{4}{17}$ hours, and the largest and smallest together can fill it in $4\frac{4}{5}$ hours. In what time could each pump alone fill it?

58.* Determinants. — If we solve the system

$$\begin{cases} ax + by = c, \\ dx + ey = f, \end{cases}$$

the solution is $$x = \frac{ce - bf}{ae - bd},\quad y = \frac{af - cd}{ae - bd}.$$

It is seen that the fractions expressing the values of x and y have the same denominator, $ae - bd$. This denominator may be written in the form

$$\begin{vmatrix} a & b \\ d & e \end{vmatrix}.$$

This symbol is called a **determinant.**

Its value is found by subtracting from the product of the upper left-hand and lower right-hand numbers the product of the upper right-hand and lower left-hand numbers.

Thus, $\begin{vmatrix} 6 & 2 \\ 5 & 7 \end{vmatrix} = 6 \times 7 - 5 \times 2 = 32.$ And $\begin{vmatrix} -8 & 3 \\ 4 & 1 \end{vmatrix} = -8 - 12 = -20.$

Since the above determinant contains *two rows* and *two columns* of numbers, it is called a determinant of the **second order.**

EXERCISES

Find the value of:

1. $\begin{vmatrix} 5 & 7 \\ 2 & 8 \end{vmatrix}$. **2.** $\begin{vmatrix} 1 & 6 \\ 6 & 1 \end{vmatrix}$. **3.** $\begin{vmatrix} 4 & 3 \\ 3 & 2 \end{vmatrix}$. **4.** $\begin{vmatrix} 10 & 3 \\ 8 & 6 \end{vmatrix}$.

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* This and the following starred sections may be omitted without interfering with the subsequent work, if desired.

5. $\begin{vmatrix} -2 & 3 \\ 4 & -1 \end{vmatrix}$. 7. $\begin{vmatrix} 8 & 0 \\ 3 & 2 \end{vmatrix}$. 9. $\begin{vmatrix} 12 & -2 \\ -9 & -1 \end{vmatrix}$. 11. $\begin{vmatrix} x & x \\ 1 & 3 \end{vmatrix}$.

6. $\begin{vmatrix} 12 & -2 \\ 1 & -1 \end{vmatrix}$. 8. $\begin{vmatrix} 0 & -2 \\ 5 & 4 \end{vmatrix}$. 10. $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. 12. $\begin{vmatrix} a^2 & 1 \\ 1 & 1 \end{vmatrix}$.

59.* Systems of Two Linear Equations solved by Determinants.† — It follows from the definition of a determinant in § 58 that the solution of the system $\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$ may be written in the form

$$x = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}.$$

Observe that the numerator of x is obtained from the denominator by replacing the coefficients of x that form the first column by the known terms in the equations, and that the numerator of y is obtained by replacing the coefficients of y that form the second column by the known terms in the equations.

Hence, instead of solving by the process of elimination, the solution of any system of two linear equations in two unknown numbers may be *written down* by determinants.

EXAMPLE. — Solve $\begin{cases} 4A - 3B = 1, \\ 3A - 4B = 6. \end{cases}$

$$A = \frac{\begin{vmatrix} 1 & -3 \\ 6 & -4 \end{vmatrix}}{\begin{vmatrix} 4 & -3 \\ 3 & -4 \end{vmatrix}} = \frac{14}{-7} = -2, \qquad B = \frac{\begin{vmatrix} 4 & 1 \\ 3 & 6 \end{vmatrix}}{\begin{vmatrix} 4 & -3 \\ 3 & -4 \end{vmatrix}} = \frac{21}{-7} = -3.$$

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† The determinant notation is an invention of modern times. It has been in use in solving systems of equations since about 1750. Leibnitz, a great German mathematician, first discovered the notation and wrote about it to a friend in 1693, but its general use did not come until much later. To-day determinants are extensively used in all fields of advanced mathematics. For a fuller treatment of determinants, see the chapter on determinants in the STONE-MILLIS HIGHER ALGEBRA.

EXERCISES

Solve by determinants:

1. $\begin{cases} 4x+y=6, \\ x+y=3. \end{cases}$

2. $\begin{cases} 3M+2N=12, \\ 2M+3N=13. \end{cases}$

3. $\begin{cases} p+5q=8, \\ 2p-q=5. \end{cases}$

4. $\begin{cases} 8a-b=4, \\ 6a+2b=14. \end{cases}$

5. $\begin{cases} R-T=1, \\ 2R+T=14. \end{cases}$

6. $\begin{cases} 12v-t=10, \\ 5v+7t=16. \end{cases}$

7. $\begin{cases} -2x+5y=4, \\ 7x-y=5. \end{cases}$

8. $\begin{cases} 4h=5-k, \\ h+3=2k. \end{cases}$

9. $\begin{cases} 2P-1=Q, \\ P+5Q=9. \end{cases}$

10. $\begin{cases} 9m-4n=20, \\ 5m-n=16. \end{cases}$

11. $\begin{cases} 2V+1=5a, \\ 3a-4=6V. \end{cases}$

12. $\begin{cases} 3-n=2m, \\ 4m+2=n. \end{cases}$

60.* Determinants of the Third Order. — Solving the system

$$\begin{cases} ax+by+cz=p, \\ dx+ey+fz=q, \\ gx+hy+iz=r, \end{cases}$$

$$x=\frac{pei+qhc+rfb-cer-fhp-iqb}{aei+dhc+gfb-ceg-fha-idb},$$

$$y=\frac{aqi+drc+gfp-cqg-fra-idp}{aei+dhc+gfb-ceg-fha-idb},$$

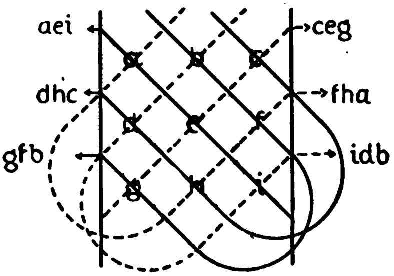
$$z=\frac{aer+dhp+gqb-peg-qha-rdb}{aei+dhc+gfb-ceg-fha-idb}.$$

The denominator $aei+dhc+gfb-ceg-fha-idb$ of each fraction may be written in the form

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$$

This symbol is called a **determinant of the third order.**

It is seen that each term of the denominator is the product of three numbers, and that these three numbers are chosen so that *one and only one is taken from each row and one and only one from each column* of the determinant. It is seen also that if the numbers of the determinant are connected by lines, as in the accompanying figure, *the three numbers on each of the three continuous lines form one of the positive terms, and the three numbers on each of the dotted lines form one of the negative terms.*



These facts will enable one to write down the value of any given determinant of the third order.

Example. —
$$\begin{vmatrix}2&6&5\\1&4&3\\8&1&7\end{vmatrix}=2\times4\times7+1\times1\times5+8\times6\times3-$$

$8\times4\times5-1\times3\times2-7\times1\times6=-3.$

EXERCISES

Find the value of:

1. $\begin{vmatrix}2&3&5\\1&2&1\\4&1&3\end{vmatrix}.$

2. $\begin{vmatrix}1&6&2\\7&8&1\\1&3&4\end{vmatrix}.$

3. $\begin{vmatrix}4&8&1\\3&1&5\\6&1&2\end{vmatrix}.$

4. $\begin{vmatrix}1&2&3\\2&1&1\\3&2&1\end{vmatrix}.$

5. $\begin{vmatrix}1&1&1\\4&4&4\\1&5&2\end{vmatrix}.$

6. $\begin{vmatrix}8&2&-3\\1&-1&4\\2&-2&1\end{vmatrix}.$

7. $\begin{vmatrix}3&-5&-1\\4&-2&3\\1&-1&1\end{vmatrix}.$

8. $\begin{vmatrix}12&1&-1\\2&-4&8\\1&-2&-1\end{vmatrix}.$

9. $\begin{vmatrix}10&0&4\\3&6&-2\\0&-1&5\end{vmatrix}.$

10. $\begin{vmatrix}a&b&c\\1&1&1\\1&2&3\end{vmatrix}.$

11. $\begin{vmatrix}x&y&z\\y&x&z\\1&1&1\end{vmatrix}.$

12. $\begin{vmatrix}1&0&n\\0&1&0\\n&1&1\end{vmatrix}.$

61.* **Systems of Three Linear Equations Solved by Determinants.** — It follows from the definition of a determinant of the third order in § 60 that the solution of the system

$$\begin{cases} ax + by + cz = p, \\ dx + ey + fz = q, \\ gx + hy + iz = r, \end{cases}$$

may be written in the form

$$x = \frac{\begin{vmatrix} p & b & c \\ q & e & f \\ r & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & p & c \\ d & q & f \\ g & r & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a & b & p \\ d & e & q \\ g & h & r \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}.$$

It is seen that the numerator of x *is obtained from the denominator by replacing the coefficients of* x *that form the first column by the known terms of the equations, and that the numerators of* y *and* z *are obtained in a similar manner.*

Hence, any system of three linear equations in three unknown numbers may be solved by merely writing down the solution in the form of determinants, and simplifying.

Example. — Solve $\begin{cases} x - 2y + z = 6, \\ x + 3y + 2z = 13, \\ 2x - y + z = 13. \end{cases}$

$$x = \frac{\begin{vmatrix} 6 & -2 & 1 \\ 13 & 3 & 2 \\ 13 & -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -2 & 1 \\ 1 & 3 & 2 \\ 2 & -1 & 1 \end{vmatrix}} = \frac{18 - 13 - 52 - 39 + 26 + 12}{3 - 1 - 8 - 6 + 2 + 2} = \frac{-48}{-8} = 6.$$

Similarly, $y = 1$, and $z = 2$.

EXERCISES

Solve by determinants:

1. $\begin{cases} x+y+z=7, \\ 2x+y-z=7, \\ 3x-y+2z=15. \end{cases}$

2. $\begin{cases} A-2B+C=2, \\ 4A+B-2C=20, \\ A+3B-5C=6. \end{cases}$

3. $\begin{cases} 3p+2q+r=5, \\ 2p-3q-r=1, \\ 5p-q+2r=14. \end{cases}$

4. $\begin{cases} M+2N-P=5, \\ 6M+5N+P=2, \\ 2M-N-8P=1. \end{cases}$

5. $\begin{cases} 3a+5b+c=1, \\ a-4b-2c=32, \\ 7a+5b-4c=36. \end{cases}$

6. $\begin{cases} 9r-s+2t=8, \\ r+3s-t=5, \\ 4r-s-3t=1. \end{cases}$

7. $\begin{cases} x-2=y+z, \\ 3x+y=z-1, \\ x+z=4-y. \end{cases}$

8. $\begin{cases} 12m-3n=4-2k, \\ m+k=2-n, \\ 5k-4n=2m+1. \end{cases}$

9. $\begin{cases} A+B=10, \\ B+C=6, \\ A+C=8. \end{cases}$

10. $\begin{cases} x-y=2, \\ y-z=3, \\ x-z=5. \end{cases}$

SUPPLEMENTARY EXERCISES

Solve:

1. $\begin{cases} \dfrac{1}{8a}+\dfrac{7}{4b}=\dfrac{5}{16}, \\ \dfrac{1}{4a}-\dfrac{3}{2b}+\dfrac{5}{65}=0. \end{cases}$

2. $\begin{cases} \dfrac{2}{5m}+\dfrac{5}{2n}=7, \\ \dfrac{5}{2m}+\dfrac{1}{n}=14\frac{1}{2}. \end{cases}$

3. $\begin{cases} \dfrac{1}{x+1}+\dfrac{1}{y+1}=\dfrac{7}{12}, \\ \dfrac{1}{x+1}-\dfrac{1}{y+1}=\dfrac{1}{12}. \end{cases}$

4. $\begin{cases} \dfrac{2}{a-2}+\dfrac{1}{b-2}=\dfrac{13}{15}, \\ \dfrac{1}{a-2}+\dfrac{2}{b-2}=\dfrac{11}{15}. \end{cases}$

5. $\begin{cases} \dfrac{8}{2m+3}-\dfrac{6}{2n-3}=5, \\ \dfrac{2}{2m+3}+\dfrac{5}{2n-3}+2=0 \end{cases}$

6. $\begin{cases} \dfrac{1}{A-5}+\dfrac{2}{B-5}+\dfrac{1}{C-5}=2, \\ \dfrac{4}{A-5}-\dfrac{8}{B-5}+\dfrac{3}{C-5}=3, \\ \dfrac{3}{A-5}-\dfrac{1}{B-5}-\dfrac{1}{C-5}=\dfrac{1}{4}. \end{cases}$

7. $\begin{cases} \dfrac{2}{3x}+\dfrac{5}{2y}+\dfrac{1}{z}=3\frac{5}{6}, \\ \dfrac{1}{x}-\dfrac{1}{3y}+\dfrac{2}{z}=2\frac{1}{6}, \\ \dfrac{1}{2x}-\dfrac{2}{3y}-\dfrac{1}{2z}=-\dfrac{11}{12}. \end{cases}$

8. In the equations $\frac{P}{R}=\frac{a}{a'}$ and $\frac{R}{Q}=\frac{b}{b'}$, eliminate R by multiplication.

9. In the formula $V=\frac{1}{6}H(B+b+4M)$ and $4M=B+b+2\sqrt{Bb}$, eliminate M, and thus derive a formula between V, H, B, and b.

10. In the formula $P_1d_1=P_2d_2$ and $P_1+P_2=w$ solve for P_1 and P_2.

Solve each of the following systems for x, y, and z:

11. $\begin{cases} x+y+z=a, \\ x+y-z=b, \\ x-y+z=c. \end{cases}$

12. $\begin{cases} ax+by=p, \\ cy+dz=q, \\ ex+fz=r. \end{cases}$

13. $\begin{cases} \frac{1}{x}+\frac{1}{y}=4, \\ \frac{1}{x}+\frac{1}{z}=3, \\ \frac{1}{y}+\frac{1}{z}=2. \end{cases}$

14. $\begin{cases} \frac{1}{x}+\frac{1}{y}=a, \\ \frac{1}{x}+\frac{1}{z}=b, \\ \frac{1}{y}+\frac{1}{z}=c. \end{cases}$

CHAPTER VI

ROOTS, SURDS, AND IMAGINARY NUMBERS

62. Roots. — The numbers whose square, cube, fourth power, fifth power, etc., are a given number are called the **square root, cube root, fourth root, fifth root,** etc., respectively, of the given number. In general, the number whose nth power is a given number is the **nth root** of that number.

Thus, since $(4x^2)^3 = 64x^6$, $4x^2$ is the cube root of $64x^6$.

A root of a number is indicated by placing before it the sign $\sqrt{}$, called the **radical sign**. Usually a vinculum is placed over the number to show how far the effect of the radical sign is to extend. Except in case of a square root, a little number called the **index** of the root is written above the radical sign to indicate what root it is. In the case of square root the index 2 is understood.

Thus, the square root of x is written $\sqrt{x}$; the cube root of n^2+1 is written $\sqrt[3]{n^2+1}$; the fourth root of a^2-b^2 is written $\sqrt[4]{a^2-b^2}$; etc.

63. Number of Roots of a Number. — Since $(+2)^2 = 4$ and $(-2)^2 = 4$, $\sqrt{4}$ is either $+2$ or -2, usually written $\sqrt{4} = \pm 2$. Similarly, since $(+3)^4 = 81$, and $(-3)^4 = 81$, $\sqrt[4]{81}$ may be either $+3$ or -3. In general,

Every positive number has two even roots that may be expressed as positive or negative numbers, these roots having the same absolute value, but opposite signs.

Since no positive or negative number raised to an even power can give a negative number, *A negative number has no even root that can be expressed as a positive or negative number.*

Since $(+2)^3=8$ and $(-2)^3=-8$, $\sqrt[3]{8}=+2$ and $\sqrt[3]{-8}=-2$. Similarly, since $(+2)^5=32$ and $(-2)^5=-32$, $\sqrt[5]{32}=+2$ and $\sqrt[5]{-32}=-2$. In general,

Any given number has one odd root that has the same sign as the given number.

The *indicated even root* of a negative number is called an **imaginary number.** All other numbers are called **real numbers**, for distinction.

Thus, $\sqrt{-9}$ and $\sqrt[4]{-1}$ are *imaginary numbers*, and 3 and $\sqrt{5}$ are *real numbers.*

It can be shown that every number has *two* square roots, *three* cube roots, *four* fourth roots, etc., expressed in terms of real and imaginary numbers.

Thus, the three cube roots of 8 are 2, $-1+\sqrt{-3}$, and $-1-\sqrt{-3}$. For the cube of each of these is 8. The four fourth roots of 81 are 3, -3, $\sqrt{-9}$, and $-\sqrt{-9}$. For the fourth power of each of these is 81. The six sixth roots of 64 are 2, -2, $1+\sqrt{-3}$, $1-\sqrt{-3}$, $-1+\sqrt{-3}$, and $-1-\sqrt{-3}$.

64. The Principal Root. — The positive real even root of a positive number, or the real odd root of a positive or negative number, is called the **principal root.**

Thus, the principal square root of 9 is 3, the principal cube root of 64 is 4, the principal cube root of -27, is -3, etc.

Unless otherwise stated, only the principal root of a number is to be understood as indicated in an expression.

Thus, in $1+\sqrt{3}$ only the principal square root of 3 is understood. Similarly, in finding the values of m and n that satisfy the equation $\sqrt{m}-\sqrt{n}=1$, only the principal square roots of m and n are allowed to be used.

65. Root of a Monomial. — As shown in the First Course,

To find the principal square root of a monomial take the square root of the numerical coefficient, and divide the exponent of each literal factor by 2.

To find the principal cube root of a monomial, take the cube root of the numerical coefficient, and divide the exponent of each literal factor by 3.

Thus, since $(3m^4n^3)^2 = 9m^8n^6$, the principal square root of $9m^8n^6$ is $3m^4n^3$, which may be obtained by the above rule.

Since $(2x^2y^4)^3 = 8x^6y^{12}$, the principal cube root of $8x^6y^{12}$ is $2x^2y^4$, which may be obtained by the above rule.

Similarly, in general,

To find the principal nth root of a monomial, take the nth root of the numerical coefficient, and divide the exponent of each literal factor by n.

EXERCISES

Find the principal square roots of:

1. 25. **3.** n^6. **5.** $16a^2b^4$. **7.** $100m^4n^8$. **9.** $4p^6q^2r^4$.

2. 81. **4.** P^{10}. **6.** $36V^4t^6$. **8.** $121A^2B^{12}$. **10.** $9x^4y^6z^8$.

Find the principal cube roots of:

11. 27. **13.** A^6. **15.** $8x^3y^6$. **17.** $125a^{12}b^{15}$. **19.** $8x^3y^6z^9$.

12. 64. **14.** s^{12}. **16.** $27P^6Q^9$. **18.** $64W^9V^{18}$. **20.** $27m^{12}n^3p^{21}$.

Find the principal fourth roots of:

21. 16. **22.** 81. **23.** a^8b^{12}. **24.** $256x^4y^{16}$. **25.** $625M^8N^{12}P^4$.

Find the principal fifth roots of:

26. 32. **27.** 243. **28.** x^5y^{10}. **29.** $32A^{15}B^{20}$. **30.** $p^{25}q^{15}r^{30}s^{20}$.

66. Square Root of a Polynomial. — As shown in the First Course (see § 135), to find the principal square root of a polynomial:

(1) *Write the given polynomial in descending or ascending powers of some letter.*

(2) *Take the square root of the first term for the first term of the root, and subtract the first term from the polynomial.*

(3) *Double the root found for the first trial divisor, divide the first term of the remainder by this, and write the quotient as the second term of the root.*

(4) *Add this quotient to the trial divisor to obtain the true divisor, multiply the true divisor by the second term of the root, and subtract the product from the preceding remainder.*

(5) *If there is still a remainder, double all of the root already found, for a new trial divisor, and proceed as before. Continue this process until all terms of the root are found.*

EXAMPLE. — Find the square root of $a^4 - 10a^3 + 29a^2 - 20a + 4$.

$$\begin{array}{l|l|l}
 & a^4 - 10a^3 + 29a^2 - 20a + 4 & a^2 - 5a + 2 \\
 & a^4 & \\
\hline
2a^2 & -10a^3 + 29a^2 - 20a + 4 & \\
2a^2 - 5a & -10a^3 + 25a^2 & \\
\hline
2a^2 - 10a & \qquad\qquad 4a^2 - 20a + 4 & \\
2a^2 - 10a + 2 & \qquad\qquad 4a^2 - 20a + 4 & \\
\hline
\end{array}$$

The square root is $a^2 - 5a + 2$.

EXERCISES

Find the square root of:

1. $A^2 - 6A + 9$.
2. $4t^2 + 12t + 9$.
3. $1 - 10V + 25V^2$.
4. $49m^4 + 28m^2f + 4f^2$.
5. $36a^2 - 48ab + 16b^2$.
6. $16xy + y^2 + 64x^2$.
7. $n^4 - 6n^3 + 13n^2 - 12n + 4$.
8. $1 + 10k + 29k^2 + 20k^3 + 4k^4$.
9. $t^6 - 4t^3 + 25t^2 + 4 + 20t - 10t^4$.
10. $16R^4 + 30R^3 - 24R^2 + 9 + 25R^6 - 40R^5$.
11. $a^4 - 2a^3b + 3a^2b^2 - 2ab^3 + b^4$.
12. $36a^2 + 12ab + b^2 - 24ac - 4bc + 4c^2$.

13. $x^2+25y^2+6xz-10xy-30yz+9z^2$.

14. $v^6-4v^5-24v^3+33v^2+14v^4-20v+4$.

Find the first four terms of the square root of:

15. $1-4x$. **16.** $1+8y$. **17.** $1-a$.

Find the fourth root of:

18. $x^4-12x^3+54x^2-108x+81$.

Suggestion. — Find the square root of the square root.

19. $m^4-8m^3n+24m^2n^2-32mn^3+16n^4$.

67. Square Root of an Arithmetical Number. — An arithmetical number is in nature a polynomial. Hence a process similar to that in § 66 may be used for finding the square root of an arithmetical number.

It is easily seen that, if the figures of an integer are marked off from right to left into groups of two, the number of figures in the integral part of the square root will be equal to the number of groups, any one figure that remains on the left being counted as a whole group.

Thus, marked off into groups, 42579 becomes 4′25′79′. Hence the integral part of the square root will contain three figures.

The similarity of the process of finding the square root of an integer to that of § 66 is shown in the following example:

Example. — Find the square root of 119,716.

Pointing off gives 11′97′16. Hence the square root has three figures — hundreds, tens, and ones. The root lies between 300 and 400, because 300^2 is less and 400^2 more than 119,716. That is, the value of the hundreds in the root is 300.

The work is as follows:

$$\begin{array}{r|r|l}
 & 11'97'16' & \underline{300+40+6=346} \\
 & 9\ 00\ 00 & \\ \hline
2\times 300=600 & 2\ 97\ 16 & \\
600+40=640 & 2\ 56\ 00 & \\ \hline
2\times 340=680 & 41\ 16 & \\
680+6=686 & \underline{41\ 16} &
\end{array}$$

Explanation. — $300^2 = 90000$. Subtracting this leaves 29716. Doubling 300 gives 600 for the first trial divisor. Dividing 29716 by 600 gives approximately 40, the value of the tens in the root. Adding 40 to 600 gives 640, the first true divisor. Multiplying 640 by 40 gives 25600. Subtracting this leaves 4116. Doubling 340, the value of the part of the root found, gives 680 for the second trial divisor. Dividing 4116 by 680 gives approximately 6. Adding 6 to 680 gives 686, the second true divisor. Multiplying 686 by 6 gives 4116. Subtracting this leaves no remainder.

Hence the exact root is $300 + 40 + 6$ or 346.

If a number that contains a decimal is marked off into groups of two figures each, beginning at the decimal point and marking both to the left and right, the number of figures in the integral and decimal parts of the square root will equal the number of periods in the integral and decimal parts, respectively, of the given number. If the decimal part has an odd number of figures, add a zero to make it even.

To find the approximate value, to any number of decimal places, of the square root of a number that is not a perfect square, mark the decimal point, annex zeros, and proceed as with a decimal.

Example. — Find to two decimal places the square root of 8.

We annex 4 zeros after the decimal point.

$$
\begin{array}{r|l|l}
 & 8.'00'00' & \underline{2 + .8 + .02 = 2.82} \\
 & \underline{4.\ 00\ 00} & \\
2 \times 2 = 4 & 4.\ 00\ 00 & \\
4 + .8 = 4.8 & \underline{3.\ 84\ 00} & \\
2 \times 2.8 = 5.6 & .\ 16\ 00 & \\
5.6 + .02 = 5.62 & \underline{.\ 11\ 24} &
\end{array}
$$

EXERCISES

Find the square root of:

1. 2209.
2. 8464.
3. 131044
4. 649636.
5. 2673225
6. 68.89.
7. 1.6129.
8. 0.1849.
9. 0.007396.

Find to three decimal places the square root of:

10. 12.	**12.** 20.	**14.** 1.61.
11. 7.	**13.** 0.2.	**15.** 0.05.

16. The legs of a right triangle are 48 inches and 64 inches. Find the hypotenuse.

17. The hypotenuse of a right triangle is 14 inches and one leg 8 inches. Find the length of the other leg.

18. A rectangle is 19 inches long and 11 inches wide. What is the length of the diagonal?

19. Find the radius of a circle whose area is 60 square inches.

20. The boiler of an engine contains 96 flues or pipes, each 3 inches in diameter. If the area of the cross section of the smokestack equals the combined areas of the cross sections of all of the flues, what must be the diameter of the smokestack?

21. Three drainage pipes, each 6 inches in diameter, empty into one larger pipe. If the carrying capacity of the larger pipe is equal to the sum of the carrying capacities of the smaller pipes, what must be its diameter?

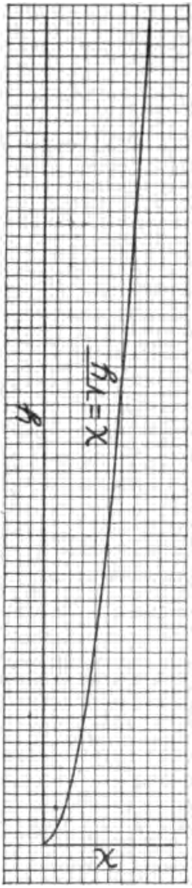
SUGGESTION. — Consider that the area of a cross section of the larger pipe must equal the sum of the cross sectional areas of the three smaller pipes.

22. The area of a triangle whose sides are a, b, and c equals $\sqrt{s(s-a)(s-b)(s-c)}$, where s is one half of the sum of the three sides. Find the area of a triangle whose sides are 12 inches, 16 inches, and 18 inches.

23. Find the area of a triangle whose sides are 46 feet, 30 feet, and 24 feet.

68. Roots of Numbers found Graphically. — If y represents any number and x the square root of that number, then $x = \sqrt{y}$. By assigning values to y and computing the corresponding values of x, we get the following sets of values of x and y that satisfy this equation:

y	0	1	4	9	16	25	36	49	64
x	0	1	2	3	4	5	6	7	8



Locating the points corresponding to these sets of values of x and y, we get the graph in the margin.

By means of this graph we can read off the square root of any number between 0 and 64, approximated to tenths.

NOTE. — If desired, the student may make such a graph on a larger piece of squared paper and with a larger scale, say 1 space to the unit along the y-axis and 5 spaces to the unit along the x-axis. The larger graph could be used for getting the roots of larger numbers, and for getting the roots with greater precision.

Similar graphs may be constructed for finding the cube roots, fourth roots, etc., of numbers.

EXERCISES

Find to tenths, by use of the above graph, the square roots of:

1. 10. 3. 20. 5. 40. 7. 60. 9. 2.5.
2. 15. 4. 28. 6. 52. 8. 6.5. 10. 12.5.

11. By constructing the graph of $x = \sqrt[3]{y}$, a curve is obtained for finding the cube roots of numbers graphically. By assigning to y the values 0, 1, 8, 27, 64, 125, 216, get the corresponding values of x, and draw the graph.

12. Use the graph in Ex. 11 for finding to tenths the cube roots of 12, 20, 45, 100, and 150.

69. Surds. — An indicated root of any number is called a **radical.**

Thus, $\sqrt{4}$, $\sqrt[3]{10}$, $\sqrt{x^2 - y^2}$, are radicals.

Any expression containing one or more radicals is called a **radical expression.**

If the exact value of a radical which is real cannot be computed, that is, cannot be exactly expressed by a whole number or common fraction, it is called a **surd.**

Thus, $\sqrt{3}$, $\sqrt[3]{12}$, $\sqrt{1-x^2}$ are surds.

The index of a surd indicates the **order** of the surd. A surd whose index is 2 is a surd of the **second order,** or a **quadratic surd.** A surd whose index is 3 is of the **third order.** In general, a surd whose index is n is of the **nth order.**

Thus, $\sqrt{5}$ is of the *second order;* $\sqrt[3]{2x}$ of the *third order;* $\sqrt[4]{a+b}$ of the *fourth order;* etc.

NOTE. — Although the exact value of a quadratic surd cannot be computed, it can be represented graphically by a line of definite length. See FIRST COURSE, § 143.

70. Simplest Form of a Surd. — The *form* of a surd may sometimes be changed without altering its value.

If a surd contains under the radical sign a factor of which the indicated root can be extracted, this factor may be removed from under the radical.

Thus, $\sqrt[3]{a^6b} = a^2\sqrt[3]{b}$, and $\sqrt{48} = 4\sqrt{3}$.

If a surd contains a fraction under the radical sign, this fraction may be removed.

Thus, $\sqrt{\frac{2}{3}} = \frac{1}{3}\sqrt{6}$, and $\sqrt[4]{\frac{m}{n^3}} = \frac{1}{n}\sqrt[4]{mn}$.

If the index of a surd and the exponents of all factors under the radical sign have a common divisor, the surd may be changed to lower order.

Thus, $\sqrt[6]{a^2b^4} = \sqrt[3]{ab^2}$, and $\sqrt[4]{9} = \sqrt{3}$.

Removing a factor or fraction from under the radical sign, or reducing the order of a surd, is called **simplifying the surd.**

A surd is in its **simplest form** when the expression under the radical sign is integral, contains no factor of which the indicated root can be extracted, and the exponents of its factors and the index have no common divisor.

71. Simplifying a Surd.—To simplify a surd by removing from under the radical sign a factor of which the indicated root can be extracted, use is made of the following principle:

The nth root of a product equals the product of the nth roots of its factors; that is:

$$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}.$$

This principle is true because $(\sqrt[n]{a}\,\sqrt[n]{b})^n = (\sqrt[n]{a})^n(\sqrt[n]{b})^n = ab.$

EXAMPLE. — Simplify $\sqrt[4]{32\,a^4b^8c^7}$.

Separating $32\,a^4b^8c^7$ into two factors so that the fourth root of one can be extracted,

$$\sqrt[4]{32\,a^4b^8c^7} = \sqrt[4]{(16\,a^4b^8c^4)(2\,c^3)} = \sqrt[4]{16\,a^4b^8c^4}\,\sqrt[4]{2\,c^3} = 2\,ab^2c\sqrt[4]{2\,c^3}.$$

To simplify a surd containing a fraction under the radical sign, first multiply both terms of the fraction by such an expression as will give a new denominator of which the indicated root can be extracted, then write the new fraction as the product of two factors, and apply the above principle.

EXAMPLE 1. — Simplify $\sqrt[3]{\frac{16}{25}}$.

Multiplying both terms of the fraction by 5,

$$\sqrt[3]{\frac{16}{25}} = \sqrt[3]{\frac{80}{125}} = \sqrt[3]{\frac{8}{125} \times 10} = \sqrt[3]{\frac{8}{125}}\;\sqrt[3]{10} = \tfrac{2}{5}\sqrt[3]{10}.$$

EXAMPLE 2. — Simplify $\sqrt[4]{\frac{5}{2\,n}}$.

$$\sqrt[4]{\frac{5}{2\,n}} = \sqrt[4]{\frac{40\,n^3}{16\,n^4}} = \sqrt[4]{\frac{1}{16\,n^4} \times 40\,n^3} = \sqrt[4]{\frac{1}{16\,n^4}}\;\sqrt[4]{40\,n^3} = \frac{1}{2\,n}\sqrt[4]{40\,n^3}.$$

To simplify a surd by reducing it to lower order, use is made of the following principle:

The value of a surd is not changed by dividing the index of the surd and the exponents of all factors under the radical sign by the same number. That is,

$$\sqrt[mx]{a^{nx}b^{px}} = \sqrt[m]{a^n b^p}.$$

To prove this principle, let $\sqrt[mx]{a^{nx}b^{px}} = r$.

Then $a^{nx}b^{px} = r^{mx}$, by definition of a root.

Hence, $\sqrt[x]{a^{nx}b^{px}} = \sqrt[x]{r^{mx}}$, or, $a^n b^p = r^m$(§ 65).

Hence, $\sqrt[m]{a^n b^p} = \sqrt[m]{r^m} = r$.

Therefore, $\sqrt[mx]{a^{nx}b^{px}} = \sqrt[m]{a^n b^p}$, because each equals r.

Example 1. — Simplify $\sqrt[12]{a^4b^8}$.
Dividing index and exponents by 4,

$$\sqrt[12]{a^4b^8} = \sqrt[3]{ab^2}.$$

Example 2. — Simplify $\sqrt[6]{125}$.

$$\sqrt[6]{125} = \sqrt[6]{5^3} = \sqrt{5}.$$

A surd sometimes may be simplified by applying both of the above principles.

Example. — Simplify $\sqrt[4]{144\,x^6y^8}$.

$$\sqrt[4]{144\,x^6y^8} = \sqrt[4]{12^2\,x^6y^8} = \sqrt{12\,x^3y^4} = \sqrt{(4\,x^2y^4)(3\,x)} = 2\,xy^2\sqrt{3\,x}.$$

EXERCISES

Reduce to the simplest form:

1. $\sqrt{8}$.
2. $\sqrt{27}$.
3. $\sqrt{12}$.
4. $\sqrt{72}$.
5. $\sqrt[3]{16}$.
6. $\sqrt[3]{24}$.
7. $\sqrt[3]{72}$.
8. $\sqrt[3]{250}$.
9. $\sqrt[4]{32}$.
10. $\sqrt[4]{162}$.
11. $\sqrt[5]{64}$.
12. $\sqrt[5]{224}$.
13. $\sqrt{\frac{1}{3}}$.
14. $\sqrt{\frac{2}{5}}$.
15. $\sqrt{\frac{8}{27}}$.
16. $\sqrt[3]{\frac{1}{2}}$.
17. $\sqrt[3]{\frac{3}{4}}$.
18. $\sqrt[3]{\frac{5}{7}}$.
19. $\sqrt[4]{\frac{1}{4}}$.
20. $\sqrt[5]{\frac{3}{16}}$.

21. $\sqrt{a^3b^2}$.

22. $\sqrt{3m^5n}$.

23. $\sqrt{8x^2y^7}$.

24. $\sqrt[3]{5v^4t}$.

25. $\sqrt[3]{16R^3r^4}$.

26. $\sqrt[4]{32A^2B^5}$.

27. $\sqrt{80x^6y^2}$.

28. $\sqrt[4]{a^2b^2}$.

29. $\sqrt[6]{a^4b^8}$.

30. $\sqrt[8]{16x^{12}y^8}$.

31. $\sqrt[6]{27w^9v^{12}}$.

32. $\sqrt[10]{32P^5Q^{15}}$.

33. $\sqrt[12]{16x^9y^6z^{12}}$.

34. $\sqrt{\frac{a}{b}}$.

35. $\sqrt{\frac{x}{y^3}}$.

36. $\sqrt{\frac{3m}{2n}}$.

37. $\sqrt[4]{\frac{1}{4x^2}}$.

38. $\sqrt[3]{\frac{a^2b}{5c^4}}$.

39. $\sqrt[3]{\frac{16x^5}{27y^2}}$.

40. $\sqrt[4]{\frac{25v^6}{49t^2}}$.

41. $\sqrt{\frac{a+b}{a-b}}$.

42. $\sqrt{\frac{m^2n}{m+n}}$.

43. $\sqrt{\frac{B^2}{4A^2}-\frac{C}{A}}$.

44. $\sqrt{\frac{x}{y^3}+\frac{y}{x^3}}$.

45. $\frac{3x}{2y}\sqrt[3]{\frac{27y^4}{x^4}}$.

46. $\frac{a^2b}{2}\sqrt[5]{\frac{a^6b^2}{27}}$.

47. $\frac{1}{K^2}\sqrt[4]{\frac{1}{K^2}-\frac{2}{K}+1}$.

48. $\sqrt{4-8\sqrt{3}}$.

49. $\sqrt{R^2-3R^2\sqrt{5}}$.

50. $\sqrt{(a-b)(a^2-b^2)}$.

72. Addition and Subtraction of Surds. — Two terms containing the same surd factor may be added or subtracted by adding or subtracting the coefficients of the surd factor.

EXAMPLE. — $4\sqrt{2}+3\sqrt{2}=7\sqrt{2}$.

Terms containing surd factors that are not of the same form may sometimes be added or subtracted by first reducing the surd factors to the same form. In such cases, all of the surds should be reduced to the simplest form.

EXAMPLE. — Simplify $\sqrt{32}+\sqrt{50}-\sqrt{8}$.

$$\sqrt{32}+\sqrt{50}-\sqrt{8}=4\sqrt{2}+5\sqrt{2}-2\sqrt{2}=7\sqrt{2}.$$

EXERCISES

Simplify:

1. $\sqrt{18}-\sqrt{8}+\sqrt{32}$.
2. $\sqrt{75}-\sqrt{3}-\sqrt{12}$.
3. $\sqrt{20}+\sqrt{45}-\sqrt{80}$.
4. $\sqrt{294}+\sqrt{486}-\sqrt{24}$.
5. $\sqrt[3]{16}-\sqrt[3]{54}+2\sqrt[3]{2}$.
6. $2\sqrt[3]{3}+\sqrt[3]{24}+5\sqrt[3]{192}$.
7. $\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{8}}$.
8. $\sqrt{\frac{2}{3}}-\sqrt{\frac{3}{8}}$.
9. $4\sqrt[3]{\frac{2}{9}}-\sqrt[3]{\frac{3}{4}}$.
10. $9\sqrt[3]{\frac{4}{9}}+\sqrt[3]{\frac{3}{2}}$.
11. $\sqrt{32}+\sqrt[4]{324}$.
12. $\sqrt{75}-\sqrt[6]{1728}$.
13. $\sqrt{4n}+\sqrt{9n}+\sqrt{16n}-\sqrt{25n}$.
14. $\sqrt[3]{8t^2}-\sqrt[3]{125t^2}-\sqrt[3]{t^2}+6\sqrt[3]{64t^2}$.
15. $\sqrt{a^3}-2a\sqrt[4]{a^2}+3a\sqrt[6]{a^3}-\sqrt[8]{a^{12}}$.
16. $5m\sqrt[6]{(m-n)^3}+\sqrt{(m-n)^3}+n\sqrt[4]{(m-n)^2}$.
17. $3\sqrt{4m^3-4m^2n}-\sqrt{mn^2-n^3}+\sqrt{(m^2-n^2)(m+n)}$.
18. $\sqrt[3]{(A+B)^4}+(A-B)\sqrt[3]{A+B}+A\sqrt[6]{A^2+2AB+B^2}$.
19. $\sqrt{\frac{5x}{y}}+\sqrt{\frac{5y}{x}}+\left(\frac{1}{x}-\frac{1}{y}\right)\sqrt{5xy}$.
20. $(p-q)\sqrt{\frac{p+q}{p-q}}+\frac{p+q}{q}\sqrt{\frac{pq^2-q^3}{p+q}}+8\sqrt{p^2-q^2}$.

73. Changing Surds to the Same Order. — It follows from the second principle in § 71 that

The value of a surd is not changed by multiplying the index of the surd and the exponents of the factors under the radical sign by the same number. That is,

$$\sqrt[m]{a^n b^p}=\sqrt[mx]{a^{nx}b^{px}}.$$

By use of this principle surds of different orders may be changed to surds of the same order, without altering their values. The lowest order to which two or more surds may be changed is the least common multiple of their given orders.

Example 1. — Reduce to the same order $\sqrt[4]{v^2t^3}$ and $\sqrt[6]{v^3t^2}$.

The l. c. m. of the indices is 12. Multiplying index and exponents of $\sqrt[4]{v^2t^3}$ by 3 and index and exponents of $\sqrt[6]{v^3t^2}$ by 2,

$$\sqrt[4]{v^2t^3} = \sqrt[12]{v^6t^9} \text{ and } \sqrt[6]{v^3t^2} = \sqrt[12]{v^6t^4}.$$

Example 2. — Reduce to the same order $\sqrt{2}$ and $\sqrt[5]{3}$.

$$\sqrt{2} = \sqrt[10]{2^5} = \sqrt[10]{32} \text{ and } \sqrt[5]{3} = \sqrt[10]{3^2} = \sqrt[10]{9}.$$

EXERCISES

Reduce to the same order:

1. $\sqrt[3]{n^2}$, $\sqrt[4]{n^3}$, $\sqrt[6]{n^5}$.
2. $\sqrt{V}$, $\sqrt[4]{V^2}$, $\sqrt[5]{V^3}$.
3. $\sqrt{2x}$, $\sqrt[3]{3x}$, $\sqrt[3]{2x^2}$.
4. $\sqrt{3t^3}$, $\sqrt[3]{2t^2}$, $\sqrt[4]{t}$.
5. $\sqrt{a^5b^3}$, $\sqrt[5]{a^4b}$, $\sqrt[4]{ab^3}$.
6. $\sqrt{2}$, $\sqrt[3]{3}$.
7. $\sqrt[3]{4}$, $\sqrt[4]{6}$, $\sqrt{2}$.
8. $\sqrt[5]{3N^3}$, $\sqrt[6]{2N^2}$.
9. $\sqrt{\frac{1}{2}}$, $\sqrt[3]{\frac{1}{3}}$, $\sqrt[4]{\frac{1}{4}}$.
10. $\sqrt[3]{5y^2z}$, $\sqrt{\frac{1}{2}yz^3}$, $\sqrt[4]{\frac{1}{3}y^3}$.

Determine which is the greater:

11. $\sqrt[3]{16}$ or $\sqrt{6}$.

Suggestion. — Change them to the same order.

12. $\sqrt[3]{11}$ or $\sqrt{5}$. 13. $\sqrt{3}$ or $\sqrt[4]{6}$. 14. $\sqrt[4]{48}$ or $\sqrt[5]{64}$.

74. Multiplication of Surds. — It follows from the first principle in § 71 that

The product of two surds of the same order is equal to the surd of that order of which the number under the radical sign equals the product of the numbers under the radical signs in the given surds. That is,

$$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}.$$

Example. — $\sqrt[3]{4}\sqrt[3]{5} = \sqrt[3]{20}$, and $\sqrt[4]{at^2}\sqrt[4]{t^3} = \sqrt[4]{at^5} = t\sqrt[4]{at}$.

The product of two surds of different orders may be obtained by first changing them to surds of the same order as in § 73, then applying the above principle.

Example. — Find the product of $\sqrt{2n}$, $\sqrt[3]{3n^2}$, and $\sqrt[6]{n^4}$.

$$\sqrt{2n} = \sqrt[6]{8n^3};\quad \sqrt[3]{3n^2} = \sqrt[6]{9n^4};\quad \sqrt[6]{n^4} = \sqrt[6]{n^4}.$$

Hence, $$\sqrt{2n}\,\sqrt[3]{3n^2}\,\sqrt[6]{n^4} = \sqrt[6]{8n^3}\,\sqrt[6]{9n^4}\,\sqrt[6]{n^4}$$
$$= \sqrt[6]{72n^{11}}, \text{ or } n\sqrt{72n^5}.$$

To multiply two polynomials involving surds, proceed as in multiplication of polynomials in Chapter I.

Example. — Multiply $2\sqrt{5} + 5\sqrt{2}$ by $3\sqrt{5} - 4\sqrt{2}$.

$$\begin{array}{rrr}
 & 2\sqrt{5} & +\ 5\sqrt{2} \\
 & 3\sqrt{5} & -\ 4\sqrt{2} \\
\hline
 & 6\sqrt{25} & +\ 15\sqrt{10} \\
 & & -\ \ 8\sqrt{10} - 20\sqrt{4} \\
\hline
 & 30\ + & 7\sqrt{10} - 40, \text{ or } 7\sqrt{10} - 10
\end{array}$$

A term consisting of a surd factor with a coefficient may be expressed as an *entire surd* by changing the coefficient to the form of a surd and then multiplying.

Example. — Write as an entire surd $3\sqrt[3]{4}$.

We may write $$3 = \sqrt[3]{3^3} = \sqrt[3]{27}.$$

Hence, $$3\sqrt[3]{4} = \sqrt{27}\ \sqrt[3]{4} = \sqrt[3]{108}.$$

EXERCISES

Find the product of:

1. $\sqrt{5}\ \sqrt{7}$.
2. $\sqrt{2}\ \sqrt{3}\ \sqrt{6}$.
3. $\sqrt{8}\ \sqrt{2}\ \sqrt{8}$.
4. $\sqrt[3]{9}\ \sqrt[3]{6}$.
5. $\sqrt[3]{21}\ \sqrt[3]{49}$.
6. $\sqrt[4]{2}\ \sqrt[4]{8}\ \sqrt[4]{5}$.

7. $\sqrt[5]{9}\,\sqrt[5]{40}$.

8. $\sqrt{\frac{1}{2}}\,\sqrt{\frac{1}{8}}$.

9. $\sqrt{\frac{4}{5}}\,\sqrt{\frac{5}{8}}$.

10. $\sqrt{\frac{2}{3}}\,\sqrt{\frac{3}{5}}\,\sqrt{\frac{5}{2}}$.

11. $\sqrt{\frac{1}{2}}\,\sqrt{\frac{1}{4}}\,\sqrt{\frac{1}{8}}$.

12. $\sqrt{2}\,\sqrt[3]{3}$.

13. $\sqrt[4]{x^3}\,\sqrt[5]{x^2}$.

14. $\sqrt{2n}\,\sqrt[3]{5n^2}$.

15. $\sqrt[4]{3t^2}\,\sqrt{2t}\,\sqrt[3]{t}$.

16. $\sqrt{2}\,\sqrt[3]{3}\,\sqrt[4]{4}$.

17. $\sqrt[6]{5y^5}\,\sqrt{2y}\,\sqrt[3]{3y^2}$.

18. $\sqrt[4]{2V^3}\,\sqrt[3]{3V}\,\sqrt{V}$.

19. $\sqrt{a}\,\sqrt{a-b}$.

20. $\sqrt{m-n}\,\sqrt{m+n}$.

21. $\sqrt{x+y}\,\sqrt{x^2-y^2}$.

22. $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})$.

23. $(5\sqrt{2}+2\sqrt{3})(\sqrt{2}+3\sqrt{3})$.

24. $(\sqrt{8}+\sqrt{5})(\sqrt{2}-\sqrt{5})$.

25. $(1+\sqrt{2}+\sqrt{3})(1-\sqrt{2}+\sqrt{3})$.

26. $(\sqrt{2}+\sqrt{3}+\sqrt{5})(\sqrt{2}+\sqrt{3}-\sqrt{5})$.

27. $(\sqrt{a}+\sqrt{b})(\sqrt{a}+2\sqrt{b})$.

28. $(m\sqrt{n}-\sqrt{mn}+n\sqrt{m})(\sqrt{m}-\sqrt{n})$.

29. $(\sqrt{x}-\sqrt[4]{xy}+\sqrt{y})(\sqrt{x}+\sqrt[4]{xy}+\sqrt{y})$.

30. $(\sqrt{n-1}+3)(\sqrt{n-1}+5)$.

31. $(\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2})(\sqrt[3]{a}+\sqrt[3]{b})$.

32. $(\sqrt{p}+\sqrt{q}+\sqrt{r})^2$.

Simplify:

33. $(\frac{1}{2}R\sqrt{2+\sqrt{2}})(\frac{1}{2}R\sqrt{2-\sqrt{2}})$.

34. $\sqrt{R^2-(\frac{1}{2}R\sqrt{2-\sqrt{2}})^2}$.

Express as entire surds:

35. $2\sqrt{3}$.

36. $5\sqrt{7}$.

37. $\frac{1}{2}\sqrt{2}$.

38. $2\sqrt[3]{2}$.

39. $\frac{1}{3}\sqrt[3]{3}$.

40. $a\sqrt{ab}$.

41. $t^3\sqrt[4]{2t}$.

42. $4mn\sqrt[3]{n^2}$.

43. $\frac{a}{b}\sqrt{\frac{b}{a}}$.

44. $\frac{3v}{2t^2}\sqrt[3]{\frac{16t}{27v^2}}$.

45. $\frac{2c}{k}\sqrt[4]{\frac{k^3}{16c^3}}$.

75. Division of Surds. — The division of two surds, or of two expressions involving surds, is most easily performed by first indicating the division in the form of a fraction, and then multiplying both dividend and divisor by such an expression as will free the divisor of surds.

This process is called **rationalizing the divisor.** The multiplier used is called the **rationalizing factor.**

When the divisor is a monomial, the rationalizing factor is a monomial.

EXAMPLE. — Divide $24\sqrt{2}$ by $\sqrt{12}$.

Multiplying both dividend and divisor by $\sqrt{3}$,

$$\frac{24\sqrt{2}}{\sqrt{12}}=\frac{24\sqrt{6}}{\sqrt{36}}=\frac{24\sqrt{6}}{6}=4\sqrt{6}.$$

When the divisor is a binomial of the form $\sqrt{a}+\sqrt{b}$, the rationalizing factor is $\sqrt{a}-\sqrt{b}$. And when the divisor is of the form $\sqrt{a}-\sqrt{b}$, the rationalizing factor is $\sqrt{a}+\sqrt{b}$. The product of $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$ is $a-b$.

Two expressions of the forms $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$ are called **conjugate expressions.**

EXAMPLE. — Divide $2\sqrt{5}+3\sqrt{2}$ by $\sqrt{5}+\sqrt{2}$.

Multiplying by $\sqrt{5}-\sqrt{2}$, conjugate of $\sqrt{5}+\sqrt{2}$,

$$\frac{2\sqrt{5}+3\sqrt{2}}{\sqrt{5}+\sqrt{2}}=\frac{(2\sqrt{5}+3\sqrt{2})(\sqrt{5}-\sqrt{2})}{(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2})}=\frac{4+\sqrt{10}}{3}=\frac{4}{3}+\frac{1}{3}\sqrt{10}.$$

When the divisor is a trinomial of the form $\sqrt{a}\pm\sqrt{b}\pm\sqrt{c}$, although it may be rationalized by a single multiplication, it is easier to perform two multiplications in succession. By multiplying by the trinomial obtained by changing the sign of one term of the divisor, a new divisor is obtained that contains one less surd than before.

Example. — Divide 2 by $\sqrt{5}+\sqrt{2}-\sqrt{3}$.

Multiplying by $\sqrt{5}+\sqrt{2}+\sqrt{3}$,

$$\frac{2}{\sqrt{5}+\sqrt{2}-\sqrt{3}}=\frac{2(\sqrt{5}+\sqrt{2}+\sqrt{3})}{(\sqrt{5}+\sqrt{2}-\sqrt{3})(\sqrt{5}+\sqrt{2}+\sqrt{3})}$$

$$=\frac{2(\sqrt{5}+\sqrt{2}+\sqrt{3})}{4+2\sqrt{10}},\text{ or }\frac{\sqrt{5}+\sqrt{2}+\sqrt{3}}{2+\sqrt{10}}.$$

Now multiplying by $2-\sqrt{10}$,

$$\frac{\sqrt{5}+\sqrt{2}+\sqrt{3}}{2+\sqrt{10}}=\frac{-3\sqrt{2}+2\sqrt{3}-\sqrt{30}}{-6}=\tfrac{1}{2}\sqrt{2}-\tfrac{1}{3}\sqrt{3}+\tfrac{1}{6}\sqrt{30}.$$

Evidently, a similar process may be used when the divisor is a polynomial of more than three terms, if the surds are of the second order. More complicated problems are not given here.

EXERCISES

Perform the indicated divisions:

1. $\dfrac{2}{\sqrt{3}}$.

2. $\dfrac{\sqrt{2}}{\sqrt{5}}$.

3. $\dfrac{8}{\sqrt{6}}$.

4. $\dfrac{12}{\sqrt{18}}$.

5. $\dfrac{9}{\sqrt{21}}$.

6. $\dfrac{\sqrt{8}}{\sqrt{10}}$.

7. $\dfrac{\sqrt{12}}{\sqrt{24}}$.

8. $\dfrac{5}{\sqrt[3]{2}}$.

9. $\dfrac{6}{\sqrt[3]{3}}$.

10. $\dfrac{\sqrt[3]{9}}{\sqrt[3]{18}}$.

11. $\dfrac{\sqrt{m}}{\sqrt{n}}$.

12. $\dfrac{10r}{\sqrt{2r}}$.

13. $\dfrac{15V}{\sqrt{5V}}$.

14. $\dfrac{\sqrt[3]{4y}}{\sqrt[3]{2y^2}}$.

15. $\dfrac{\sqrt{x}}{\sqrt[3]{x}}$.

16. $\dfrac{36}{\sqrt{3}+1}$.

17. $\dfrac{10}{1-\sqrt{5}}$.

18. $\dfrac{\sqrt{2}}{1+\sqrt{2}}$.

19. $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$.

20. $\dfrac{\sqrt{2}+\sqrt{5}}{2\sqrt{2}+\sqrt{3}}$.

21. $\dfrac{6\sqrt{7}-5\sqrt{6}}{2\sqrt{7}+3\sqrt{6}}$.

22. $\dfrac{1}{\sqrt{m}-\sqrt{n}}$.

23. $\dfrac{a-b}{\sqrt{a}+\sqrt{b}}$.

24. $\dfrac{x^3-y^3}{\sqrt{x^3}-\sqrt{y^3}}$.

25. $\dfrac{1-\sqrt{2}}{1+\sqrt{2}+\sqrt{3}}$.

27. $\dfrac{\sqrt{2}-\sqrt{3}+\sqrt{5}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

26. $\dfrac{5}{\sqrt{7}-\sqrt{5}+\sqrt{3}}$.

28. $\dfrac{\sqrt{p}}{\sqrt{p}+\sqrt{q}+\sqrt{r}}$.

29. The volume of a frustum of a pyramid whose altitude is H, lower base B, and upper base b is

$$\frac{\frac{1}{3}H(B\sqrt{B}-b\sqrt{b})}{\sqrt{B}-\sqrt{b}}.$$

Show that this may be written in the simpler form

$$\tfrac{1}{3}H(B+b+\sqrt{Bb}).$$

76. Powers of Surds. — Powers of surds are obtained by use of the principle

$$(\sqrt[m]{a})^n=\sqrt[m]{a^n}.$$

This principle follows from § 74, thus:

$$\begin{aligned}(\sqrt[m]{a})^n &= \sqrt[m]{a}\,\sqrt[m]{a}\,\sqrt[m]{a}\cdots \text{ to } n \text{ factors}\\ &= \sqrt[m]{a\cdot a\cdot a\cdots \text{ to } n \text{ factors}}\\ &= \sqrt[m]{a^n}.\end{aligned}$$

Example. — Raise $2a\sqrt[3]{2a^2b}$ to the fourth power.

$$(2a\sqrt[3]{2a^2b})^4=16a^4\sqrt[3]{(2a^2b)^4}=16a^4\sqrt[3]{16a^8b^4}=32a^6b\sqrt[3]{2a^2b}.$$

77. Roots of Surds. — Roots of surds are obtained by use of the principle

$$\sqrt[m]{\sqrt[n]{a}}=\sqrt[mn]{a}.$$

This principle follows from the fact that the mth root of the nth root of a number is one of the m equal factors of one of the n

equal factors of the number, which is one of the mn equal factors of the number.

Example. — Find the fifth root of $\sqrt[4]{3x^3y^2}$.

$$\sqrt[5]{\sqrt[4]{3x^3y^2}} = \sqrt[20]{3x^3y^2}.$$

EXERCISES

Simplify:

1. $(\sqrt{3})^3$.
2. $(\sqrt{5})^4$.
3. $(2\sqrt{2})^3$.
4. $(4\sqrt[3]{2})^2$.
5. $(3\sqrt[3]{5})^4$.
6. $(\sqrt[3]{x^2})^2$.
7. $(\sqrt[4]{N^3})^3$.
8. $(t\sqrt[5]{t})^6$.
9. $(2ab\sqrt{ab})^3$.
10. $(6x^2\sqrt[4]{x^2})^2$.
11. $(10m^3\sqrt[5]{mn^3})^4$.
12. $(3p^3q^2\sqrt[5]{p^2q^4})^5$.
13. $(2\sqrt{a^2-b^2})^4$.
14. $\sqrt[3]{\sqrt{5}}$.
15. $\sqrt[3]{\sqrt{8}}$.
16. $\sqrt{\sqrt[3]{25}}$.
17. $\sqrt{\sqrt[3]{64}}$.
18. $\sqrt[5]{\sqrt[3]{32}}$.
19. $\sqrt{\sqrt[3]{\sqrt[4]{x^6}}}$.
20. $\sqrt[4]{\sqrt[3]{625n^8}}$.
21. $\sqrt[3]{\sqrt[4]{8x^6y^{12}}}$.
22. $\sqrt{49r^6s^4\sqrt[3]{8s^{12}}}$.
23. $\sqrt{\sqrt[3]{(a^2-b^2)^2}}$.
24. $\sqrt[m]{\sqrt[n]{\sqrt[p]{a^{mn}}}}$.

78. Imaginary Numbers. — Imaginary numbers have been defined (see § 63) as the indicated even roots of negative numbers. By means of more advanced mathematics, any imaginary number may be expressed in terms of imaginary numbers of the second order. Hence the discussion is limited to the latter. Imaginary numbers will be involved in the roots of equations in the next and succeeding chapters.

A binomial consisting of a real and an imaginary term is called **a complex expression.**

Thus, $2-\sqrt{-5}$ is a complex expression.

79. Typical Form of Imaginary Numbers. — By applying the same process as that of simplifying a surd by removing a factor from under the radical sign, any imaginary number may be reduced to the form $n\sqrt{-1}$.

Thus, $\sqrt{-4}=\sqrt{4}\,\sqrt{-1}=2\sqrt{-1}$; $\sqrt{-10}=\sqrt{10}\,\sqrt{-1}$.

When an imaginary number is expressed in the form $n\sqrt{-1}$, it is said to be in the **typical form.**

The factor $\sqrt{-1}$ of any imaginary number expressed in the typical form is called the **imaginary unit.** It is represented usually by the more convenient symbol i.

80. Addition and Subtraction of Imaginary Numbers. — Imaginary numbers to be added or subtracted must first be reduced to the typical form, and then combined as in the addition and subtraction of surds.

EXAMPLE. — Simplify $\sqrt{-16}+\sqrt{-9}-\sqrt{-25}$.

$$\begin{aligned}\sqrt{-16}+\sqrt{-9}-\sqrt{-25}&=4i+3i-5i\\&=2i,\text{ or }2\sqrt{-1}.\end{aligned}$$

To add or subtract two complex expressions, combine their real terms and combine their imaginary terms.

EXERCISES

Reduce to the typical form:

1. $\sqrt{-36}$.
2. $\sqrt{-64}$.
3. $\sqrt{-81}$.
4. $\sqrt{-100}$.
5. $\sqrt{-625}$.
6. $\sqrt{-6}$.
7. $\sqrt{-12}$.
8. $\sqrt{-20}$.
9. $\sqrt{-\frac{1}{4}}$.
10. $\sqrt{-\frac{1}{9}}$.
11. $\sqrt{-a^4b^2}$.
12. $\sqrt{-16\,m^2n^6}$
13. $\sqrt{-a^2-2\,ab-b^2}$.
14. $\sqrt{24\,xy-9\,x^2-16\,y^2}$.

Simplify:

15. $\sqrt{-9}-\sqrt{-4}+\sqrt{-36}$.
16. $\sqrt{-81}+\sqrt{-100}-\sqrt{-1}$.
17. $\sqrt{-64}-\sqrt{-4}-\sqrt{-16}$.
18. $\sqrt{-49}-2\sqrt{-1}+\sqrt{-121}$.
19. $\sqrt{-\frac{2}{3}}+\sqrt{-\frac{3}{2}}+\sqrt{-1}$.
20. $a\sqrt{-a^2}+\sqrt{-a^4}+a^2\sqrt{-1}$.

21. Add $x + 2y\sqrt{-1}$ and $2x - y\sqrt{-1}$.

22. Add $10 - 3i$ and $6 + 5i$.

23. From $3m - 2ni$ take $m + 3ni$.

24. Simplify $(8 - 5\sqrt{-9}) - (12 + 2\sqrt{-16})$.

25. Write in the form of a complex expression $(a - bi) - (c + di)$.

81. Powers of i. — The successive powers of i are:

$$i = \sqrt{-1} = i$$

$$i^2 = (\sqrt{-1})^2 = -1$$

$$i^3 = (\sqrt{-1})^3 = i \times i^2 = -i$$

$$i^4 = (\sqrt{-1})^4 = i \times i^3 = -i^2 = +1$$

$$i^5 = (\sqrt{-1})^5 = i \times i^4 = i$$

$$i^6 = (\sqrt{-1})^6 = i \times i^5 = i^2 = -1$$

etc.

Hence, the successive powers of i have only the four different values, i, -1, $-i$, $+1$, *repeating in regular order.*

It follows that the value of any power of i may be determined by expressing it in terms of the four lowest powers.

Thus, $i^{11} = (i^8)(i^3) = (i^4)^2(i^3) = (+1)^2(-i) = -i$;

$$i^{16} = (i^4)^4 = (+1)^4 = +1.$$

82. Multiplication of Imaginary Numbers. — The principle $\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$, used in the multiplication of surds, does not hold for the multiplication of imaginary numbers.

Thus, by that principle, $\sqrt{-4}\,\sqrt{-9} = \sqrt{36} = 6$.

But, $\sqrt{-4}\sqrt{-9} = 2i \times 3i = 6i^2 = -6$.

Hence, use of the principle gives the wrong value.

Imaginary numbers to be multiplied should first be reduced to the typical form, and then the product formed by using the values of the powers of i in § 81.

Example 1. — Simplify $\sqrt{-16}\,\sqrt{-9}\,\sqrt{-25}$.

$$\begin{aligned}\sqrt{-16}\,\sqrt{-9}\,\sqrt{-25} &= (4i)(3i)(5i)\\ &= 60i^3\\ &= -60i, \text{ or } -60\sqrt{-1}.\end{aligned}$$

Example 2. — Multiply $\sqrt{-16}-1$ by $\sqrt{-9}+2$.

$$\sqrt{-16}-1=4i-1;\quad \sqrt{-9}+2=3i+2.$$

$$\begin{array}{r} 4i-1\\ \underline{3i+2}\\ 12i^2-3i\qquad\\ \underline{+8i-2}\\ -12+5i-2, \text{ or } 5\sqrt{-1}-14. \end{array}$$

EXERCISES

Find the value of:

1. i^9.
2. i^{12}.
3. i^{19}.
4. i^{21}.
5. i^{36}.
6. i^{42}.
7. i^{51}.
8. i^{57}.

Simplify:

9. $\sqrt{-25}\,\sqrt{-4}$.
10. $\sqrt{-9}\,\sqrt{-16}$.
11. $\sqrt{-36}\,\sqrt{-49}$.
12. $\sqrt{-81}\,\sqrt{-64}$.
13. $\sqrt{-100}\,\sqrt{-1}$.
14. $\sqrt{-a^2}\,\sqrt{-b^2}$.
15. $\sqrt{-10}\,\sqrt{-6}$.
16. $\sqrt{+4}\,\sqrt{-9}\,\sqrt{-25}$.
17. $\sqrt{-16}\,\sqrt{-49}\,\sqrt{-\frac{1}{4}}$.
18. $\sqrt{-\frac{1}{9}}\,\sqrt{-81}\,\sqrt{-36}$.
19. $\sqrt{-5}\,\sqrt{-7}\,\sqrt{-2}$.
20. $\sqrt{-n}\,\sqrt{-n^3}\,\sqrt{-n^5}$.
21. $\sqrt{-25}\,\sqrt{-4}\,\sqrt{-36}\,\sqrt{-49}\,\sqrt{-1}$.
22. $\sqrt{-16}\,\sqrt{-9}\,\sqrt{-25}\,\sqrt{-81}\,\sqrt{-2}\,\sqrt{-2}$.
23. $(1+\sqrt{-2})(1-\sqrt{-2})$.
24. $(\sqrt{-4}+3)(\sqrt{-4}-3)$.

25. $(\sqrt{-9}-\sqrt{-16})(\sqrt{-4}+\sqrt{-1})$.

26. $(\sqrt{-2}+\sqrt{-3})(\sqrt{-2}-\sqrt{-3})$.

27. $(2\sqrt{-4}+3\sqrt{-9})(2\sqrt{-4}+3\sqrt{-9})$.

28. $(\sqrt{-a}+\sqrt{-b})(\sqrt{-a}-\sqrt{-b})$.

29. $(x\sqrt{-x}-y\sqrt{-y})(x\sqrt{-x}-y\sqrt{-y})$.

30. Show that the product of two complex expressions of the forms $a+bi$ and $a-bi$ is real.

83. Division of Imaginary Numbers.—In division involving imaginary numbers it is best to first reduce each imaginary number to the typical form, then multiply both dividend and divisor by such an expression as will make the divisor real and free of radicals. The process is similar to that in the division of surds.

EXAMPLE 1.—Divide 6 by $\sqrt{-4}$.

$$\frac{6}{\sqrt{-4}}=\frac{6}{2i}=\frac{6i}{2i^2}=\frac{6i}{-2}=-3i, \text{ or } -3\sqrt{-1}.$$

EXAMPLE 2.—Divide $2-\sqrt{-9}$ by $3+\sqrt{-16}$.

$$\frac{2-\sqrt{-9}}{3+\sqrt{-16}}=\frac{2-3i}{3+4i}=\frac{(2-3i)(3-4i)}{(3+4i)(3-4i)}=\frac{-6-17i}{25}=-\frac{6}{25}-\frac{17}{25}i.$$

EXERCISES

Simplify:

1. $\dfrac{5}{\sqrt{-1}}$

2. $\dfrac{3}{2\sqrt{-1}}$.

3. $\dfrac{10}{\sqrt{-4}}$

4. $\dfrac{21}{\sqrt{-9}}$.

5. $\dfrac{n}{\sqrt{-n}}$.

6. $\dfrac{4\sqrt{-1}}{\sqrt{-4}}$

7. $\dfrac{\sqrt{-16}}{\sqrt{-9}}$.

8. $\dfrac{\sqrt{-100}}{\sqrt{-25}}$.

9. $\dfrac{-t}{\sqrt{-9t^2}}$.

10. $\dfrac{2\sqrt{-25a^4}}{3\sqrt{-16a^2}}$.

11. $\dfrac{\sqrt{-25v^8}}{\sqrt{-49v^6}}$.

12. $\dfrac{\sqrt{-\frac{1}{4}x^2}}{\sqrt{-\frac{1}{9}x^4}}$.

13. $\dfrac{2}{1+\sqrt{-9}}$.

14. $\dfrac{1+\sqrt{-3}}{1-\sqrt{-2}}$.

15. $\dfrac{3-\sqrt{-5}}{3+\sqrt{-5}}$.

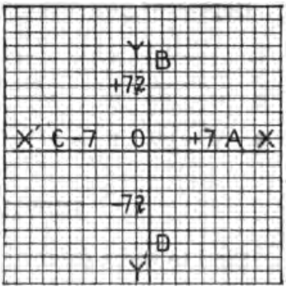
16. $\dfrac{2+\sqrt{-6}}{1-\sqrt{-8}}$

17. $\dfrac{1}{5+2\sqrt{-5}}$. **18.** $\dfrac{\sqrt{-2}+2\sqrt{-3}}{2\sqrt{-5}-3\sqrt{-2}}$. **19.** $\dfrac{x+\sqrt{-y}}{x-\sqrt{-y}}$.

20. Show that the quotient of two complex expressions of the forms $a+bi$ and $a-bi$ is a complex expression.

84. Graphical Representation of Imaginary Numbers. — Imaginary numbers were so named centuries ago because when first discovered they had no meaning, and were for centuries uninterpreted. It is now seen that they are truly numbers, and that they may be represented graphically.

All positive and negative real numbers may be represented, as we have seen before, by means of distances measured to the right or left of a starting point O on a line XX'.



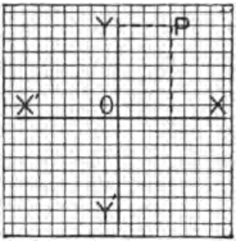
If the line OA, which represents $+7$, is revolved counterclockwise about O, it may be made to take the position of OB, OC, OD, or any other line through O.

Now, multiplying $+7$ by i gives $+7i$. Multiplying again by i gives -7, which is represented by OC. Hence multiplying $+7$ twice by i gives the same result as revolving OA into the position OC. Therefore, multiplying $+7$ once by i may be interpreted as revolving OA only half as far, *i.e.* into the position OB. Hence, $+7i$ is represented graphically by OB. Similarly, multiplying $+7$ three times by i would revolve it into the position OD. Hence, $-7i$ is represented graphically by OD. The same interpretation would hold in case of any other real number that might be chosen in place of $+7$. Hence, the following principle:

If all real numbers are represented by distances measured along the line XX' from O, positive numbers to the right and negative numbers to the left, then all imaginary numbers will be represented by distances measured from O along the line YY' perpendicular to XX', those with positive coefficients above O, and those with negative coefficients below.

Example. — $\sqrt{-9}=3i$. Hence $\sqrt{-9}$ is represented by a distance of 3 spaces measured upward on YY' from O.

85. Graphical Representation of Complex Expressions. — Since a complex expression is composed of a real term and an imaginary term, it may be represented graphically.



Thus, to represent $5+\sqrt{-49}$, or $5+7i$, we measure the 5 real units along OX and the 7 imaginary units along OY. As in other graphical work, this locates the point P. The point P represents graphically the complex expression $5+\sqrt{-49}$.

EXERCISES

Represent graphically:

1. $\sqrt{-4}$. 4. $-\sqrt{-25}$. 7. $8+\sqrt{-9}$. 10. $-6+\sqrt{-25}$.
2. $\sqrt{-16}$. 5. $-\sqrt{-81}$. 8. $12-\sqrt{-16}$. 11. $-4-\sqrt{-81}$.
3. $\sqrt{-36}$. 6. $-\sqrt{-64}$. 9. $7+\sqrt{-100}$. 12. $-12-\sqrt{-36}$.

SUPPLEMENTARY EXERCISES

Find to three decimal places the square root of:

1. $\frac{2}{5}$.

SUGGESTION. — Either change a common fraction to a decimal, or rationalize the denominator, before extracting the square root.

2. $\frac{3}{7}$. 3. $\frac{5}{11}$. 4. $\frac{7}{19}$. 5. $\frac{5}{6}$. 6. $\frac{3}{10}$. 7. $\frac{13}{20}$.

Simplify:

8. $\sqrt[n]{a^{2n}b^{3n}}$. 10. $\sqrt[n+1]{A^{2n+2}B^{n+1}}$. 12. $\sqrt[n]{R^n-3R^n\sqrt{5}}$.
9. $\sqrt[2n]{x^{6n}y^n}$. 11. $x\sqrt[2n]{x^{3n}}$. 13. $(\sqrt[n]{a}-\sqrt[n]{b})^2$.

14. Does $-3+\sqrt{11}$ satisfy $x^2+6x-2=0$?

15. Does $4-\sqrt{17}$ satisfy $n^2-8n-1=0$?

16. Is $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$ a cube root of 1?

17. Is $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$ a cube root of 1?

18. Find the square root of $7+2\sqrt{10}$.

Solution. — Since $(\sqrt{a}\pm\sqrt{b})^2=a\pm 2\sqrt{ab}+b$, the square root of any expression of the form $x\pm 2\sqrt{y}$ may be found, if it can be written in the form $a\pm 2\sqrt{ab}+b$, by taking the sum, or difference, of the square roots of the terms a and b.

Now, $7+2\sqrt{10}$ may be written $5+2\sqrt{5\times 2}+2$. Hence its square root is $\sqrt{5}+\sqrt{2}$.

Find the square root of:

19. $10-\sqrt{84}$.

Suggestion. — This may be written $10-2\sqrt{21}$, or $7-2\sqrt{7\times 3}+3$.

20. $11+2\sqrt{30}$.

21. $14+2\sqrt{45}$.

22. $13-\sqrt{48}$.

23. $6+\sqrt{32}$.

24. $26-\sqrt{640}$.

25. $18+\sqrt{308}$.

Rationalize the denominators of:

26. $\dfrac{3}{2-\sqrt{-3}}$.

27. $\dfrac{2}{\sqrt{3}+\sqrt{-1}}$.

28. $\dfrac{2-\sqrt{-3}}{2+\sqrt{-3}}$.

29. $\dfrac{3}{\sqrt{-3}-\sqrt{2}}$.

30. $\dfrac{x+y\sqrt{-1}}{x-y\sqrt{-1}}$.

31. $\dfrac{a+b\sqrt{-1}}{a\sqrt{-1}-b}$.

Simplify:

32. $(-\frac{1}{2}-\frac{1}{2}\sqrt{-5})(-\frac{1}{2}-\frac{1}{2}\sqrt{-5})^2$.

33. $(2\sqrt{3}+3\sqrt{-7})(5\sqrt{3}-4\sqrt{-7})$.

CHAPTER VII

QUADRATIC EQUATIONS

86. Quadratic Equations. — Every **quadratic equation** (see § 42 for definition) in one unknown number whose unknown number is denoted by x can be written in the form $Ax^2 + Bx + C = 0$, by simplifying, transposing, and combining similar terms.

Thus, $(5x-2)(2x+1) = 7x^2 - 4x$ may be written $3x^2 + 5x - 2 = 0$.

If a quadratic equation, when reduced to the above form, contains a term involving the first power of the unknown number, it is called a **complete quadratic** equation. One that does not contain a term involving the first power of the unknown number is called a **pure quadratic equation.**

Thus, $5x^2 - 7x + 2 = 0$ is a *complete* quadratic equation. And $2N^2 + 9 = 0$ is a *pure* quadratic equation.

EXERCISES

Express each of the following equations in the form $Ax^2 + Bx + C = 0$, and determine whether it is a complete or pure quadratic:

1. $x(2x-3) = 10 - 3x$.
2. $5n(n+2) = n^2 + 2$.
3. $12t^2 = 7(t-3)$.
4. $10(V-1) = 3V^2 + 2$.
5. $(2a-1)^2 = 3a - 5$.
6. $(y+2)^3 = (y-1)^3$.
7. $(6r-5)(r+2) = (2r+7)(2r-1)$.
8. $(2P+1)(2P-1) = (3P-2)(3P+2)$.
9. $(a-1)(a-2)(a-3) - (a+1)(a+2)(a+3) = 0$.
10. $\dfrac{r^2+r+1}{r-1} - 6 = \dfrac{r^2-r+1}{r+1}$.
11. $2T + 1 + \dfrac{(3T+1)(T-2)}{T+2} = 0$.

87. Quadratics solved by Factoring. — Some quadratic equations may be solved by factoring, as illustrated in the following example.

EXAMPLE. — Solve $2t^2 + t = 10$.

Transposing, $2t^2 + t - 10 = 0.$

Factoring, $(2t + 5)(t - 2) = 0.$

Now, this equation is satisfied by any value of t that will make either of the factors $2t + 5$ or $t - 2$ zero, since then their product would be zero. Such values of t may be found by setting each factor equal to zero, and solving the resulting linear equations. Setting each factor equal to zero,

$$2t + 5 = 0;\text{ whence } t = -2\tfrac{1}{2}.$$

$$t - 2 = 0;\text{ whence } t = 2.$$

To solve a quadratic equation by factoring:

(1) *Write the equation in the form* $Ax^2 + Bx + C = 0$.

(2) *Factor the resulting first member.*

(3) *Set each of the factors equal to zero, and solve the resulting linear equations.*

Since a quadratic equation that can be solved by factoring is *equivalent* to the two linear equations to which it leads, and each of these has always one root, the quadratic equation has two roots. It can be shown that in general,

Every quadratic equation has two roots.

EXERCISES

Solve by factoring:

1. $n^2 - 5n + 4 = 0.$
2. $k^2 = 12k - 32.$
3. $P^2 + 6P = 16.$
4. $2A^2 = A + 3.$
5. $5v + 6 = 4v^2.$
6. $18t^2 + 61t = 7.$
7. $10k^2 = 31k + 14.$
8. $81N + 24 = 60N^2.$
9. $x(x + 2) = 4x + 15.$
10. $5(p + 25) = 6p^2.$
11. $(4d - 1)^2 = 10d^2 + 31.$
12. $(x + 6)^2 = (3x - 2)^2.$

13. $(n-1)(n+1)(n-2)=n^2(n+2)-103.$

14. $(2R-3)(R+4)=(3R-14)(2R+3)+R^2-6.$

15. $6-\frac{4}{y^2}=\frac{23}{y}.$

16. $\frac{h+3}{h+15}=\frac{1}{h+8}.$

17. $2D+1=\frac{3D^2-4}{D+2}.$

18. $\frac{1}{2a-2}=\frac{a-7}{a+1}+\frac{24+a}{5-5a^2}.$

19. $\frac{x^3+x+1}{x+1}=\frac{x^2-x+1}{x-1}-\frac{2}{3}.$

20. $\frac{5}{g+5}-\frac{2}{g+2}=\frac{6}{g+23}.$

88. Quadratics solved by Square Root.—Any pure quadratic equation may be solved by square root.

EXAMPLE 1.—Solve $4n^2-1=0$.

Transposing, $4n^2=1.$

Dividing by 4, $n^2=\frac{1}{4}.$

Now since $n^2=\frac{1}{4}$, n must be the square root of $\frac{1}{4}$.

Hence, $n=\pm\frac{1}{2}.$

EXAMPLE 2.—Solve $2V^2+3=0$.

Transposing, $2V^2=-3.$

Dividing by 2, $V^2=-\frac{3}{2}.$

Taking square root, $V=\pm\sqrt{-\frac{3}{2}}$, or $\pm\frac{1}{2}\sqrt{-6}$, *imaginary* roots.

To solve a pure quadratic equation by square root:

(1) *By transposing, etc., solve for the square of the unknown number.*

(2) *Find the square roots of the value obtained.*

EXERCISES

Solve:

1. $5y^2-20=0.$

2. $12F^2-125=7F^2.$

3 $k^2=63-6k^2.$

4. $16A^2=9.$

5. $3p^2-7=0.$

6. $8T^2+1=0.$

7. $5n^2 + 6 = 0.$

8. $12G^2 + 3 = 0.$

9. $7b^2 - 21 = 0.$

10. $16 + 5R^2 = 0.$

11. $x(x-2) = 9 - 2x.$

12. $(N+3)(2N+5) = 11(N+4).$

13. $\dfrac{1}{a+2} = \dfrac{a-2}{12}.$

14. $\dfrac{z^2+7}{9} = 3.$

15. $\dfrac{A+1}{A-1} + \dfrac{A-1}{A+1} = 1\tfrac{5}{6}.$

16. $\dfrac{6n^2-5}{n^2-4} - \dfrac{n-2}{n+2} = \dfrac{n+2}{n-2}.$

17. $\dfrac{t+9}{9t+1} - \dfrac{9t+1}{t+9} = 0.$

18. $3a+2 - \dfrac{(a+4)(4a-3)}{2a+3} = 0.$

19. $\dfrac{K+7}{5} - \dfrac{9}{K-2} = 1.$

20. $\dfrac{4m+3}{4m-3} + \dfrac{4m-3}{4m+3} = 2\tfrac{4}{63}.$

21. $\dfrac{12}{5}\left(B - 1 + \dfrac{2}{B+1}\right) = \dfrac{5}{B+1}.$

Find approximate to hundredths the roots of:

22. $x^2 = 6.$

23. $4M^2 - 7 = 0.$

24. $15t^2 - 9 = 0.$

25. $1 - 5t^2 = 0.$

26. $3D^2 = 28.$

27. $8s^2 - 16 = 0.$

28. $\dfrac{H+5}{6} - \dfrac{7}{H+1} = 1.$

89. Complete Quadratics solved by Completing the Square. — A trinomial that is a perfect square of a binomial can be written in either the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$. (See § 15.) Hence, if we have given either $a^2 + 2ab$ or $a^2 - 2ab$, it is made a perfect square by adding b^2, *i.e. the square of one half of the coefficient of a in the term 2 ab.*

This process is called **completing the square.**

Thus, to complete the square of $k^2 - 12k$, we add the square of $\frac{1}{2}$ of 12, or 36. This gives $k^2 - 12k + 36$, which is the square of $k - 6$.

By use of this process of completing the square, any complete quadratic equation can be solved.

EXAMPLE. — Solve $4y^2 = 6y - 3.$

Transposing, $4y^2 - 6y = -3.$

Dividing by 4, $y^2 - \frac{3}{2}y = -\frac{3}{4}.$

Adding square of $\frac{1}{2}$ of $-\frac{3}{2}$, $y^2 - \frac{3}{2}y + \frac{9}{16} = -\frac{3}{16}.$

Taking square root, $y - \frac{3}{4} = \pm \frac{1}{4}\sqrt{-3}.$

Transposing, $y = \frac{3}{4} \pm \frac{1}{4}\sqrt{-3}.$

To solve a complete quadratic by completing the square:

(1) *Reduce the equation to the form* $x^2 + px = q$, *where* x *is the unknown number.*

(2) *Add to each member the square of one half the coefficient of the term in the first power of the unknown number.*

(3) *Take the square root, attaching the double sign to the second member.*

(4) *Solve the resulting linear equations.*

EXERCISES

Solve by completing the square:

1. $n^2 + 10n = 24.$
2. $4A^2 + 11A = 3.$
3. $3x^2 = 20 - 7x.$
4. $6v^2 = 49v - 8.$
5. $12T^2 + 31T - 15 = 0.$
6. $3y^2 + 8y + 4 = 0.$
7. $6N^2 - 2 = N.$
8. $7k = 10k^2 - 12.$
9. $7D^2 = 12D + 3.$
10. $4s^2 - 3 = 11s.$
11. $10z^2 - 29z + 10 = 0.$
12. $12H^2 - 14H + 3 = 0.$
13. $6t^2 + 6 = 13t.$
14. $2R^2 + R = 1.$
15. $3V^2 = 17V + 28.$
16. $9m^2 - 26 = 6m.$
17. $5 + G = 6G^2.$
18. $2E - 7 = 3E^2.$
19. $(3x - 4)^2 - (x - 2)^2 = 1.$
20. $\frac{1}{n} + \frac{1}{2 + n} - \frac{3}{4} = 0.$
21. $\frac{3v - 1}{v + 2} - \frac{4v - 1}{v + 3} = 1.$
22. $\frac{a + 5}{2a - 1} + \frac{3}{a - 4} = \frac{1}{3}.$
23. $\frac{1}{m + 2} - \frac{2m - 1}{4m + 3} + \frac{4}{5} = 0.$
24. $\frac{4}{d - 4} - 2 = \frac{d + 3}{d^2 - 5d + 4}.$
25. $\frac{y + 5}{3 - y} + \frac{2y}{2 - y} = \frac{8 - 3y}{y}.$
26. $\frac{5}{N - 2} + \frac{2}{N - 4} = \frac{3}{6 - N}.$
27. $\frac{2}{2t - 6} + \frac{8}{8 - 3t} + \frac{7}{t + 3} = 0.$
28. $\frac{x^2 - 7x + 2}{x^2 + 9x + 8} = \frac{x + 12}{x - 1}.$

Find approximate to hundredths the roots of:

29. $a^2+8a+4=0$.

30. $V^2+5V+1=0$.

31. $2n^2-n-2=0$.

32. $6D^2+3D=5$.

90. Quadratics solved by Formula. — Any quadratic equation whose unknown number is x may be written in the form

$$Ax^2+Bx+C=0.$$

If this general equation is solved by completing the square, the roots obtained are

$$x=\frac{-B\pm\sqrt{B^2-4AC}}{2A}.$$

Now, the roots of any given particular quadratic equation may be written down at once, without performing all of the steps in the solution by completing the square, by substituting for A, B, and C, in the above formula for the roots of the general equation Ax^2+Bx+C, the particular values which they have in the given equation.

Example. — Solve $2x^2=1-5x$.

Transposing, $2x^2+5x-1=0$.

Now, comparing this equation to $Ax^2+Bx+C=0$, it is seen that $A=2$, $B=5$, and $C=-1$. Substituting these values for A, B, and C in the formula for the roots, we get

$$x=\frac{-5\pm\sqrt{25+8}}{4}$$

$$=-\tfrac{5}{4}\pm\tfrac{1}{4}\sqrt{33}.$$

The student should master this formula as thoroughly as the multiplication table, because by it the roots of complete quadratics can be obtained much more quickly than by completing the square.

EXERCISES

Solve by formula:

1. $Q^2-8Q+15=0.$
2. $z^2-2z-63=0.$
3. $a^2+16a+48=0.$
4. $2x^2-3x-5=0.$
5. $6v^2+37v+6=0.$
6. $6N^2+17N=14.$
7. $15b^2=8-2b.$
8. $21y^2=43y-20.$
9. $t^2-7t+2=0.$
10. $8B^2+B=5.$
11. $12n-7=2n^2.$
12. $20V^2=V-13.$
13. $3P^2-8=2P.$
14. $11k+3k^2-4=0.$
15. $9R^2-R-6=0.$
16. $(x-5)(x+2)=6.$
17. $x(x+8)+8(x-8)=0.$
18. $(p-4)^2=(3-p)(1+2p).$
19. $n^2+n+1=0.$
20. $n^2-n+1=0.$
21. $v^2=2v.$
22. $\frac{y-1}{y+1}=\frac{y+1}{y-1}-1.$
23. $\frac{1}{g}+g=1\frac{1}{2}.$
24. $R-\frac{1}{R}=3-\frac{1}{3}.$
25. $\frac{F-1}{F-4}+\frac{F+9}{F+3}=0.$
26. $b+2=\frac{3b^2-4}{2b+1}.$
27. $H=\frac{1}{H}-1.$
28. $\frac{n-2}{2+n}+\frac{2+n}{n-2}=\frac{2(5n-4)}{4-n^2}.$
29. $\frac{2}{5(c-2)}=\frac{1-c}{2+c}+\frac{c+1}{c^2-4}.$
30. $\frac{2r+3}{4r-2}-\frac{7-r}{2+2r}=\frac{7-3r}{4-3r}.$
31. $\frac{8+V}{8-V}+\frac{4-2V}{4+2V}=\frac{8}{3}.$
32. $\frac{1}{8}+\frac{1}{4(n+1)}=\frac{3}{2n^2-2}.$
33. $\frac{A+3}{2A+8}-\frac{A}{1+A}=\frac{1}{18}.$
34. $5Q-\frac{3Q-3}{Q-3}=2Q+\frac{3Q-6}{2}.$
35. $\frac{s-5}{s+3}-\frac{80}{s^2-9}=\frac{1}{2}+\frac{s-8}{3-s}.$

Find approximate to hundredths the roots of:

36. $f^2 + 6f + 5 = 0.$

37. $2S^2 - S - 2 = 0.$

38. $5v^2 = 2v + 2.$

39. $6n^2 + 5n = 2.$

40. $-x^2 + 4x + 6 = 0.$

41. $-2T^2 - T + 4 = 0.$

91. Literal Equations: Formulæ. — If a literal equation, or formula, is of the second degree in the number solved for, it may be solved by one of the foregoing methods.

Example. — Solve $V^2 + 2aV = b^2 + 2ab$ for V.

Transposing and grouping, $V^2 + 2aV - (b^2 + 2ab) = 0.$

Using the formula, $A = 1$, $B = 2a$, and $C = -(b^2 + 2ab)$.

Hence,
$$V = \frac{-2a \pm \sqrt{4a^2 + 4(b^2 + 2ab)}}{2}$$
$$= \frac{-2a \pm (2a + 2b)}{2}$$
$$= b, \text{ or } -2a - b.$$

EXERCISES

1. Solve $av^2 - a^3 = 0$ for v.
2. Solve $x^2 - nx = 2n^2$ for x; for n.
3. Solve $x^2 - ax + b = 0$ for x.
4. Solve $v^2 + 2sv + t = 0$ for v.
5. Solve $m^2 - am - bm + ab = 0$ for m.
6. Solve $t^2 - (2m - n)t = 2mn$ for t.
7. Solve $y^2 + 2b + b^2 = 2by + 2y$ for y.
8. Solve $P^2 + 2P + 1 = nP^2 + nP$ for P.
9. Solve $\dfrac{a}{a - n} + \dfrac{n - a}{a} = 2$ for n.
10. Solve $\dfrac{R^2 + 2mR}{m - n} = \dfrac{m}{n} + \dfrac{R}{2n}$ for R.

11. Solve $\frac{at+b}{bt+a} - \frac{ct-d}{dt-c} = 0$ for t.

12. Solve $\frac{m}{r+m} + \frac{n}{r+n} + \frac{p}{r+p} = 0$ for r.

13. Solve $\left(\frac{V}{n+1} + n\right)^2 - \left(\frac{V}{n-1} - n\right)^2 = 0$ for V.

14. Solve $\frac{k^2}{d+N} - \frac{k^2}{d-N} = f$ for N.

15. The formula $E^2 = \frac{JWhR}{t}$ is used in electrical work. Solve for E.

16. $H = 0.24\ C^2Rt$ expresses a law of heat. Solve for C.

17. $E = \frac{4\ n^2L^2w}{g}$ expresses a law of sound. Solve for L.

18. $\frac{F}{f} = \frac{d^2}{D^2}$ expresses a law of attraction between two bodies. Solve for D.

19. The horse power of automobile engines is computed by the formula $H = KND(D-1)(R+2)$. Solve for D.

20. The elevation of a point above sea level is obtained by use of the formula $H = 513\,t + t^2$. Solve for t.

21. The area of a ring is computed by the formula

$$A = \pi(r^2 - r'^2).$$

Solve for r; for r'.

22. The volume of a frustum of a cone is computed by the formula $V = \frac{1}{3}H(R^2 + r^2 + Rr)$. Solve for R; for r.

92. Problems solved by Quadratics. — In solving problems by means of quadratic equations, since the roots of such equations may be fractions, negative numbers, surds, or even imaginary numbers, the nature of the roots obtained must be observed, and roots that do not conform to the requirements of the problems must be discarded.

EXERCISES

1. Separate 30 into two parts so that the square of one part exceeds the square of the other part by 160.

2. The hypotenuse and longer leg of a right triangle are together 18 inches long. The shorter leg is 6 inches long. Find the hypotenuse and longer leg.

3. The area of a rectangle is 192 square inches. The length is 4 inches more than the width. Find the length and width.

4. The area of a triangle is 240 square inches. The altitude is 4 inches less than the base. Find the altitude and base.

5. One side of a rectangle is 8 inches longer than the other. The diagonal is 40 inches. Find the area.

6. When the sides of a square are increased 4 inches, the area is increased 176 square inches. Find the side of the square before it is increased.

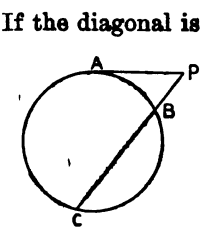
7. A rectangular piece of tin was twice as long as it was wide. A 4-inch square was cut from each corner and the sides and ends were turned up so as to make a box containing 1536 cubic inches. Find the dimensions of the piece of tin.

8. The length of a rectangular prism containing 240 cubic inches is 10 inches. Its entire surface is 248 square inches. Find the other two dimensions.

9. It is 180 rods around a rectangular field containing 11¼ acres. Find the dimensions.

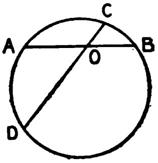
10. A room is 6 feet longer than it is wide. If the diagonal is 30 feet, find the length and width of the floor.

11. It is shown in geometry that if from a point P without a circle a tangent PA and a secant PC be drawn as in the margin, $\overline{PA}^2 = PB \times PC$. If $PA = 10$ and $BC = 4$, find PC.



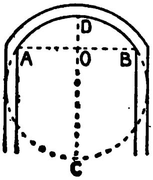
12. Using 4000 miles for the radius of the earth, and making use of the principle stated in Problem 11, find how far a man can

see over a smooth body of water when standing in a boat so that his eye is 10 feet above the level of the water.



13. It is shown in geometry that if two chords, AB and CD, intersect at O, $OA \times OB = OC \times OD$. If $OC = 2$ inches, $OD = 7$ inches, and $AB = 8$ inches, find the lengths of OA and OB.

14. A window 5 feet wide is to be surmounted by a circular stone arch, cut to an arc of a 12-foot circle. Find the rise by use of the principle stated in Problem 13.



SUGGESTION. — $OD =$ rise. $AO = OB$. $DC = 12$ feet.

15. A farmer has a field of wheat 40 rods wide and 80 rods long. How wide a strip must be cut around the field in order to have one half of it cut?

16. A tree 100 feet high is broken off by a storm so that the top reaches the ground 30 feet from the foot of the tree. Find the height of the part standing.

17. By increasing the edges of a cube each 2 inches, the volume was increased 1016 cubic inches. What was the length of the cube before it was increased?

18. A rectangular field is 3 times as long as it is wide. If its width is increased by 20 rods, its area is doubled. Find the area of the field.

19. A farmer sold his crop of oats for $600. If he had held them for 4 months, he could have gotten 10 cents per bushel more for them, in which case he could have received as much for them and have had 300 bushels left. How many bushels had he?

20. A man bought a farm for $12,600. He then sold all but 40 acres of it at a gain of $20 per acre over cost, receiving the entire cost of the farm. How many acres did he sell?

21. A party engaged a yacht for two weeks for $168. Had there been two less, the cost to each would have been $2 more. How many were there?

22. A tree was broken over by a storm so that the top touched the ground 50 feet from the foot of the stump. The stump was $\frac{3}{8}$ of the height of the tree. What was the height of the tree?

23. A tree which stood at the edge of the bank of a stream fell with its top in the water. The tree was 60 feet high, the bank on which it stood was 15 feet above the water level, and the body of the tree passed under the surface of the water at a point 20 feet from the bank. What portion of the tree was under water?

24. The hypotenuse of a right triangle is 4 inches longer than one leg and 2 inches longer than the other. Find the sides of the triangle.

25. One pipe can fill a cistern in 6 minutes less time than is required for another pipe to fill it. The two together can fill it in $10\frac{2}{7}$ minutes. Find the time required for each pipe alone to fill the cistern.

26. One of two pipes can fill a tank in 28 minutes, and the time required for the other pipe to fill it is $19\frac{4}{9}$ minutes longer than is required for the two pipes together to fill it. Find the time required for the two pipes together to fill the tank.

27. Two trains on the same road start at the same time from stations 225 miles apart. One takes $\frac{2}{3}$ minute longer than the other to run a mile, and they meet in 3 hours. Find the speed of each train.

28. A teamster having 12 miles to drive increased his speed one mile an hour after he had traveled 2 miles. He thus finished the distance in half an hour less time than it would have taken had he not increased his speed. How long did it take to drive the 12 miles?

SUGGESTION. — Let his first rate be x miles an hour.

29. A merchant bought a certain number of mirrors for $36, and, after breaking one, sold the rest for 50 cents apiece more than they cost him, thus making $2.50 by the transaction. How many did he buy?

30. The rate at which a man can row in still water is twice the rate of the current of a river. He rows 6 miles down the river and back in 4 times as many hours as he could row miles per hour in still water. Find the rate of the current and the rate of his rowing in still water.

31. The number of square inches in the area of a square exceeds the number of inches in its perimeter by 32. What is its area?

32. The side of a square is the same as the diameter of a circle. The area of the square exceeds that of the circle by approximately 3.4336 square inches. Find the diameter of the circle.

33. The base of a triangle exceeds its altitude by 2 inches, and its area is 112 square inches. Find its altitude.

34. What is the area of a square field whose diagonal is 10 rods longer than one of its sides?

35. By adding a border to a rug 6 feet by 9 feet its area is increased 34 square feet. Find the width of the border.

93. Character of the Roots of a Quadratic. — In the use of the formula

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A},$$

in § 90, it was observed that the character of the roots depended upon the expression under the radical sign. This expression, $B^2 - 4AC$, for that reason, is called the **discriminant** of the equation $Ax^2 + Bx + C = 0$.

A study of the formula shows that:

(1) *When the discriminant is positive and not a perfect square, the roots are real, unequal, and involve surds.*

(2) *When the discriminant is positive and a perfect square, the roots are real, unequal, and free of surds.*

(3) *When the discriminant is zero, the roots are real and equal.*

(4) *When the discriminant is negative, the roots are imaginary.*

From these observations we can determine the character of the roots of any given quadratic equation, when desired, without solving the equation.

EXAMPLE. — Determine the nature of the roots of $P^2 + 2P + 5 = 0$.

Here $A = 1$, $B = 2$, $C = 5$. Hence, $B^2 - 4AC = -16$. Hence, the roots are imaginary.

EXERCISES

By use of the discriminant, determine the character of the roots of the following:

1. $n^2 - 5n + 4 = 0$.
2. $a^2 + 6a - 1 = 0$.
3. $t^2 + t + 3 = 0$.
4. $3x^2 + 2x - 1 = 0$.
5. $2A^2 - 7A + 8 = 0$.
6. $12p^2 + 5p + 2 = 0$.
7. $3V^2 = 4V - 2$.
8. $9r^2 = 4r + 5$.
9. $4x^2 = 12x - 9$.
10. $7x - 11 = 4x^2$.
11. $2K^2 + 9K = 4$.
12. $16b^2 + 25 = 40b$.
13. $R^2 - 6R + 6 = 0$.
14. $n(n - 4) = n - 2$.

For what values of c will the following have equal roots?

15. $cx^2 + 3x - 5 = 0$.

SOLUTION. — The discriminant is $9 + 20c$. For equal roots, $9 + 20c = 0$. Solving, $c = -\frac{9}{20}$.

16. $cx^2 - 6x = 4$.
17. $4x^2 + cx + 25 = 0$.
18. $2t^2 + ct + 32 = 0$.
19. $3A^2 - 2A + c = 0$.
20. $2n^2 + 8n + c = 0$.
21. $3V^2 + 10V + c + 3 = 0$.

Determine the relation between m and n for which the following will have equal roots:

22. $mx^2 + 5x + n = 0$.
23. $my^2 - ny + 4 = 0$.
24. $t^2 + mt + n = 0$.
25. $nP^2 - 2mP + 8 = 0$.

94. Relation between the Roots and Coefficients. — By dividing both members of the general quadratic $Ax^2 + Bx + C = 0$ by A, it may be written in the form $x^2 + px + q = 0$.

Solving this equation by the formula, and calling the roots r_1 and r_2, we get

$$r_1 = \frac{-p+\sqrt{p^2-4q}}{2} \text{ and } r_2 = \frac{-p-\sqrt{p^2-4q}}{2}.$$

Adding these roots, $r_1 + r_2 = -p.$

Multiplying them, $r_1 r_2 = q.$

Hence, if an equation is written in the form $x^2 + px + q = 0$,

(1) *The sum of the roots is the coefficient of x with its sign changed;*

(2) *The product of the roots is the term free of x.*

The principles (1) and (2) enable one to write down the quadratic equation whose roots are given.

Example. — Form the equation whose roots are 7 and -3.

The sum of the roots is $7 - 3$, or 4, and their product is -21.

Hence, the equation is $x^2 - 4x - 21 = 0$.

In testing the roots of an equation to see if the solution is correct, especially in case the roots obtained are imaginary or involve surds, so that substitution of the roots in the original equation is not convenient, it is advantageous to use principles (1) and (2).

Example. — Solve and check $7y^2 - 2y + 3 = 0$.

By the formula, $y = \frac{1}{7} \pm \frac{1}{7}\sqrt{-20}.$

The sum of these roots is $\frac{2}{7}$. Their product is $(\frac{1}{7})^2 - (\frac{1}{7}\sqrt{-20})^2$, or $\frac{3}{7}$. Hence, since the equation may be written in the form $y^2 - \frac{2}{7}y + \frac{3}{7} = 0$, the roots check.

EXERCISES

Form the equations whose roots are the following:

1. 6, 2.
2. 9, 1.
3. -5, -3.
4. 8, -2.
5. -7, 6.
6. -12, -2.

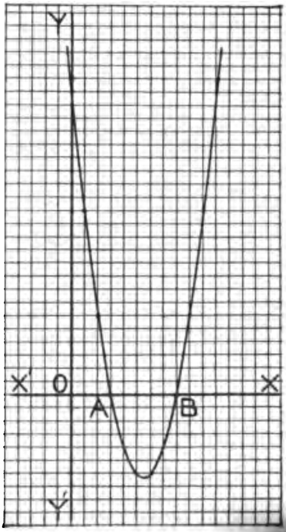
7. $10, -5$.
8. $-3, -20$.
9. $9, -11$.
10. $\frac{1}{2}, \frac{1}{3}$.
11. $\frac{4}{5}, -\frac{1}{5}$.
12. $\frac{3}{8}, -\frac{1}{3}$.
13. $-\frac{1}{4}, -\frac{5}{6}$.
14. $a, 3a$.
15. $-2n, 5m$.
16. $1+\sqrt{5}, 1-\sqrt{5}$.
17. $-4+\sqrt{3}, -4-\sqrt{3}$.
18. $1+\sqrt{-6}, 1-\sqrt{-6}$.
19. $-3+\sqrt{-4}, -3-\sqrt{-4}$.

Solve and check by use of principles (1) and (2):

20. $n^2+n-6=0$.
21. $t^2-5t+7=0$.
22. $3a^2-2a-5=0$.
23. $5V^2=V-1$.
24. $N(N-3)=4$.
25. $2r^2-5(r-5)=0$.
26. $(A+3)(A-2)=3$.
27. $x^2-ax+3b=0$.

95. Graphical Interpretation of Roots. — Any value of x which satisfies both equations of the system

$$\begin{cases} y = Ax^2+Bx+C, & (1) \\ y = 0 & (2) \end{cases}$$



satisfies the equation $Ax^2+Bx+C=0$.

The graph of (1) is some kind of curve, and the graph of (2) is the x-axis (see § 51). Hence, since the coördinates of the points where the graphs of (1) and (2) cross must satisfy both (1) and (2) (see § 53), the roots of $Ax^2+Bx+C=0$ may be obtained by observing the x-distances of the points where the graph of (1) crosses the x-axis.*

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* For another method of solving a quadratic graphically, see FIRST COURSE, pages 292 and 293.

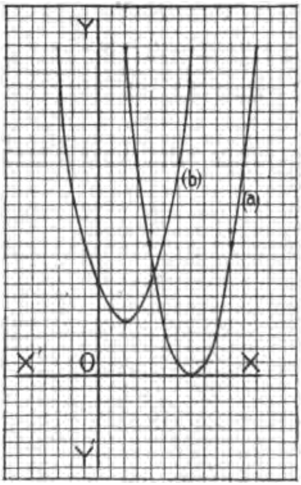
Example. — Solve graphically $x^2 - 11x + 24 = 0$.

Let $$y = x^2 - 11x + 24.$$

By assigning values to x, and computing the corresponding values of y, the following sets of values are obtained:

x	0	1	2	3	5	7	10	11
y	24	14	6	0	-6	-4	14	24

Locating the points whose coördinates are these sets of values of x and y, and connecting them by a smooth line, gives the curve on the preceding page. It is seen that this curve crosses the x-axis at two points, A and B, whose x-distances are 3 and 8, respectively. Hence, the roots of $x^2 - 11x + 24 = 0$ are 3 and 8.



If the roots of a quadratic are *equal*, the graph of (1) will just touch the x-axis at one point.

Example. — The roots of $x^2 - 14x + 49 = 0$ are each $+7$. The graph of $y = x^2 - 14x + 49$ is the curve (a) in the figure, which just touches the x-axis at a point whose x-distance is the root $+7$.

If the roots of a quadratic are *imaginary*, the graph of (1) will not cross or touch the x-axis at all.

Example. — The roots of $x^2 - 4x + 8 = 0$ are $2 \pm 2i$. The graph of $y = x^2 - 4x + 8$ is the curve (b) in the figure, which lies entirely above the x-axis.

EXERCISES

Solve graphically:

1. $x^2 - 12x + 35 = 0$.
2. $x^2 + x - 72 = 0$.
3. $x^2 + 14x + 48 = 0$.
4. $2x^2 - 23x + 56 = 0$.
5. $3x^2 + 14x = 40$.
6. $5x^2 + 3x - 110 = 0$.

Find, by the discriminant, the nature of the roots of the following, and give their graphical interpretation:

7. $4x^2 - 33x + 8 = 0.$
8. $2x^2 + 3x = 54.$
9. $x^2 + 16x + 64 = 0.$
10. $x^2 + x + 5 = 0.$
11. $9x^2 - x + 2 = 0.$
12. $4x^2 + 25 = 20x.$

SUPPLEMENTARY EXERCISES

If r_1 and r_2 are the roots of $Ax^2 + Bx + C = 0$, then

$$x^2 + \frac{B}{A}x + \frac{C}{A} = x^2 - (r_1 + r_2)x + r_1r_2 = (x - r_1)(x - r_2),$$

by § 94. Hence,

$$Ax^2 + Bx + C = A(x - r_1)(x - r_2).$$

Therefore, the three factors of an expression of the form $Ax^2 + Bx + C$ are A, $x - r_1$, and $x - r_2$, where r_1 and r_2 are the roots of the equation formed by putting $Ax^2 + Bx + C$ equal to zero.

Hence any trinomial of the form $Ax^2 + Bx + C$ may be factored, if factors containing surds or imaginary numbers are accepted. It sometimes is desirable to find such factors of a trinomial.

Evidently, by use of the discriminant $B^2 - 4AC$ we can tell whether the factors of such a trinomial are *real* or *imaginary*, or *free of surds*.

Find the nature of the factors of:

1. $x^2 - 15x + 54.$
2. $24t^2 - t - 10.$
3. $3n^2 - 9n + 28.$
4. $2y^2 + y + 3.$
5. $2V^2 + 5V - 7.$
6. $x^2 - 9x + 3.$
7. $4x^2 + 2x + 1.$
8. $6a^2 + 11a - 40.$
9. $10R^2 - R + 2.$
10. $6E^2 - E - 35.$

Factor:

11. $2x^2 + 2x - 5.$

SOLUTION. — Let $2x^2 + 2x - 5 = 0$.

Solving, $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{11}$.

Hence, $2x^2 + 2x - 5 = 2(x + \frac{1}{2} + \frac{1}{2}\sqrt{11})(x + \frac{1}{2} - \frac{1}{2}\sqrt{11})$

12. $2s^2 - s + 2$.

13. $5P^2 + 8P + 4$.

14. $3b^2 - b - 1$.

15. $9d^2 + 4d + 1$.

16. $n^2 + 2n - 143$.

17. $10y^2 + 11y - 8$.

18. $7A^2 - A + 16$.

19. $2T + 15 + T^2$.

CHAPTER VIII

HIGHER EQUATIONS

96. Higher Equations. — In preceding chapters we have worked out general methods of solving any linear or any quadratic equation in one unknown number. General algebraic methods of solving any equation of the third or of the fourth degree also have been worked out by mathematicians. But in *elementary algebra* we shall undertake to solve only two special kinds of equations of degree higher than the second.

97. Higher Equations solved by Factoring. — Some equations of higher degree than the second may be solved by factoring. The method is the same as that of solving quadratics by factoring.

Example 1. — Solve $n^3 + 1 = 0$.

Factoring, $(n + 1)(n^2 - n + 1) = 0.$

This equation is satisfied by any values of n that will make either of the factors $n + 1$ or $n^2 - n + 1$ equal to zero, for then their product would be zero. Hence, equating each factor to zero, we get

$$n + 1 = 0 \text{ and } n^2 - n + 1 = 0.$$

Solving, $n + 1 = 0,\ n = -1.$

Solving, $n^2 - n + 1 = 0,\ n = \frac{1}{2} \pm \frac{1}{2}\sqrt{-3}.$

Example 2. — Solve $t^6 = 64$.

Transposing, $t^6 - 64 = 0.$

Factoring, $(t + 2)(t - 2)(t^2 + 2t + 4)(t^2 - 2t + 4) = 0.$

Putting each factor equal to zero,

$$t+2=0,\ t-2=0,\ t^2+2t+4=0,\ t^2-2t+4=0.$$

Solving each of these,

$$t=-2,\ 2,\ -1\pm\sqrt{-3}, \text{ or } 1\pm\sqrt{-3}.$$

EXAMPLE 3. — Solve $A^3+5A=6A^2-12$.

Transposing, $$A^3-6A^2+5A+12=0.$$

Factoring by remainder theorem,

$$(A+1)(A-3)(A-4)=0.$$

Putting each factor equal to zero, and solving,

$$A=-1,\ 3, \text{ or } 4.$$

As appears in each of the above examples, so it may be shown in general that

The number of roots of an equation is equal to the degree of the equation.

To solve any equation in one unknown number by factoring,

(1) *Transpose all terms to the left member, making the right member zero.*

(2) *Factor the resulting left member, and equate each factor to zero.*

(3) *Solve the resulting equations.*

EXERCISES

Solve:

1. $n^3-1=0.$
2. $a^6-1=0.$
3. $V^3=27.$
4. $x^4-1=0.$
5. $y^4-81=0.$
6. $t^4-5t^2+4=0.$
7. $k^4-16=0.$
8. $N^4-2N^2=15.$
9. $8s^3-27=0.$
10. $64A^3+125=0.$

11. $2g^4 = 5g^2 + 12.$
12. $m^6 + 7m^3 = 8.$
13. $4E^4 = 20E^2 - 21.$
14. $9c^4 + 18c^2 + 8 = 0.$
15. $x^3 + 2x^2 = x + 2.$
16. $z^3 - z^2 - 4z + 4 = 0.$
17. $T^3 + T = 3T^2 + 3.$
18. $a^3 + 5a^2 = 4a + 20.$
19. $R^4 + 4R^3 + R + 4 = 0.$
20. $h^5 + h^3 = h^2 + 1.$
21. $3P^4 + P^3 + 2P^2 + 24P = 0.$
22. $n^4 - 3n^3 + 3n^2 - n = 0.$
23. Find the three cube roots of 64.

SUGGESTION. — Let n be the cube root of 64.

Then $$n^3 = 64.$$

24. Find the four fourth roots of 81.
25. Find the six sixth roots of 1.
26. Find the six sixth roots of 64.

98. Higher Equations written in the Form of Quadratics. — Some equations of higher degree than the second may be written in the form of quadratics, and solved by the methods of solving quadratics.

EXAMPLE 1. — Solve $r^4 - 13r^2 + 36 = 0.$

This has the form of a quadratic in r^2. Solving for r^2 by the formula,

$$r^2 = \frac{13 \pm \sqrt{169 - 144}}{2}$$

$$= 9 \text{ or } 4.$$

Now, solving, $r^2 = 9,\ r = \pm 3.$

Solving, $r^2 = 4,\ r = \pm 2.$

Hence, the four roots are 2, -2, 3, -3.

EXAMPLE 2. — Solve $\dfrac{k^2}{k+1} + \dfrac{2(k+1)}{k^2} = 3.$

Putting n for $\dfrac{k^2}{k+1}$, the equation may be written

$$n + \frac{2}{n} = 3.$$

Solving, $n = 1$ or 2.

Hence, the given equation is equivalent to the equations

$$\frac{k^2}{k+1} = 1, \text{ and } \frac{k^2}{k+1} = 2.$$

Solving $\frac{k^2}{k+1} = 1$, $k = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$.

Solving $\frac{k^2}{k+1} = 2$, $k = 1 \pm \sqrt{3}$.

EXAMPLE 3. — Solve $2v^2 + 2v + 1 = \frac{10}{v^2 + v}$.

This may be written $2(v^2 + v) + 1 = \frac{10}{v^2 + v}$.

Putting n for $v^2 + v$,

$$2n + 1 = \frac{10}{n}.$$

Solving, $n = 2$ or $-\frac{5}{2}$.

Hence, the given equation is equivalent to the equations

$$v^2 + v = 2 \text{ and } v^2 + v = -\frac{5}{2}.$$

Solving these, $v = 1, -2, -\frac{1}{2} + \frac{3}{2}\sqrt{-1}, -\frac{1}{2} - \frac{3}{2}\sqrt{-1}$.

EXERCISES

Solve:

1. $x^4 + 2x^2 = 48$.
2. $6a^4 + a^2 = 15$.
3. $24V^4 = V^2 + 10$.
4. $n^6 + 9n^3 + 8 = 0$.
5. $8p^6 = 37p^3 + 216$.
6. $y^8 - 2y^4 + 1 = 0$.
7. $(x^2 - 1)^2 + 24 = 11(x^2 - 1)$.
8. $(t^2 + 2t)^2 - 2(t^2 + 2t) = 3$.
9. $(n^2 - n)^2 - 8(n^2 - n) + 12 = 0$.
10. $(2P^2 + 7P)^2 - 3(2P^2 + 7P) = 10$.

11. $\frac{x}{1-x^2}+\frac{2-2x^2}{x}=3.$

12. $\frac{m}{2m^2+1}+\frac{2m^2+1}{m}=2.$

13. $\frac{2d+1}{d^2}+\frac{d^2}{2d+1}=2.$

14. $A^2+1=\frac{4}{A^2+1}.$

15. $c^2+c-1=\frac{2}{c^2+c}.$

16. $H^2+3H=1-\frac{1}{H^2+3H+1}$

17. $4n^2-10n+3=\frac{1}{2n^2-5n+1}.$

18. $(2v^2+6v)^2=1-(v^2+3v-2).$

19. $(R^2-R+4)^2+(R^2-R)=2.$

20. $(b^2-3b+1)(b^2-3b+2)=12.$

21. $(W^2+5W-1)(W^2+5W+1)+1=0.$

22. $\left(x+\frac{8}{x}\right)^2+\left(x+\frac{8}{x}\right)=42.$

23. $\left(n-\frac{5}{n}\right)^2-\left(n-\frac{5}{n}\right)=30.$

24. $a^2+a+\frac{1}{a}+\frac{1}{a^2}=4.$

Suggestion. — $a^2+2+\frac{1}{a^2}=\left(a+\frac{1}{a}\right)^2.$

25. $y^4+3y^3-2y^2-3y+1=0.$

Suggestion. — Divide by y^2, and arrange as in Ex. 24.

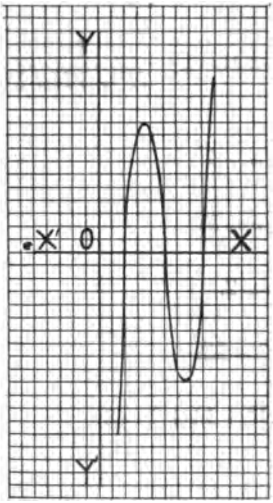
26. $D^4-3D^3+4D^2-3D+1=0.$

SUPPLEMENTARY EXERCISES

The method of § 95 may be used to represent graphically the roots of an equation of any degree in one unknown number.

Represent graphically the roots of:

1. $x^3 - 15x^2 + 66x - 80 = 0.$



Solution. — Let $y = x^3 - 15x^2 + 66x - 80.$ By assigning values to x, computing the corresponding values of y, locating the points whose coördinates are the sets of values thus found, and joining the consecutive points by a smooth line, the curve shown in the figure is obtained.

This curve cuts the x-axis at three points, whose x-distances are the roots 2, 5, and 8, respectively.

2. $x^3 - 15x^2 + 71x - 105 = 0.$

3. $x^3 - 31x - 30 = 0.$

4. $x^4 - 29x^2 + 100 = 0.$

5. $x^4 - 2x^3 - 72x^2 + 128x + 512 = 0.$

Determine graphically the nature of the roots of:

6. $x^3 - 6x^2 + 16x - 96 = 0.$

7. $x^3 - 22x^2 + 145x - 300 = 0.$

CHAPTER IX

EQUATIONS INVOLVING SURDS

99. Equations involving Surds. — To solve an equation in which the unknown number is involved in one or more surds, it is necessary first to free the unknown number of radical signs. To do this, both members of the equation must be raised to the same power.

EXAMPLE. — Solve $\sqrt{n+2}+\sqrt{3-n}=3.$

Transposing, $\sqrt{n+2}=3-\sqrt{3-n}.$

Squaring both members, $n+2=9-6\sqrt{3-n}+3-n.$

Transposing and combining, $2n-10=-6\sqrt{3-n}.$

Dividing by -2, $5-n=3\sqrt{3-n}.$

Squaring both members, $25-10n+n^2=27-9n.$

Transposing and combining, $n^2-n-2=0.$

Solving, $n=2 \text{ or } -1.$

The above example illustrates the following general rule for freeing an equation of radical signs:

(1) *By transposing, if necessary, change the equation into an equivalent one of which one member consists of a single surd.*

This is called **isolating the surd.**

(2) *Raise both members to such a power as will free this isolated surd of the radical sign.*

(3) *If surds still remain, repeat the operation.*

In checking the solution of an equation involving surds by substituting the values of the roots found, it must be remembered that *only the principal roots of the radicals involved may be used.* (See § 64.)

Thus, 5 is not a root of $\sqrt{3t+1}+\sqrt{t-1}=2$, although when $t=5$, it would become $4-2=2$ if the negative square root of $t-1$ were taken for the second term. The equation of which 5 is a true root is $\sqrt{3t+1}-\sqrt{t-1}=2$.

100. Extraneous Roots. — When both members of an equation are raised to the same power, roots may be introduced (extraneous roots) that will not satisfy the given equation.

For, let any equation be represented by $A=B$.

Squaring both members, $A^2=B^2$.

Transposing and factoring, $(A-B)(A+B)=0$.

This equation is equivalent to $A-B=0$ and $A+B=0$.

Of these two equations, $A-B=0$ alone is equivalent (see § 44) to the given equation. Hence, all of the roots of $A+B=0$ are introduced by squaring.

Hence, in solving an equation which involves surds, it is necessary to test all roots obtained and to discard those which do not satisfy the equation in its original form.

EXAMPLE. — Solve $\sqrt{x+1}-\sqrt{8x}+5=0$. (1)

Isolating a surd, $\sqrt{x+1}=\sqrt{8x}-5$.

Squaring, $x+1=8x-10\sqrt{8x}+25$. (2)

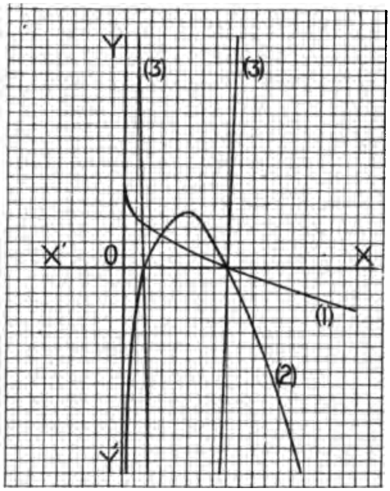
Isolating the surd, $10\sqrt{8x}=7x+24$.

Squaring, $800x=49x^2+336x+576$. (3)

Solving, $x=1\frac{23}{49}$ or 8.

By substitution it is seen that 8 satisfies (1), but $1\frac{23}{49}$ does not. The extraneous root $1\frac{23}{49}$ was introduced by squaring.

What took place in the preceding solution is shown graphically in the accompanying figure. The graphs of equations (1), (2), and (3) are the curves (1), (2), and (3), respectively (the two lines marked (3) are parts of one continuous curve that turns upward at a point 522+ spaces below the x-axis). Squaring equation (1) changed the curve (1) into curve (2), and squaring equation (2) changed the curve (2) into curve (3). Curve (1) shows that equation (1) has only one root, 8. Curve (2) shows that equation (2) has, in addition to the root 8, another root between 1 and 2. And curve (3) shows that equation (3) has the same roots as equation (2).



An equation involving surds may not be satisfied by any value of the unknown number. Such an equation is called an **impossible equation.**

Example.—Solve $\sqrt{2s+8}+2+2\sqrt{s+5}=0$.

Isolating a surd, $\sqrt{2s+8}+2=-2\sqrt{s+5}$.

Squaring, $2s+8+4\sqrt{2s+8}+4=4s+20$.

Hence, $2\sqrt{2s+8}=s+4$.

Squaring again, $8s+32=s^2+8s+16$.

Solving, $s=4$ or -4.

Neither of these roots satisfies the given equation.

EXERCISES

Solve:

1. $\sqrt{x+1}=2.$

2. $8-\sqrt{y-1}=6.$

3. $\sqrt{t+3}=8.$

4. $\sqrt{2x-3}=5.$

5. $5+\sqrt{n+7}=n.$

6. $\sqrt[3]{13+a}=2.$

7. $\sqrt[3]{2v-1}=3.$

8. $4\sqrt{m+5}=3\sqrt{3m+4}.$

9. $5\sqrt{R+2}-3\sqrt{4R-3}=0.$

10. $\sqrt{k+2}-1=\sqrt{k-3}.$

11. $\sqrt{15+3D}+2=\sqrt{23-D}.$

12. $2\sqrt{9+x}=3\sqrt{x+24}-10.$

13. $\sqrt{M^2+4M+9}=M+7.$

14. $\sqrt{5t}+4=\sqrt{12t+21}.$

15. $\sqrt{A}=\sqrt{A+25}-1.$

16. $\sqrt{d-12}=\sqrt{d}-2.$

17. $\sqrt{s}-7+\sqrt{s+7}=0.$

18. $2\sqrt{n+2}=\sqrt{n}.$

19. $w-3=\sqrt{2w^2-w+10}-5.$

20. $\sqrt{n+1}+\sqrt{n-2}=0.$

21. $\sqrt{h^2-7}=7-h.$

22. $\sqrt{r}+\sqrt{32+r}+16=0.$

23. $\sqrt{y+4}+\sqrt{y}+2=0.$

24. $\sqrt{n+2}-\sqrt{3-n}=3.$

25. $\sqrt{a-1}+\sqrt{3a+1}=2.$

26. $1+\sqrt{z+2}=\sqrt{z}.$

27. $\sqrt{2p-7}+\sqrt{2p+9}=8.$

28. $\sqrt[4]{a^2-5a+6}=\sqrt{a+4}.$

29. $\sqrt{2x-5}=\sqrt[4]{x^2-7x+12}.$

30. $\sqrt[3]{t+1}=\sqrt{t+1}.$

31. $\sqrt{y^2+1}=\sqrt[3]{y^2+1}.$

32. $\sqrt{x}-\sqrt{x-3}=\dfrac{2}{\sqrt{x}}.$

33. $\dfrac{2}{\sqrt{x}}=\sqrt{x}+\sqrt{x-1}.$

34. $\sqrt{n}+\sqrt{n-6}=\dfrac{3}{\sqrt{n-6}}.$

35. $\sqrt{a}+\sqrt{1+a}=\dfrac{2}{\sqrt{1+a}}.$

36. $\sqrt{s}-\sqrt{s-8}=\dfrac{2}{\sqrt{s-8}}.$

37. $\dfrac{20}{\sqrt{15+k}}-\sqrt{k}=\sqrt{15+k}.$

38. $\dfrac{12}{\sqrt{R^2+8}}=\sqrt{R^2+8}+R.$

39. $\dfrac{\sqrt{1+n}+\sqrt{1-n}}{\sqrt{1+n}-\sqrt{1-n}}=2.$

40. $\dfrac{1}{1-\sqrt{a}}-\dfrac{1}{1+\sqrt{a}}-\dfrac{8}{1-a}=0.$

41. $\dfrac{\sqrt{A+16}}{\sqrt{4-A}}+\dfrac{\sqrt{4-A}}{\sqrt{A+16}}-2\frac{1}{2}=0.$

42. $\sqrt{5N+2\sqrt{N^2-N+7}}-4=0.$

43. $\sqrt{13+3\sqrt{r+5+4\sqrt{r+1}}}-5=0.$

44. $3x^2-4x+\sqrt{3x^2-4x-6}=18.$

SUGGESTION. — Let $\sqrt{3x^2-4x-6}=y.$

Then $y^2+6+y=18.$

45. $\sqrt{n^2+3n+1}=1-2n^2-6n.$

46. $2\sqrt{4t^2-10t-2}=10t+10-4t^2.$

47. $r+22=3r^2-6\sqrt{3r^2-r-6}.$

48. $\sqrt{2a-4}+\sqrt{3a+4}-\sqrt{10a-4}=0.$

49. $\sqrt{2a-4}+\sqrt{3a+4}+\sqrt{10a-4}=0.$

50. $t=\pi\sqrt{\dfrac{l}{g}}$ expresses a law of the vibration of pendulums. Solve for l.

51. $v=\sqrt{2gs}$ gives the velocity acquired by a falling body. Solve for s.

52. $\dfrac{N}{n}=\dfrac{\sqrt{T}}{\sqrt{t}}$ expresses a law of vibration of strings, in musical instruments. Solve for t; for T.

53. $t=\dfrac{u-\sqrt{u^2-2gs}}{g}$ expresses a law of motion of projectiles fired vertically. Solve for s.

SUPPLEMENTARY EXERCISES

Solve the following, and show graphically the introduction of extraneous roots, as in the first example of § 100:

1. $\sqrt{x-1}+\sqrt{3x+1}=2$.
2. $\sqrt{3x+1}+\sqrt{x-1}=6$.
3. $\sqrt{2x+4}-\sqrt{x+3}=1$.
4. $\sqrt{4x}-\sqrt{2x+1}=1$.

CHAPTER X

SYSTEMS INVOLVING QUADRATIC OR HIGHER EQUATIONS

101. Systems involving Quadratic or Higher Equations. — The solution of a system of two equations in two unknown numbers in which one or both of the equations is of the second or higher degree cannot always be obtained by the methods of elementary algebra. Consequently in this chapter we shall only undertake to solve certain special types of systems.

As in the case of a system of two linear equations, to solve a system involving higher equations requires first the *elimination* of one of the unknown numbers. The method of elimination adapted to any given system depends upon the character of the equations in the system.

102. One Equation Linear and One Quadratic. — If one equation of a system is *linear* and the other *quadratic*, either unknown number may be eliminated by **substitution**.

EXAMPLE. — Solve $\begin{cases} m - 2n = 6, & (1) \\ m^2 + n^2 = 17. & (2) \end{cases}$

Solving (1) for m, $\quad m = 6 + 2n.$

Substituting $6 + 2n$ for m in (2), $\quad 36 + 24n + 5n^2 = 17.$

Solving this quadratic, $\quad n = -1$ or $-\frac{19}{5}.$

Substituting -1 for n in (1), $\quad m + 2 = 6.$

Hence, $\quad m = 4.$

Substituting $-\frac{19}{5}$ for n in (1), $\quad m + \frac{38}{5} = 6.$

Hence, $\quad m = -\frac{8}{5}.$

Therefore there are two solutions:

$$m = 4,\ n = -1;\ \text{and}\ m = -\tfrac{8}{5},\ n = -\tfrac{12}{5}.$$

To solve a system containing one linear and one quadratic equation:

(1) *Solve the linear equation for one unknown number, expressed in terms of the other.*

(2) *Substitute the value found in the quadratic equation, and solve the resulting quadratic in one unknown number.*

(3) *Substitute each value found for the one unknown number in the linear equation, and solve for the corresponding value of the other.*

It is evident that:

Any system of one linear and one quadratic equation has two solutions.

EXERCISES

Solve:

1. $\begin{cases} r - s = 1, \\ r^2 + s^2 = 5. \end{cases}$

2. $\begin{cases} h + k = 6, \\ h^2 + k^2 = 20. \end{cases}$

3. $\begin{cases} a - 2b = 3, \\ a^2 - b^2 = 24. \end{cases}$

4. $\begin{cases} 4x + y = 7, \\ 2x^2 + xy = 5. \end{cases}$

5. $\begin{cases} m + n = 9, \\ m^2 + mn + n^2 = 67. \end{cases}$

6. $\begin{cases} x^2 - xy + y^2 = 12, \\ 3x = y + 10. \end{cases}$

7. $\begin{cases} Rr + 10 = 0, \\ r + 4R = 6. \end{cases}$

8. $\begin{cases} vt + 40 = 0, \\ 2v = 3t + 3. \end{cases}$

9. $\begin{cases} 2P + 2N = 5PN, \\ 2N + 2P = 5. \end{cases}$

10. $\begin{cases} a^2 + b^2 + 4a + 6b = 40, \\ a = 10 + b. \end{cases}$

11. $\begin{cases} 3V = 2T, \\ 3VT + 2V + T = 485. \end{cases}$

12. $\begin{cases} 2x^2 + 3y^2 = 11 + 4xy, \\ x - 5 = 3y. \end{cases}$

13. $\begin{cases} (6 - m)(7 - n) = 80, \\ m + n = 5. \end{cases}$

14. $\begin{cases} k-3=n, \\ k^2+19+n^2=3\,kn. \end{cases}$

15. $\begin{cases} x+y=4, \\ \frac{1}{x}+\frac{1}{y}=1. \end{cases}$

16. $\begin{cases} A+B=2, \\ \frac{3}{B}+\frac{2}{A}=6. \end{cases}$

17. $\begin{cases} p^2+pq+q^2=97, \\ 1-\frac{q}{p}=\frac{19}{p}. \end{cases}$

18. $\begin{cases} \frac{t}{u}+\frac{u}{t}=3\frac{1}{8}, \\ 3\,t-2\,u+12=0. \end{cases}$

19. $\begin{cases} 1-\frac{P}{Q}+\frac{P^2}{Q^2}=\frac{21}{Q^2}, \\ 1+\frac{P}{Q}=\frac{9}{Q}. \end{cases}$

Solve for x and y:

20. $\begin{cases} x-y=a, \\ xy=a^2+b^2. \end{cases}$

21. $\begin{cases} x+y=m, \\ 4\,xy=m^2-n^2. \end{cases}$

22. $\begin{cases} x-y=h, \\ xy=t^2. \end{cases}$

23. $\begin{cases} x+y=a, \\ x^2+y^2=b. \end{cases}$

24. In any circle whose radius is r, circumference c, and area S, $c=2\,\pi r$ and $S=\pi r^2$. By eliminating r between these two equations, find the equation between S and c.

103. Two Quadratic Equations, each of the Form $ax^2+by^2=c$. — If each equation of a system is a quadratic of the form $ax^2+by^2=c$, in which x and y are the unknown numbers, either unknown number may be eliminated by **addition or subtraction.**

EXAMPLE. — Solve $\begin{cases} x^2+2\,y^2=22, & (1) \\ 2\,x^2+y^2=17. & (2) \end{cases}$

Multiplying (1) by 2, $2\,x^2+4\,y^2=44.$

Subtracting (2) from this, $3\,y^2=27.$

Solving, $y=\pm 3.$

Substituting 3 for y in (1), $x^2+18=22.$

Solving, $x=\pm 2.$

Substituting -3 for y in (1), $x^2+18=22.$

Hence, $x=\pm 2.$

Therefore there are four solutions:

$x=2,\ y=3$; $x=-2,\ y=3$; $x=2,\ y=-3$; $x=-2,\ y=-3.$

To solve a system of two quadratic equations, each of the form $ax^2 + by^2 = c$:

(1) *Eliminate one unknown number by addition or subtraction, as in a system of linear equations.*

(2) *Solve the resulting quadratic in one unknown number.*

(3) *Substitute each value of the unknown number thus found in either of the given equations, and find the corresponding values of the other unknown number.*

It is evident that for each of the two values found for the first unknown number there are two corresponding values of the other. Hence:

Any system of two quadratic equations, each of the form $ax^2 + by^2 = c$, *has four solutions.*

It has been observed that in the case of every system solved in this and the preceding sections,

The number of solutions is equal to the product of the degrees of the two equations.

It may be shown that this rule is true in general. If one equation is of the *m*th degree and the other of the *n*th degree, the number of solutions will be *mn*. Exceptions to this rule in certain peculiar systems will be noted later.

EXERCISES

Solve:

1. $\begin{cases} 3x^2 - 8k^2 = 40, \\ 5x^2 + k^2 = 81. \end{cases}$

2. $\begin{cases} 3A^2 - 2B^2 = 0, \\ 5A^2 - 3B^2 = 1. \end{cases}$

3. $\begin{cases} r^2 - s^2 = 7, \\ r^2 + s^2 = 25. \end{cases}$

4. $\begin{cases} 4m^2 + 9n^2 = 36, \\ m^2 + n^2 = 25. \end{cases}$

5. $\begin{cases} 4W^2 - 9V^2 = 36, \\ W^2 + V^2 = 16. \end{cases}$

6. $\begin{cases} 3x^2 - 5y^2 = 30, \\ x^2 + 5y^2 = 70. \end{cases}$

7. $\begin{cases} v^2 + 4t^2 = 40, \\ 2v^2 - 9t^2 = 63. \end{cases}$

8. $\begin{cases} R^2 - 2D^2 = 17, \\ 2R^2 + D^2 = 54. \end{cases}$

9. $\begin{cases} 4a^2 - 9b^2 = 0, \\ 6a^2 + b^2 = 3\frac{2}{9}. \end{cases}$

10. $\begin{cases} 4P^2 + 3Q^2 = 43, \\ 3P^2 - Q^2 = 3. \end{cases}$

11. $\begin{cases} 9x^2 + y^2 = 29, \\ 27x^2 - 2y^2 + 38 = 0. \end{cases}$

12. $\begin{cases} E^2 - 25R^2 = 20, \\ 25R^2 = 88 - 2E^2. \end{cases}$

13. $\begin{cases} 3C^2 = y^2 + 27, \\ C^2 - y^2 + 45 = 0. \end{cases}$

14. $\begin{cases} 2r^2 - 3r'^2 = 20, \\ r^2 + 5r'^2 = 36. \end{cases}$

15. $\begin{cases} \dfrac{x}{y} + \dfrac{y}{x} = \dfrac{5}{16xy}, \\ x - \dfrac{y^2}{x} = \dfrac{3}{16x}. \end{cases}$

104. Two Quadratic Equations, One Homogeneous. — A **homogeneous equation** is one all of whose terms are of the same degree in the unknown numbers.

Thus, $2v^2 - vt + t^2 = 0$ is homogeneous, every term being of the second degree.

The process of solving a system of two quadratic equations, one of which is homogeneous, may be reduced to the process of § 102.

Example 1. — Solve $\begin{cases} m^2 - 5mn + 6n^2 = 0, & (1) \\ m^2 - mn + n^2 = 12. & (2) \end{cases}$

Factoring (1), $(m - 2n)(m - 3n) = 0.$

Hence, (1) is equivalent to the two equations

$$m - 2n = 0, \qquad (3)$$

and

$$m - 3n = 0. \qquad (4)$$

Therefore the given system is equivalent to the two systems

$A \begin{cases} m - 2n = 0, \\ m^2 - mn + n^2 = 12; \end{cases}$ $B \begin{cases} m - 3n = 0, \\ m^2 - mn + n^2 = 12. \end{cases}$

The systems A and B are each solved by substitution, as in § 102.

The solutions of A are $m = 4$, $n = 2$ and $m = -4$, $n = -2$.

The solutions of B are $m = \frac{6}{7}\sqrt{21}$, $n = \frac{2}{7}\sqrt{21}$ and $m = -\frac{6}{7}\sqrt{21}$, $n = -\frac{2}{7}\sqrt{21}$.

Hence, these four solutions are the solutions of the given system.

Example 2. — Solve $\begin{cases} 5x^2 + 4xy - y^2 = 0, & (1) \\ x^2 + x + y = 5. & (2) \end{cases}$

Solving (1) for x by the quadratic formula,

$$x = \frac{-4y \pm \sqrt{16y^2 + 20y^2}}{10}$$
$$= -y \text{ or } \tfrac{1}{5}y.$$

Hence, the given system is equivalent to the two systems

$$A \begin{cases} x = -y, \\ x^2 + x + y = 5; \end{cases} \qquad B \begin{cases} x = \frac{1}{5}y, \\ x^2 + x + y = 5. \end{cases}$$

Solving system A by the method of § 102,

$$x = -3 + \sqrt{14},\ y = -15 + 5\sqrt{14};\ x = -3 - \sqrt{14},\ y = -15 - 5\sqrt{14}.$$

Similarly, the solutions of B are found to be

$$x = \sqrt{5},\ y = -\sqrt{5};\ x = -\sqrt{5},\ y = \sqrt{5}.$$

These four solutions are the solutions of the given system.

The method of Example 1 may be used only in case the first member of the homogeneous equation may be factored. The method of Example 2 may be used in case of any homogeneous equation, whether the first member can be factored or not, and hence is the more general method of the two.

To solve a system of two quadratic equations, one being homogeneous, find by factoring or by solving by the quadratic formula two linear equations to which the homogeneous equation is equivalent. Use each of these with the other given equation to form a new system. Solve these two systems by the method of § 102.

EXERCISES

Solve:

1. $\begin{cases} x^2 - 3xy + 2y^2 = 0, \\ 2x^2 + xy - y^2 = 9. \end{cases}$

2. $\begin{cases} a^2 - 5ab + 6b^2 = 0, \\ a^2 - b^2 = 27. \end{cases}$

3. $\begin{cases} 3m^2 - 2mn - n^2 = 0, \\ m + n + n^2 = 32. \end{cases}$

4. $\begin{cases} v^2 + 7vt + 10t^2 = 0, \\ v^2 + 3v + vt - 4t = 33. \end{cases}$

5. $\begin{cases} x^2 - y^2 = 0, \\ 2x^2 + xy = 48. \end{cases}$

6. $\begin{cases} 2V^2 + 3Vg = 0, \\ V^2 - Vg + g^2 = 19. \end{cases}$

7. $\begin{cases} 3PQ + 5Q^2 = 0, \\ 3P^2 + 4PQ = 15. \end{cases}$

8. $\begin{cases} a^2 + ab = 0, \\ a^2 - ab + b^2 = 27. \end{cases}$

105. Quadratics with All Unknown Terms of Second Degree. — A system of two quadratic equations of which neither is homogeneous, but in each of which all unknown terms are of the second degree, may be solved by first combining the given equations so as to obtain a homogeneous equation.

EXAMPLE. — Solve

$$\begin{cases} p^2 - 7pq - 9q^2 = 9, & (1) \\ p^2 + 5pq + 11q^2 = 5. & (2) \end{cases}$$

Evidently we may get a homogeneous equation by eliminating the terms free of unknown numbers.

Multiplying (1) by 5, $5p^2 - 35pq - 45q^2 = 45.$ (3)

Multiplying (2) by 9, $9p^2 + 45pq + 99q^2 = 45.$ (4)

Subtracting (3) from (4), $4p^2 + 80pq + 144q^2 = 0,$

or $p^2 + 20pq + 36q^2 = 0.$ (5)

Now the given system is equivalent to the new system formed of (5) and either (1) or (2). Let us form the system

$$\begin{cases} p^2 + 20pq + 36q^2 = 0, & (5) \\ p^2 + 5pq + 11q^2 = 5. & (2) \end{cases}$$

This system is solved by the method of § 104.
The solutions are:

$p = 2,\ q = -1;\quad p = -2,\ q = 1;\quad p = 2\frac{4}{7},\ q = -\frac{1}{7};\quad p = -2\frac{4}{7},\ q = \frac{1}{7}.$

By eliminating the known terms, derive from the given equations a homogeneous equation. Use this homogeneous equation with either of the given equations to form a new system. Solve this system by the method of § 104.

EXERCISES

Solve:

1. $\begin{cases} x^2 + xy = 21, \\ 2\,xy - y^2 = 8. \end{cases}$

2. $\begin{cases} 2\,m^2 - mn + n^2 = 16, \\ m^2 + mn + 2\,n^2 = 44. \end{cases}$

3. $\begin{cases} A^2 - 3\,AB + 2\,B^2 = 3, \\ 2\,A^2 + B^2 = 6. \end{cases}$

4. $\begin{cases} h^2 + k^2 = 10, \\ 3\,k^2 + hk = 6. \end{cases}$

5. $\begin{cases} w^2 + wv + v^2 = 4, \\ w^2 - 2\,wv = 12. \end{cases}$

6. $\begin{cases} R^2 - Rt - 5\,t^2 = 15, \\ R^2 - 6\,t^2 = 1. \end{cases}$

7. $\begin{cases} x^2 + xy - 2\,y^2 + 44 = 0, \\ xy + 3\,y^2 - 80 = 0. \end{cases}$

8. $\begin{cases} M^2 + MN + N^2 = 39, \\ 2\,M^2 + 3\,MN + N^2 = 63. \end{cases}$

9. $\begin{cases} a^2 - 2\,ab = -16, \\ 2\,ab + 4\,b^2 = 120. \end{cases}$

10. $\begin{cases} p^2 + pq + 2\,q^2 = 74, \\ 2\,p^2 + 2\,pq + q^2 = 73. \end{cases}$

11. $\begin{cases} F^2 - 2\,FC + 15 = 0, \\ C^2 - FC + F^2 = 21. \end{cases}$

12. $\begin{cases} g^2 - gv - 12\,v^2 = 8, \\ g^2 + gv - 10\,v^2 = 20. \end{cases}$

106. Symmetrical Equations. — A **symmetrical equation** is one that is unchanged when the unknown numbers are interchanged.

Thus, $m^2 + n^2 = 5$, $xy = 2$, and $a + b = 8$ are symmetrical.

Some systems involving only symmetrical equations, one linear and one quadratic, or both quadratic, may be solved by the methods of the foregoing sections. But another good method of solving such systems consists of so combining the equations as to find the values of the *sum* and the *difference* of the unknown numbers, and thus derive two or more systems of linear equations to which the given system is equivalent.

EXAMPLE. — Solve $\begin{cases} x^2 + y^2 = 40, & (1) \\ \qquad xy = 12. & (2) \end{cases}$

Multiplying (2) by 2, $\qquad 2\,xy = 24. \qquad (3)$

Adding (3) to (1), $\qquad x^2 + 2\,xy + y^2 = 64.$

Taking square root, $\qquad x + y = \pm 8.$

Subtracting (3) from (1), $x^2 - 2\,xy + y^2 = 16.$

Taking square root, $\qquad x - y = \pm 4.$

Hence, the given system is equivalent to the four systems:

$$A\begin{cases}x+y=8,\\x-y=4;\end{cases}\quad B\begin{cases}x+y=-8,\\x-y=4;\end{cases}\quad C\begin{cases}x+y=8,\\x-y=-4;\end{cases}\quad D\begin{cases}x+y=-8,\\x-y=-4.\end{cases}$$

The solution of A is $x=6$, $y=2$; of B, $x=-2$, $y=-6$; of C, $x=2$, $y=6$; of D, $x=-6$, $y=-2$.

When, except for sign, the equations are symmetrical, the system may be solved by the same method.

EXAMPLE. — Solve $\begin{cases}m-n=2, & (1)\\ m^2+n^2=74. & (2)\end{cases}$

Squaring (1), $m^2-2mn+n^2=4.$ (3)

Subtracting (3) from (2), $2mn=70.$ (4)

Adding (4) and (2), $m^2+2mn+n^2=144.$

Taking square root, $m+n=\pm 12.$

Hence, the given system is equivalent to the two systems:

$$A\begin{cases}m-n=2,\\m+n=12;\end{cases}\quad B\begin{cases}m-n=2,\\m+n=-12.\end{cases}$$

The solution of A is $m=7$, $n=5$; of B, $m=-5$, $n=-7$.

NOTE. — In this and the preceding sections we have discussed methods of solving systems containing linear and quadratic equations of a few special types. Except in special cases such as those in the foregoing sections, to solve a system of two quadratic equations generally involves the solution of an equation of the *fourth degree* in one unknown number.

Thus, consider the system $\begin{cases}w+2v^2=7. & (1)\\ w^2+3v=6. & (2)\end{cases}$

Solving (1) for w, $w=7-2v^2.$

Substituting in (2), $49-28v^2+4v^4+3v=6,$

an equation of the fourth degree.

Consequently, we cannot undertake to solve all systems involving two quadratic equations.

EXERCISES

Solve:

1. $\begin{cases} x + y = 1, \\ x^2 + y^2 = 5. \end{cases}$

2. $\begin{cases} p + q = 3, \\ p^2 + q^2 = 29. \end{cases}$

3. $\begin{cases} a^2 + b^2 = 37, \\ ab = 6. \end{cases}$

4. $\begin{cases} m + n = 7, \\ mn = 6. \end{cases}$

5. $\begin{cases} P^2 + Q^2 = 13, \\ PQ = 6. \end{cases}$

6. $\begin{cases} w - v = 4, \\ w^2 + v^2 = 40. \end{cases}$

7. $\begin{cases} x - y = 8, \\ xy + 15 = 0. \end{cases}$

8. $\begin{cases} xy + 5 = 0, \\ x^2 + y^2 = 26. \end{cases}$

9. $\begin{cases} A^2 + AB + B^2 = 7, \\ A^2 + B^2 = 10. \end{cases}$

10. $\begin{cases} m^2 + mn + n^2 = 19, \\ m - n = 1. \end{cases}$

11. $\begin{cases} s + t = 4, \\ s^2 - st + t^2 = 7. \end{cases}$

12. $\begin{cases} x^2 + xy + y^2 = 4, \\ x^2 - xy + y^2 = 2. \end{cases}$

13. $\begin{cases} a + ab + b = 29, \\ a^2 + ab + b^2 = 61. \end{cases}$

14. $\begin{cases} W^2 - WV + V^2 = 63, \\ W - V + 3 = 0. \end{cases}$

15. $\begin{cases} r^2 + rr' + r'^2 = 31, \\ r^2 - rr' + r'^2 = 21. \end{cases}$

16. $\begin{cases} x^2 - 2\,xy + y^2 = 4, \\ x^2 + 2\,xy + y^2 = 64. \end{cases}$

107. Systems involving Higher Equations solved by Division. — We are able to solve some special systems involving equations of higher degree than the second. If the members of one equation are divisible by the members of the other, one or more simpler systems equivalent to the given system may be derived.

If A, B, M, and N represent integral expressions involving two unknown numbers, then the system

$$\begin{cases} MA = NB, & (1) \\ M = N & (2) \end{cases}$$

is equivalent to the two systems

$$(a)\ \begin{cases} A = B, & (3) \\ M = N, & (2) \end{cases} \quad \text{and} \quad (b)\ \begin{cases} M = 0, & (4) \\ N = 0. & (5) \end{cases}$$

For, by substituting M for N in (1), transposing, and factoring,

$$M(A-B)=0. \tag{6}$$

Also, from (2),

$$M-N=0. \tag{7}$$

Now, the system composed of equations (6) and (7) is equivalent to the two systems (*a* and (*b*), because any solution of (*a*) or (*b*) will satisfy the equations (6) and (7), and any values of the unknown numbers satisfying (6) and (7) will satisfy the equations of (*a*) and (*b*).

Example. — Solve $\begin{cases} x^3+y^3=28x-7y-49, & (1) \\ x+y=4x-y-7. & (2) \end{cases}$

Dividing (1) by (2),

$$x^2-xy+y^2=7.$$

Hence, the given system is equivalent to the two systems

$$A\begin{cases} x^2-xy+y^2=7, \\ x+y=4x-y-7, \end{cases} \text{ and } B\begin{cases} x+y=0, \\ 4x-y-7=0. \end{cases}$$

Solving A by the method of § 102,

$$x=1,\ y=-2, \text{ or } x=3,\ y=1.$$

Solving B,

$$x=1\tfrac{2}{5},\ y=-1\tfrac{2}{5}.$$

If either M or N is an arithmetical number (not zero), the *system* (*a*) *alone* is equivalent to the given system, because in that case either equation (4) or (5) of system (*b*) would be impossible.

Example. — Solve $\begin{cases} v^3-t^3=56, & (1) \\ v^2+vt+t^2=28. & (2) \end{cases}$

Dividing (1) by (2),

$$v-t=2.$$

Hence, the given system is equivalent to the single system

$$\begin{cases} v-t=2, \\ v^2+vt+t^2=28. \end{cases}$$

Solving this system by the method of § 102 or § 106,

$$v=4,\ t=2, \text{ or } v=-2,\ t=-4.$$

Observe that the system in the above example has only *two solutions*, whereas we expected *three solutions*. Such a system, in which the number of solutions is *less than the product of the degrees of the equations*, is called a **defective system.**

EXERCISES

Solve:

1. $\begin{cases} m^3 + n^3 = 35, \\ m + n = 5. \end{cases}$

2. $\begin{cases} v^3 - t^3 = 27, \\ v - t = 3. \end{cases}$

3. $\begin{cases} a^3 - b^3 = 133, \\ a - b = 7. \end{cases}$

4. $\begin{cases} x^3 + y^3 = 65, \\ x + y = 5. \end{cases}$

5. $\begin{cases} P^3 + Q^3 = 28, \\ P + Q = 4. \end{cases}$

6. $\begin{cases} a^2 - ab + b^2 = 7, \\ a^3 + b^3 = 28. \end{cases}$

7. $\begin{cases} w^3 - v^3 = 56, \\ w^2 + wv + v^2 = 28. \end{cases}$

8. $\begin{cases} M^3 - N^3 = 19, \\ M^2 + MN + N^2 = 19. \end{cases}$

9. $\begin{cases} x^4 + x^2y^2 + y^4 = 243, \\ x^2 - xy + y^2 = 9. \end{cases}$

10. $\begin{cases} r^2 + rs + s^2 = 13, \\ r^4 + r^2s^2 + s^4 = 91. \end{cases}$

11. $\begin{cases} a^2b + ab^2 = 30, \\ a + b = 5. \end{cases}$

12. $\begin{cases} m^3 - n^3 = 6\,m, \\ m - n = 3\,m. \end{cases}$

13. $\begin{cases} 1 - v^3 = t^2, \\ 1 - v = t. \end{cases}$

14. $\begin{cases} x^3 + y^3 = 14\,x + 21\,y - 49, \\ x + y = 2\,x + 3\,y - 7. \end{cases}$

15. $\begin{cases} m^3 - n^3 = 62\,m - 186\,n, \\ m - n = 2\,m - 6\,n. \end{cases}$

16. $\begin{cases} a^4 + a^2 + 1 = \frac{3}{7}\,b^2, \\ a^2 + a + 1 = b. \end{cases}$

108. Other Systems involving Higher Equations. — Other systems involving equations of higher degree than the second may be solved by special devices.

EXAMPLE 1. — Solve $\begin{cases} x^4 + y^4 = 641, & (1) \\ x + y = 3. & (2) \end{cases}$

Let $\quad x = a + b$ and $y = a - b.$

Substituting in (2), $\quad a + b + a - b = 3.$

Hence, $\quad a = \frac{3}{2}. \quad (3)$

Substituting in (1), $(a+b)^4+(a-b)^4=641,$

or $2\,a^4+12\,a^2b^2+2\,b^4=641.$ (4)

From (3) and (4), $\frac{81}{8}+27\,b^2+2\,b^4=641,$

or $16\,b^4+216\,b^2-5047=0.$ (5)

Solving (5) by quadratic formula, $b^2=\frac{49}{4}$ or $-\frac{103}{4}.$

Hence, $b=\pm\frac{7}{2}$ or $\pm\frac{1}{2}\sqrt{-103}.$

Using the value of a and each of the four values of b, we get

$$x=a+b=5,\ -2,\ \tfrac{3}{2}+\tfrac{1}{2}\sqrt{-103},\text{ or }\tfrac{3}{2}-\tfrac{1}{2}\sqrt{-103}.$$

$$y=a-b=-2,\ 5,\ \tfrac{3}{2}-\tfrac{1}{2}\sqrt{-103},\text{ or }\tfrac{3}{2}+\tfrac{1}{2}\sqrt{-103}.$$

EXAMPLE 2. — Solve $\begin{cases}a^4b^4+a^2b^2=20, & (1)\\ a+b=3.\end{cases}$

Factoring (1), $(a^2b^2+5)(a^2b^2-4)=0.$ (2)

Hence, $a^2b^2+5=0,$ (3)

or $a^2b^2-4=0.$ (4)

Solving (3), $ab=\pm\sqrt{-5}.$ Solving (4), $ab=\pm 2.$

Hence, the given system is equivalent to the four systems:

$A\begin{cases}a+b=3,\\ ab=\sqrt{-5};\end{cases}$ $B\begin{cases}a+b=3,\\ ab=-\sqrt{-5};\end{cases}$ $C\begin{cases}a+b=3,\\ ab=2;\end{cases}$ $D\begin{cases}a+b=3,\\ ab=-2.\end{cases}$

These systems will give in all eight solutions.

EXERCISES

Solve:

1. $\begin{cases}m^4+n^4=17,\\ m+n=3.\end{cases}$

2. $\begin{cases}w^4+v^4=257,\\ w-v=3.\end{cases}$

3. $\begin{cases}A^4+B^4=82,\\ A+B=4.\end{cases}$

4. $\begin{cases}x+y=5,\\ x^4+y^4=81.\end{cases}$

5. $\begin{cases}n+m=5,\\ n^4m^4-37\,n^2m^2+36=0.\end{cases}$

6. $\begin{cases}(a+b)^4-(a+b)^2=12,\\ a-b=0.\end{cases}$

7. $\begin{cases} h^6 - k^6 = 0, \\ h + 2k = 6. \end{cases}$

Suggestion. — By factoring find the linear and quadratic equations to which the first equation is equivalent. Use each of these with the second equation to form a new system.

8. $\begin{cases} x^6 - y^6 = 0, \\ x^2 + 2xy = 27. \end{cases}$

9. $\begin{cases} p^4 + p^2q^2 + q^4 = 0, \\ p^2 - 2pq + q^2 = 0. \end{cases}$

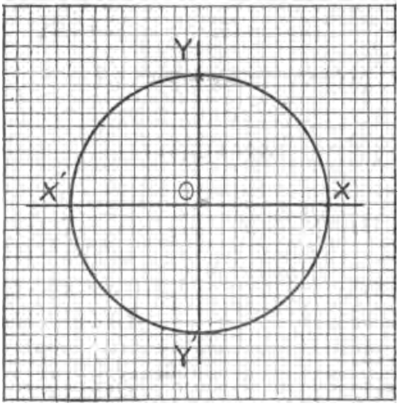
109. Graphs of Quadratic and Higher Equations. — The *graph* of an equation of the second or higher degree is in general some kind of *curve*. See First Course, § 153 and § 154.

Example 1. — Draw the graph of $x^2 + y^2 = 100$.

By assigning values to x and computing the corresponding values of y, the following sets of values are obtained :

x	0	2	4	6	8	10	− 2	− 4	− 6	− 8	− 10
y	± 10	± 9.8	± 9.2	± 8	± 6	0	± 9.8	± 9.2	± 8	± 6	0

If x is given a value greater than + 10 or less than − 10, the corresponding value of y obtained is imaginary. This shows that the graph does not extend farther than 10 spaces to the right or left of the y-axis. Similarly, it does not extend farther than 10 spaces above or below the x-axis.



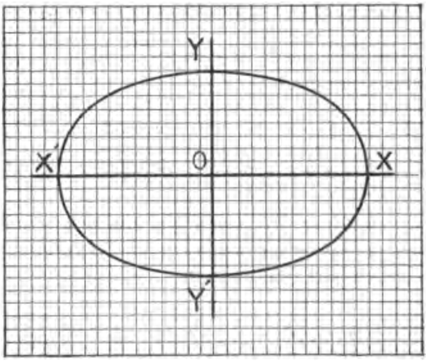
By locating the points whose coördinates are the above sets of values, and joining the consecutive points by a smooth curve, we get the **circle** in the figure.

EXAMPLE 2. — Draw the graph of $4x^2 + 9y^2 = 576$.

By assigning values to x we get the following sets of values of x and y:

x	0	3	6	9	12	-3	-6	-9	-12
y	± 8	± 7.7	± 6.9	± 5.3	0	± 7.7	± 6.9	± 5.3	0

Marking the points whose coördinates are these sets of values, and connecting the consecutive points by a smooth line, gives the oval in the figure. This curve is called an **ellipse**.



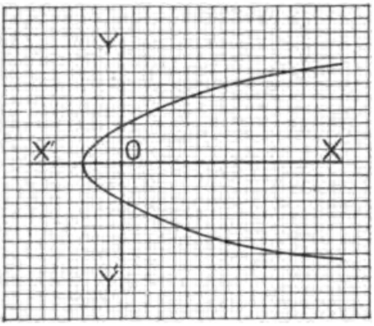
EXAMPLE 3. — Draw the graph of $y^2 = 3x + 9$.

Proceeding as above, we get:

x	0	3	6	9	12	15	-3	Greater negative values
y	± 3	± 4.2	± 5.2	± 6	± 6.7	± 7.3	0	Imaginary

By locating the points whose coördinates are these sets of values, and connecting the consecutive points by a smooth line, we get the curve in the figure.

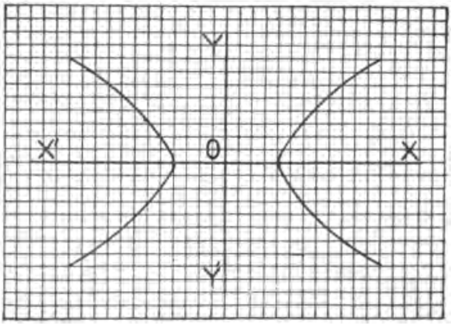
This curve is called a **parabola**. It is not a closed figure like the circle and ellipse, but its branches continue farther and farther apart as they recede from the y-axis.



EXAMPLE 4. — Draw the graph of $x^2 - 2y^2 = 16$.
Proceeding as above, we get:

x	4	6	8	10	-4	-6	-8	-10
y	0	± 3.2	± 4.9	± 6.5	0	± 3.2	± 4.9	± 6.5

It is seen that for values of x between -4 and $+4$, y is imaginary. This shows that no part of the curve lies in the strip extending 4 spaces to the right and 4 spaces to the left of the y-axis.

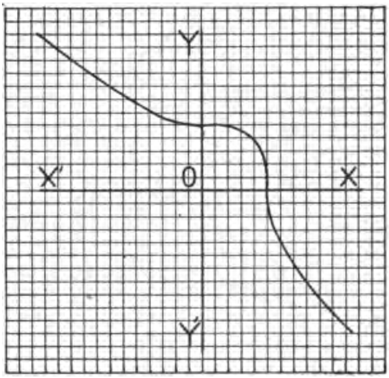


Drawing the curve, it is found to consist of two branches, as in the figure. This curve is called an **hyperbola**.

Note. — The curves obtained in the above examples play a very important part in our everyday life. Since the time of the early Greeks it has been known that they are the sections obtained by cutting a cone by a plane. The great astronomer Kepler (1571–1630) showed that the orbit in which the earth moves about the sun is an *ellipse*. It is now known that the orbits of all the planets are ellipses. Also, those comets that reappear periodically move in ellipses. Other comets that make their appearance only once move in *parabolas*. It is known also that all projectiles, such as an object thrown up in the air, or a shot fired from a cannon, move in parabolas. Thus, a knowledge of the parabola is used in gunnery in the navy, etc.

Example 5. — Draw the graph of $x^3 + y^3 = 125$.

Proceeding as in the above examples, we get the curve shown in the figure.



It is seen that a straight line may be drawn cutting this curve in *three points*. Likewise, a straight line may be drawn cutting either of the above graphs of equations of the second degree in *two points*. Similarly, in general the graph of an equation of the fourth degree is such that it may be cut by a straight line in *four* points and only four, etc.

EXERCISES

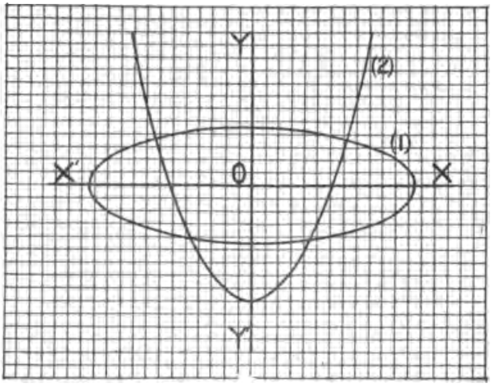
Draw the graphs of:

1. $x^2 + y^2 = 64$.
2. $x^2 + 4y^2 = 64$.
3. $9x^2 + 16y^2 = 576$.
4. $9x^2 - 4y^2 = 36$.
5. $x^2 = 2y$.
6. $xy = 8$.
7. $y^2 + 4x = 16$.
8. $x^3 + y^3 = 512$.
9. $x^3 - y^3 = 216$.

110. Systems solved Graphically. — A system involving two equations, of which one or both are of the second or higher degree, may be solved by drawing the graphs of the equations, and noting the coördinates of the points where the graphs cross. These coördinates must satisfy both equations, and hence form the solutions of the system. See § 53.

EXAMPLE. — Solve $\begin{cases} x^2 + 8y^2 = 160, & (1) \\ x^2 = 4y + 36. & (2) \end{cases}$

The graph of (1) is the ellipse (1), and the graph of (2) is the parabola (2). These curves meet at four points, whose coördinates are approximately $x = 7.1$, $y = 3.7$; $x = -7.1$, $y = 3.7$; $x = 4.4$, $y = -4.2$; $x = -4.4$, $y = -4.2$, the four solutions of the system.



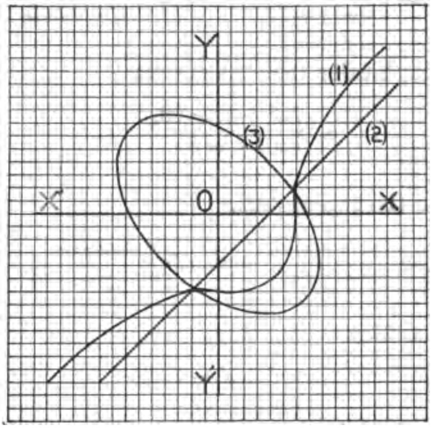
What takes place in solving a system when a new system equivalent to the given one is found and solved may be shown graphically.

EXAMPLE. — Interpret graphically the solution of $\begin{cases} x^3 - y^3 = 208, & (1) \\ x - y = 4. & (2) \end{cases}$

Dividing (1) by (2), $x^2 + xy + y^2 = 52$. (3)

Now (2) and (3) form a new system whose solutions are found to be $x = 6$, $y = 2$ and $x = -2$, $y = -6$.

The graphs of (1), (2), and (3) are the curve (1), straight line (2), and ellipse (3) in the figure. It is seen that the only points where (1) and (2) cross are points where (2) and (3) cross, and conversely. Hence, to solve the system (1) and (2) is equivalent to solving the system (2) and (3), which has only the two solutions.



EXERCISES

Solve graphically:

1. $\begin{cases} x + 2y = 12, \\ x^2 + y^2 = 45. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 100, \\ x - y = 5. \end{cases}$

3. $\begin{cases} 4x^2 + 9y^2 = 324, \\ y^2 = x + 16. \end{cases}$

4. $\begin{cases} x^2 - y^2 = 4, \\ y^2 - x = 12. \end{cases}$

5. $\begin{cases} x^2 + 4y^2 = 144, \\ 4x^2 + y^2 = 144. \end{cases}$

6. $\begin{cases} x^2 + y^2 = 81, \\ x^2 - y^2 = 16. \end{cases}$

7. $\begin{cases} x^2 - y^2 = 4, \\ x^2 - y = 16. \end{cases}$

8. $\begin{cases} x^2 + y^2 = 100, \\ x^2 + 2y^2 = 144. \end{cases}$

9. Show graphically that all solutions of the system

$$\begin{cases} x^2 + y^2 = 16, \\ x^2 - y^2 = 64, \end{cases}$$

are imaginary.

10. Show graphically that the following systems are equivalent:

$$A\begin{cases}x^2-y^2=16,\\x-y=2,\end{cases}\text{ and }B\begin{cases}x+y=8,\\x-y=2.\end{cases}$$

11. Show graphically that the system

$$\begin{cases}x^2+y^2=40,\\x+y=8,\end{cases}$$

is equivalent to the two systems

$$A\begin{cases}x+y=8,\\x-y=4,\end{cases}\text{ and }B\begin{cases}x+y=8,\\x-y=-4.\end{cases}$$

12. Show graphically that the system

$$\begin{cases}x^3+y^3=152,\\x+y=8,\end{cases}$$

is defective.

111. Problems. — Some problems may be solved by use of systems of equations in two unknown numbers, in which one or both equations are of the second or higher degree.

EXERCISES

1. Find two numbers whose sum is 13 and the sum of whose squares is 97.

2. The area of a right triangle is 50 square feet. The hypotenuse is 15 feet. Find the legs.

3. A man has two fields, each square. It takes 40 rods more to fence one than the other. The larger contains $5\frac{5}{8}$ acres more than the smaller. Find the length of each.

4. The area of a rectangular floor is 1200 square feet. The diagonal of the floor is 50 feet. Find the dimensions.

5. The perimeter of a rectangle is 100 yards. Its area is 600 square yards. Find its dimensions.

6. The diagonal of a rectangle is 30 feet. If its length were increased 6 feet, its area would be increased 108 square feet. What are its dimensions?

7. The perimeter of a right triangle is 48 inches. Its area is 96 square inches. Find the three sides.

8. The floor of a room consists of a plain center surrounded by a parquetry border 3 feet wide. There are 140 square feet in the plain center and 180 square feet in the border. Find the dimensions of the room.

9. The area of a rectangle is 36 square inches. If its length be increased by 3 inches and its width by 2 inches, its area will be doubled. Find its dimensions.

10. A rectangle is 20 inches long and 16 inches wide. How much must be added to its width, and how much must be taken from its length, in order that its area may be increased by 22 square inches and its perimeter by 2 inches?

11. The hypotenuse of a right triangle is 10. If one leg be increased by 3 and the other leg by 4, the hypotenuse will become 15. Find the sides of the triangle.

12. A rectangular lot which contains 880 square yards is 2.2 times as long as it is wide. How much will it cost to build a fence around it at 25 cents a linear foot?

13. A guy rope to a derrick is attached to a stake 30 feet from the foot of the derrick. If the rope were $16\frac{2}{3}$ feet longer, it would reach to a stake $53\frac{1}{3}$ feet from the foot of the derrick. Find the height of the derrick and the length of the rope.

14. A flower garden contains 1000 square feet, and is surrounded by a path 5 feet wide. The area of the path is 750 square feet. What are the dimensions of the garden?

15. A laborer received $32.50 wages. If he had worked 5 days longer, and had received 50 cents a day less, he would ha received \$41.25. How long did he work, and what were his wages a day?

16. A certain number of men do a piece of work in a certain number of days. If there were two fewer men, it would take 4 days longer to do the work; and if there were twice as many men, it would take 4 days less to do the work. Find the number of men and the number of days required for them to do the work.

17. A grocer bought apples and potatoes for \$54. He sold the apples for \$36.80 and the potatoes for \$18.70. He gained as many per cent on the apples as he lost on the potatoes. How much did he pay for each?

18. A certain sum of money placed at simple interest for one year amounted to \$265. If the principal had been \$50 more and the rate 1% less, the interest would have been the same. Find the principal and the rate.

19. The sides of a triangle are 10, 12, and 14. Find the altitude on the side 14.

SUGGESTION. — The altitude divides the given triangle into two right triangles. Call the base of one of these triangles x. Then the other side will be $14 - x$. Draw the figure.

20. The sides of a triangle are 12, 16, and 20. Find the altitude on side 20, and also find the area of the triangle.

21. One season an agent sold a lot of reapers at a uniform price, receiving \$4800 for all. Another agent sold one more reaper, but received \$8 less for each. They found that each had received the same amount for all. Find the number sold by each, and the price of each.

22. A and B start simultaneously from towns 247 miles apart and travel towards each other until they meet. A travels 9 miles per day. The number of days before they meet is 3 greater than the number of miles B travels per day. How far does each travel?

23. A and B both start from the same place and travel in the same direction. But A starts 2 hours before B, and is overtaken by him 30 miles from the starting point. Had each traveled half a mile more per hour, B would have overtaken A just 42 miles from the starting point. Find the rate of travel of each.

24. A rectangular field contains 1536 square rods. The length and width of another field are each just 5 rods less, and its area is 375 square rods less. Find the dimensions of the first field.

25. The sum of two numbers is 60. The sum of their squares divided by their product is 2.5. Find the numbers.

26. The fore wheel of a carriage makes 6 revolutions more than the hind wheel in going 120 yards. But if the circumference of each wheel is increased by 3 feet, the fore wheel will make but 4 more revolutions than the hind wheel in going 120 yards. Find the circumference of each.

SUPPLEMENTARY EXERCISES

Solve:

1. $\begin{cases} m^2+n^2-m-n=22, \\ m+n+mn=-1. \end{cases}$

SUGGESTION. — Multiply members of second equation by 2, and add to first.

2. $\begin{cases} x^2-y^2=x+y, \\ x^2-3\,xy=5\,x-15\,y. \end{cases}$

SUGGESTION. — By factoring, find linear equations equivalent to each equation.

3. $\begin{cases} A^4+A^2B^2+B^4=91, \\ A^2+AB+B^2=13. \end{cases}$

4. $\begin{cases} p^2q-p=50, \\ p^4q^2-p^2=2475. \end{cases}$

5. $\begin{cases} x^2+y^2+7\,xy=171, \\ xy=2(x+y). \end{cases}$

6. $\begin{cases} r^2+s^2-(r-s)=20, \\ rs+r-s=1. \end{cases}$

7. $\begin{cases} M^2N+MN^2=180, \\ M^2N^2=400. \end{cases}$

8. $\begin{cases} (x+y)(x-4\,y)=4\,y-x, \\ x^2-2\,xy+y^2=3(x-y). \end{cases}$

9. $\begin{cases} w^2v^2+wv=6, \\ w^2+v^2=5. \end{cases}$

10. $\begin{cases} d^4=9\,t^4+48, \\ d^2=3\,t^2+2. \end{cases}$

11. $\begin{cases} P^2+Q=7, \\ Q^2+P=11. \end{cases}$

12. $\begin{cases} a^2+ab+b=0, \\ a^2+ab+a=0. \end{cases}$

13. $\begin{cases} h^2=8h+6k, \\ k^2=6h+8k. \end{cases}$

14. $\begin{cases} \sqrt{x}+\sqrt{y}=8, \\ x^2+y^2=706. \end{cases}$

Suggestion. — Let

$$\sqrt{x}=v, \quad \sqrt{y}=w.$$

15. $\begin{cases} m-n=19, \\ \sqrt[3]{m}-\sqrt[3]{n}=1. \end{cases}$

Suggestion. — Let

$$\sqrt[3]{m}=v, \quad \sqrt[3]{n}=w.$$

16. $\begin{cases} \sqrt{a}+\sqrt{b}=2, \\ \sqrt{a^3}+\sqrt{b^3}=26. \end{cases}$

17. $\begin{cases} x+y+2\sqrt{x+y}=24, \\ x-y+3\sqrt{x-y}=10. \end{cases}$

18. $\begin{cases} w+\sqrt{wv}+v=14, \\ w^2+wv+v^2=84. \end{cases}$

19. $\begin{cases} \dfrac{1}{x}-\dfrac{1}{y}=1, \\ \dfrac{1}{x^3}-\dfrac{1}{y^3}=19. \end{cases}$

CHAPTER XI

PROPORTION. VARIABLES

112. Ratio. — The **ratio** of two numbers of the same kind is the quotient of the first divided by the second. It is usually written in the form of a fraction.

Thus, the ratio of 12 feet to 20 feet is written $\frac{12 \text{ feet}}{20 \text{ feet}}$. The ratio of m to n is written $\frac{m}{n}$. An old form of writing the ratio of m to n, now little used, is $m : n$.

All rules applicable to fractions apply also to ratios, such as the rule for reduction to lower terms, etc.

The dividend or numerator of a ratio is sometimes called the **antecedent**, and the divisor or denominator the **consequent**.

Two quantities are called **commensurable** when their ratio can be expressed by two whole numbers, *i.e.* when a unit of measure can be found that is contained an integral number of times in each. Two quantities that are not commensurable are called **incommensurable**.

EXERCISES

1. Express the ratio of 6 pounds to 8 pounds as a fraction in its lowest terms.

2. The tax rate in a certain community is \$3 to every \$100 of property. Express this rate or ratio as a common fraction. As a decimal.

3. The death rate in the United States in a recent year, due to street car accidents, was given as 4 to 100,000 population. Express this rate or ratio as a common fraction in its lowest terms. As a decimal.

4. What is the ratio of the circumference to the diameter of a circle? Are the circumference and the diameter commensurable? Why not?

5. One cubic foot of iron weighs 450 pounds, and one cubic foot of water weighs 62.5 pounds. Express the ratio of the weight of iron to the weight of an equal volume of water as a common fraction in its lowest terms. As a decimal.

6. Express the ratio of $m + n$ to $m^3 + n^3$ as a fraction in its lowest terms.

7. Express the ratio of $\frac{2}{3}$ to $\frac{5}{6}$ as a simple fraction.

8. Express the ratio of $\frac{v}{t}$ to $\frac{t'}{v'}$ as a simple fraction.

9. Show that $\frac{7}{8}$ and $\frac{9}{10}$ are commensurable.

10. Show that any two fractions whose terms are whole numbers are commensurable.

11. Show that 4 and $\sqrt{2}$ are not commensurable.

12. Are $\sqrt{32}$ and $\sqrt{8}$ commensurable?

13. Show that the diagonal and side of a square are not commensurable.

14. Which ratio is the greater, $\frac{5}{8}$ or $\frac{29}{47}$?

15. Divide a profit of $2400 between two partners so that their shares shall be in the ratio of 2 to 3.

113. Proportion. — An equation each of whose members consists of a ratio is called a **proportion**.

Thus, $\frac{3 \text{ lb.}}{8 \text{ lb.}} = \frac{6 \text{ mi.}}{16 \text{ mi.}}$, $\frac{2m}{5n} = \frac{6m}{15n}$, and $\frac{a}{b} = \frac{c}{d}$ are proportions.

The proportion $\frac{a}{b} = \frac{c}{d}$ is read "*a* is to *b* as *c* is to *d*," as well as "*a* divided by *b* equals *c* divided by *d*."

In any proportion $\frac{a}{b} = \frac{c}{d}$, the numbers a, b, c, and d are called the **terms**. The first and fourth terms, a and d, are called the

extremes. The second and third terms, b and c, are called the **means.**

The four terms of a proportion are sometimes said to be "proportional," or "in proportion."

If $\frac{a}{b}=\frac{c}{d}$, the fourth term d is called the **fourth proportional** to a, b, and c.

A proportion such as $\frac{a}{n}=\frac{n}{b}$, in which the means are equal, is called a **mean proportion.** The mean term n is called a **mean proportional** between a and b. And the fourth term b is called the **third proportional** to a and n.

Since a proportion is an equation, all processes which may be performed upon an equation may be performed upon a proportion.

Since the ratio of any two concrete numbers is the same as that of the corresponding abstract numbers, in forming a proportion confusion may be avoided by writing only the abstract numbers.

Thus, the ratio $\frac{2 \text{ oz.}}{7 \text{ oz.}}$ may be expressed by $\frac{2}{7}$.

EXERCISES

Find the value of the unknown term in each of the following proportions:

1. $\frac{x}{5}=\frac{9}{15}$.
2. $\frac{t}{8}=\frac{1}{12}$.
3. $\frac{6}{v}=\frac{16}{9}$.
4. $\frac{12}{20}=\frac{8}{r}$.
5. $\frac{7}{9}=\frac{P}{27}$.
6. $\frac{8}{21}=\frac{4}{k}$.
7. $\frac{1}{y}=\frac{10}{36}$.
8. $\frac{84}{32}=\frac{F}{42}$.

Find the value of n in each of the proportions:

9. $\frac{x}{y}=\frac{n}{z}$.
10. $\frac{v}{2n}=\frac{t}{s}$.
11. $\frac{1}{k}=\frac{k}{n}$.
12. $\frac{a}{b}=\frac{cn}{a}$.
13. $\frac{r-n}{r}=\frac{d+n}{d}$.
14. $\frac{2\pi r}{n}=\frac{2\pi r'}{n+r}$.

Find the fourth proportional to:

15. 3, 4, and 6.

16. 9, 2, and 12.

17. 1, 20, and $\frac{1}{2}$.

18. $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{8}$.

19. 1, x^2, and $2x$.

20. 1, $m+n$, and $m-n$.

Find the third proportional to:

21. 3 and 12.

22. 8 and 1.

23. $\frac{3}{4}$ and $\frac{1}{6}$.

24. w and wv.

25. p and q.

26. $a-b$ and a^2-b^2.

Find the mean proportional between:

27. 1 and 16.

28. 9 and 25.

29. 6 and 54.

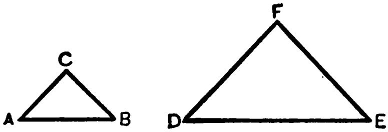
30. 5 and 7.

31. a and b.

32. V and $V-2V^2+V^3$.

33. What number must be added to each of the four numbers 1, 17, 5, 29 to make the results proportional?

34. In a city whose population was 96,000 the number of Germans was 12,000. When the total population increased to 120,000, the number of Germans increased proportionally. Of the 120,000 population how many were Germans?



35. It is known in geometry that in two similar triangles the corresponding sides are proportional; that is,

$$\frac{AB}{DE}=\frac{AC}{DF}.$$

If $AB=3$ inches, $AC=2$ inches, and $DE=7$ inches, find the length of DF.

36. In two similar triangles the sides of the smaller are 8 inches, 10 inches, and 12 inches, and the longest side of the larger is 40 inches. Find the other sides of the larger triangle.

37. A tree casts a shadow 60 feet long when a post 4 feet high casts a shadow 5 feet 8 inches long. How high is the tree?

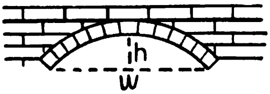
38. The line which divides an angle of a triangle into equal parts divides the opposite side into two parts which are proportional to the adjacent sides. The sides of a triangle are 12 inches, 16 inches, and 18 inches. The angle formed by the 12-inch and 16-inch sides is divided by a line into two equal parts. Find the lengths of the parts into which it divides the 18-inch side.

39. The sides of a triangle are 10 inches, 14 inches, and 15 inches. Find the lengths of the parts into which the 14-inch side is divided by the line which divides the opposite angle into equal parts.

40. The problem to divide a given line into two parts such that the longer part shall be a mean proportional between the whole line and the shorter part has been called since the time of the ancient Greeks the Problem of the Golden Section. A line is 16 inches long. Find the lengths of the parts into which it is divided by the golden section.

41. If a line is 18 feet long, find the lengths of the parts into which it is divided by the golden section.

42. If the radius of a circular arch is r, the width or span w, and the height or rise h, as shown in the figure, then the half-span is a mean proportional between the rise and the difference between the diameter and rise; that is,



$$\frac{h}{\frac{1}{2}w}=\frac{\frac{1}{2}w}{2r-h}.$$

If the stone arch of a doorway is to have a span of 10 feet and rise of 4 feet, find the radius of the circle to which the stones of the arch must be cut.

43. The span of a bridge is 80 feet and the rise of the arch 16 feet. Find the radius of the arch.

44. A lever is a stiff bar by means of which a great resistance may be overcome through the application of a small force or effort. The bar revolves on a pivot called the fulcrum, the weight or resistance is applied at one end, and the effort at the other.

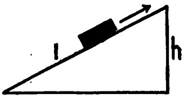
If the resistance is W, its distance from the fulcrum D, the effort w, and its distance from the fulcrum d, then

$$\frac{w}{W}=\frac{D}{d}.$$

If $W = 1200$ pounds, $D = 4$ inches, and $d = 60$ inches, find w.

45. What effort is required to lift a stone weighing 720 pounds by means of a crowbar whose entire length is 54 inches, if the fulcrum is placed 3 inches from the stone.

46. If an object is moved up an inclined plane or slope, neglecting friction, the force applied parallel to the incline required to move the object is to the weight of the object as the height of the incline is to its length. That is, if W is the weight of the object, F the force required to move it, h the height of the incline, and l the length of the incline, then



$$\frac{F}{W}=\frac{h}{l}.$$

If $W = 1200$ pounds, $l = 40$ feet, and $h = 8$ feet, find F.

47. What force is required to roll a barrel weighing 216 pounds up a board and into a wagon, if the board is 10 feet long, and the wagon bed 30 inches from the ground?

48. What force is required to move a block of ice weighing 200 pounds up an incline 80 feet long and 36 feet high?

114. General Principles in Proportion. — The following principles, which are often used in the applications of proportion, were nearly all established in the First Course, § 119. Let the student give the proof of each of them.

(1) *If four numbers are in proportion, the product of the extremes equals the product of the means.* That is,

$$\text{if } \frac{a}{b}=\frac{c}{d}, \text{ then } ad = bc.$$

(2) *If the product of two numbers equals the product of two other numbers, the four numbers are in proportion.* That is,

$$\text{if } ad = bc, \text{ then } \frac{a}{b}=\frac{c}{d}.$$

(3) *If four numbers are in proportion, they are in proportion by inversion.* That is,

$$\text{if } \frac{a}{b}=\frac{c}{d}, \text{ then } \frac{b}{a}=\frac{d}{c}.$$

(4) *In any proportion, the means may be interchanged, or the extremes interchanged, without destroying the proportion.* That is,

$$\text{if } \frac{a}{b}=\frac{c}{d}, \text{ then } \frac{a}{c}=\frac{b}{d} \text{ and } \frac{d}{b}=\frac{c}{a}.$$

(5) *The terms of any proportion are in proportion by addition.* That is,

$$\text{if } \frac{a}{b}=\frac{c}{d}, \text{ then } \frac{a+b}{b}=\frac{c+d}{d}.$$

(6) *The terms of any proportion are in proportion by subtraction.* That is,

$$\text{if } \frac{a}{b}=\frac{c}{d}, \text{ then } \frac{a-b}{b}=\frac{c-d}{d}.$$

(7) *Like powers of the terms of a proportion are in proportion.* That is,

$$\text{if } \frac{a}{b}=\frac{c}{d}, \text{ then } \frac{a^2}{b^2}=\frac{c^2}{d^2}, \frac{a^3}{b^3}=\frac{c^3}{d^3}, \text{ etc.}$$

(8) *Like principal roots of the terms of a proportion are in proportion.* That is,

$$\text{if } \frac{a}{b}=\frac{c}{d}, \text{ then } \frac{\sqrt{a}}{\sqrt{b}}=\frac{\sqrt{c}}{\sqrt{d}}, \ \frac{\sqrt[3]{a}}{\sqrt[3]{b}}=\frac{\sqrt[3]{c}}{\sqrt[3]{d}}, \text{ etc.}$$

(9) *If two or more ratios are equal, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent* That is,

$$\text{if } \frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\text{etc., then } \frac{a+c+e+\text{etc.}}{b+d+f+\text{etc.}}=\frac{a}{b}=\text{etc.}$$

Suggestion. — Let each of the given ratios equal r. Thus, $\frac{a}{b}=r$, etc. Clear the equations obtained of fractions, add, factor, etc.

By use of the above principles new proportions may be derived from any given proportion.

Example. — If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}=\frac{\sqrt[3]{a^3-b^3}}{\sqrt[3]{c^3-d^3}}$.

By (7), $$\frac{a^2}{b^2}=\frac{c^2}{d^2}.$$

By (5), $$\frac{a^2+b^2}{b^2}=\frac{c^2+d^2}{d^2}.$$

By (4), $$\frac{a^2+b^2}{c^2+d^2}=\frac{b^2}{d^2}.$$

By (8), $$\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}=\frac{b}{d}.$$

Similarly, by applying in succession (7), (6), (4), and (8) to $\frac{a}{b}=\frac{c}{d}$,

$$\frac{\sqrt[3]{a^3-b^3}}{\sqrt[3]{c^3-d^3}}=\frac{b}{d}.$$

Hence, $$\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}=\frac{\sqrt[3]{a^3-b^3}}{\sqrt[3]{c^3-d^3}}, \text{ since each equals } \frac{b}{d}.$$

EXERCISES

If $\frac{a}{b}=\frac{c}{d}$, show that:

1. $\frac{a}{a+b}=\frac{c}{c+d}$.

2. $\frac{a^2-b^2}{a^2}=\frac{c^2-d^2}{c^2}$.

3. $\frac{a+b}{c+d}=\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$.

4. $\frac{a-b}{c-d}=\frac{\sqrt[3]{a^3-b^3}}{\sqrt[3]{c^3-d^3}}$.

5. $\frac{a-2b}{b}=\frac{c-2d}{d}$.

6. $\frac{2a^2+3b^2}{b^2}=\frac{2c^2+3d^2}{d^2}$.

7. $\frac{na-b}{nc-d}=\frac{a}{c}$.

8. $\frac{a+kb}{a-kb}=\frac{c+kd}{c-kd}$.

9. $\frac{a^2+b^2}{2ab}=\frac{c^2+d^2}{2cd}$.

10. $\frac{a^4-4b^4}{b^4}=\frac{c^4-4d^4}{d^4}$.

11. $\frac{a^3+b^3}{3a^2b+3ab^2}=\frac{c^3+d^3}{3c^2d+3cd^2}$.

12. $\frac{a}{b}=\frac{\sqrt{ac}}{\sqrt{bd}}$.

13. $\frac{a^2c+ac^2}{b^2d+bd^2}=\frac{(a+c)^3}{(b+d)^3}$.

14. $\frac{a}{a+c}=\frac{a+b}{a+b+c+d}$.

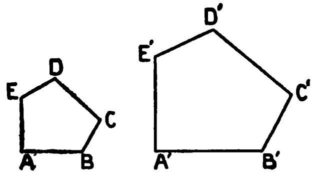
15. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, show that $\frac{a^2+c^2+e^2}{b^2+d^2+f^2}=\frac{c^2}{d^2}$.

16. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, show that $\frac{a-c+e}{b-d+f}=\frac{a}{b}$

17. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, show that $\frac{a^2+2c^2+3e^2}{b^2+2d^2+3f^2}=\frac{a^2}{b^2}$.

18. In the similar polygons shown in the figure it is known that

$$\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}=\text{etc.}$$



Show that the perimeters are to each other as any two corresponding sides; that is,

$$\frac{AB + BC + CD + \text{etc.}}{A'B' + B'C' + C'D' + \text{etc.}} = \frac{AB}{A'B'}.$$

115. Variables. — Any quantity which is continuously changing is called a **variable.** Any quantity which does not change is called a **constant.**

Thus, the temperature at a given place, a man's age, the cost of living, etc., are variables.

Some variables are so related to each other that the values of one depend upon the values of the others.

The relation between such variables may be expressed by an equation.

Thus if an object is let fall toward the ground, the velocity at which it falls gradually increases until the object strikes the ground. If the object falls from a great height, its velocity at the end of 1 second is 32 feet per second; at the end of 2 seconds it is 64 feet per second; at the end of 3 seconds it is 96 feet per second; etc.

Evidently, the velocity that the object may have at any point of its fall depends upon the length of time that it has been falling. If v represents the velocity in feet per second when the object has been falling for t seconds, then

$$v = 32\,t.$$

NOTE. — The above law holds for an object falling in a vacuum. If an object falls through the open air, the resistance of the air tends to retard the fall.

116. Direct Variation. — If two variables so change that their *ratio is constant*, one is said to **vary directly as the other.** As one increases the other also increases, and as one decreases the other also decreases.

If x and y are the variables, then $\frac{x}{y} = k$, where k is some constant.

The fact that x varies directly as y is sometimes expressed also by the symbol $\propto$, thus:

$$x \propto y.$$

It is evident that when one quantity varies directly as another the ratio of any two corresponding values of the variables equals the ratio of any other two corresponding values, since each ratio equals the same constant.

Thus, if x varies directly as y, and x' and y', and x'' and y'' are any two pairs of corresponding values of x and y, then

$$\frac{x'}{y'}=\frac{x''}{y''}.$$

Hence, if one quantity varies directly as another, any two pairs of corresponding values of the quantities form a proportion.

Example. — The weight of an object on or below the surface of the earth varies directly as its distance from the center of the earth. If an object weighs 120 pounds at the surface of the earth, find its weight 1000 miles below the surface.

Let w be the required weight in pounds.

Then $$\frac{w}{120}=\frac{3000}{4000}.^*$$

Solving, $$w=90.$$

117. Inverse Variation. — One variable is said to **vary inversely** as another when, during all of their changes, their *product remains constant.* As one increases, the other must decrease.

If x and y are the variables, then $xy=k$, where k is some constant.

It is evident that the product of any two corresponding values of the variables equals the product of any other two corresponding values, since each product equals the same constant.

Thus, if x varies inversely as y, and x' and y' and x'' and y'' are any two pairs of corresponding values, then

$$x'y'=x''y''.$$

This principle may be applied as in the following example.

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[*] 4000 miles is taken to be the radius of the earth.

Example. — It is a law of gases that the volume of a given quantity of a gas varies inversely as the pressure upon it. Gas in a cylinder occupies 60 cubic inches when the pressure upon it is 8 pounds. What will be the volume if the pressure is increased to 42 pounds?

Let V be the required volume, in cubic inches.

Then $$42V = 8 \times 60.$$

Solving, $$V = 11\tfrac{3}{7}.$$

118 Other Forms of Variation. — One variable is said to vary **jointly** as two or more other variables when it varies as their product.

If x varies jointly as y and z, then $\frac{x}{yz} = k$, or $x = kyz$, where k is some constant.

Thus, the area of a triangle varies jointly as its altitude and base. If A = area, h = altitude, and b = base, then $A = \frac{1}{2}hb$.

One variable is said to **vary directly as a second and inversely as a third** when it varies directly as the quotient of the second divided by the third.

One variable may also vary directly or inversely as the square, cube, square root, etc., of another. In general, one variable may vary as any algebraic expression, involving one or more other variables.

Thus, $V = \pi R^2 H$ is the formula for computing the volume of a cylinder. It shows that the volume varies jointly as the altitude and the square of the radius.

Two quantities x and y may so vary that $y = 2x^2 - 3x + 8$, or $y = x^3 - 4$, etc. For any value that x takes, y assumes a corresponding value.

EXERCISES

1. Write $v \propto t$ as an equation; also express this variation of v and t by a proportion between two sets of corresponding values of v and t.

2. In a bicycle pump the volume v of air confined varies inversely as the pressure p upon the piston. Express the relation between v and p in three ways.

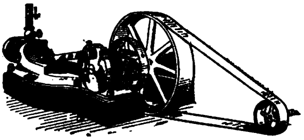
3. If n varies directly as d, and $n = 10$ when $d = 2$, find n when $d = 9$.

4. If R varies inversely as t, and $R = \frac{1}{2}$ when $t = 8$, find R when $t = 60$.

5. If W varies directly as F and inversely as d, and $W = 120$ when $F = 8$ and $d = 36$, find W when $F = 20$ and $d = 12$.

6. When air is pumped into an automobile tire, the pressure required varies inversely as the volume. If the pressure is 40 pounds when the volume is 140 cubic inches, what is the pressure when the volume is 100 cubic inches?

7. The number of revolutions per minute made by a pulley driven by a belt, as in the figure, varies inversely as the diameter of the pulley. If a pulley which is 24 inches in diameter and makes 60 revolutions per minute were replaced by a pulley 16 inches in diameter, how many revolutions per minute would the latter make, if the belt runs at the same speed?



8. If an engine driving a 12-inch pulley gives it 120 revolutions per minute, how many revolutions per minute would it cause a 20-inch pulley to make when running at the same speed?

9. If an object weighs 180 pounds at the surface of the earth, what would be its weight 500 miles below the surface of the earth? 2000 miles? 3000 miles? See Example, § 116.

10. The weight of an object above the surface of the earth varies inversely as the square of its distance from the center of the earth. If an object weighs 150 pounds at the surface of the earth, what would it weigh 1000 miles above the surface of the earth? 4000 miles? 10,000 miles?

11. The distance in feet that an object will fall from rest varies directly as the square of the time in seconds. A body will fall 16 feet the first second. How far will it fall in 4 seconds? In 10 seconds? In 60 seconds?

12. The pressure of the wind against a flat surface, such as a wall, varies jointly as the area of the surface and the square of the velocity of the wind in miles per hour. The pressure on one square foot when the velocity is 20 miles per hour is 2 pounds. Find the pressure on a wall 10 feet by 30 feet when the velocity is 40 miles per hour.

13. The side of a house is 48 feet long and 20 feet high. Find the pressure against it when the wind is blowing 30 miles per hour.

14. The intensity or brightness of light upon a surface varies inversely as the square of the distance of the surface from the source of the light. If the intensity of light on a surface from a lamp 4 feet away is 8 candle power, what would be the intensity if the lamp were removed to a distance of 16 feet from the surface?

15. An oil lamp and a 16 candle power electric light illuminate a surface with equal intensity where the oil lamp is 4 feet and the electric light 8 feet from the surface. Find the candle power of the lamp.

16. Three standard candles one foot from a wall and an arc light 5 feet from the wall illuminate it equally. Find the candle power of the arc light.

17. A light shines on the page of a book 4 feet away. How far from the light must the book be held to receive 4 times as much light?

18. The time required for the pendulum of a clock to make a vibration varies directly as the square root of its length. A pendulum 100 centimeters long makes a vibration in 1 second. Find the time of vibration of a pendulum 81 centimeters long. Of one 64 centimeters long. Of one 16 centimeters long.

19. Find the length of a pendulum which vibrates in 2 seconds. In 4 seconds. In 10 seconds.

20. The amount of water delivered by a pipe per minute varies directly as the square of the diameter of the pipe. If a 3-inch pipe delivers 200 gallons per minute, how much would a 4-inch pipe deliver?

21. The squares of the times of revolution of the planets about the sun vary directly as the cubes of their distances from the sun. The earth makes a revolution about the sun in approximately 365 days, and is 93,000,000 miles from the sun. Mercury is 36,000,000 miles from the sun. How long does it take Mercury to make a revolution about the sun?

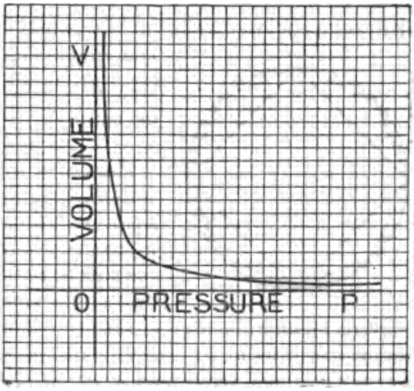
22. Jupiter is 483,300,000 miles from the sun. How long does it take Jupiter to make a revolution about the sun?

119. Variation shown Graphically. — Variation of two quantities may be depicted graphically to the eye by expressing the relation between the variables by an equation, and drawing the graph of the equation. The values of one variable corresponding to values assumed by the other may then be determined at sight from the graph.

Example. — The volume of a gas varies inversely as the pressure upon it. (See Example, § 117.) When the pressure is 5 pounds the volume is 2 cubic feet. Represent graphically the variation of the volume and pressure.

If the volume is denoted by V and the pressure by P,

$$VP = 10.$$



Since, from the nature of the quantities, only positive values of V and P may be used, the graph is found to be the branch of an hyperbola shown in the figure.

This curve shows that as P increases V decreases, that is, the curve approaches nearer and nearer to the P-axis. And as P decreases V increases.

The value of V corresponding to any value of P may be seen at a glance from the curve. Thus, when $P = 20$, $V = \frac{1}{2}$.

EXERCISES

1. Show that the fact that x varies directly as y is represented graphically by a straight line.

2. If x varies directly as y, and $x = 4$ when $y = 2$, draw the graph showing the variation of x and y.

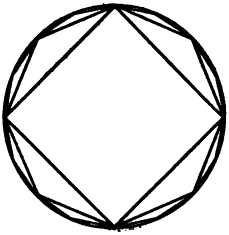
3. From the graph in Ex. 2, tell the value of x when y is 20; 40; 45; 18.

4. The distance a body falls from rest varies directly as the square of the time of falling. A body falls 16 feet the first second. Draw the graph showing the variation of the distance and time.

5. An object is thrown upward at an angle of 45° with horizontal, and with such a force that its horizontal velocity is 16 feet per second. If x represents the distance that the object moves horizontally in any time and y the distance that it moves vertically in the same time, then $y = x - \frac{1}{16}x^2$. By drawing the graph of this equation, determine the path of the object.

6. Draw the graph showing the variation of the length and the time of vibration of a pendulum. See Ex. 18, § 118.

120. The Limit of a Variable. — If the value of a variable changes according to some law, taking on a series of values that approach nearer and nearer to some constant value in such a manner that the difference between the variable and the constant *becomes and remains* less than any assigned value, however small, then the constant is called the **limit of the variable.**



Thus, if a square is drawn with its corners or vertices in a circle; then this square replaced by a polygon with eight equal sides, then this polygon replaced by another with sixteen equal sides, etc., the area of the eight-sided polygon is greater than that of the square, the area of the sixteen-sided polygon greater than that of the eight-sided polygon, etc. Evidently, as the number of sides of the polygon is thus increased, the area of the polygon grows larger and larger, and if the number of sides is indefinitely increased, the difference between the area of the polygon and that of the circle becomes and remains less than any fixed quantity, however small. Hence, as the number of sides is indefinitly increased, the *area of the polygon approaches the area of the circle as a limit.*

The sign $\doteq$ placed between a variable and a constant indicates that the variable approaches the constant as a limit.

Thus, $x \doteq a$ means that x approaches a as a limit.

EXERCISES

1. What is the limit approached by the perimeter of the polygon in § 120 as the number of sides of the polygon is increased indefinitely?

2. What is the limit of the sum of the fractions $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, etc., as the number of fractions taken is increased indefinitely?

SUGGESTION. — Find the sum of the first two fractions; then the sum of the first three; etc. See what number the sum approaches but never quite equals, as more and more fractions are taken.

3. What is the limit of 0.3333 ⋯ as the number of decimal places taken is increased indefinitely?

4. If $n \doteq 0$, what limit does $2n$ approach? n^2? $n + 6$?

5. If $t \doteq 2$, what limit does $t^2 - 4$ approach?

6. If x increases indefinitely, what limit does $\frac{1}{x}$ approach?

121. Infinity. — Since the series of numbers 1, 2, 3, 4, 5, ⋯, may be continued indefinitely, if a variable x takes in succession all of these values, there is no final value that x may reach. It *increases without limit.* It may thus become greater than any fixed or constant number, however great.

If a variable becomes and remains greater than any constant number, however great, it is said to become **infinite.** A variable that has become infinite is called **infinity.** It is represented by the symbol ∞.

Thus, time is infinite. It extends without limit into the past, and without limit into the future. Similarly space is infinite. It reaches without limit in every direction.

Care has to be used in dealing with infinity in mathematical work. The symbol ∞ may not be operated upon as ordinary numbers may be, because, since it denotes a quantity greater than any fixed number, however great, it may represent widely different values.

For example, the sum of the series of whole numbers, 1, 2, 3, 4, 5, 6, 7, $\cdots$, is infinite, since it is greater than any fixed number that may be assigned. But the sum of the odd numbers, 1, 3, 5, 7, $\cdots$, alone is infinite also. Hence each sum is represented by ∞. Yet one sum must be greater than the other, because the whole of a thing is greater than a part of it.

Again, if we add a number to infinity we get infinity. Thus, $\infty + 6 = \infty$.

122. Values of $\frac{x}{n}$ when $x = \infty$, and when $x \doteq 0$. — If x in the fraction $\frac{1}{x}$ takes in succession the values 1, 2, 3, 4, $\cdots$, without limit there is no final value that $\frac{1}{x}$ may reach, but it becomes smaller and smaller as x takes larger and larger values. As x becomes infinite, $\frac{1}{x}$ approaches zero as a limit. Similarly, in general,

If n in the fraction $\frac{n}{x}$ is any constant not zero, and x a variable which increases without limit, then $\frac{n}{x}$ approaches zero as a limit.

That is, if $x = \infty$, then $\frac{n}{x} \doteq 0$.

Again, if x in the fraction $\frac{1}{x}$ takes in succession the values $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots$, without limit, $\frac{1}{x}$ takes in succession larger and larger values and becomes infinite. And in general,

If n in the fraction $\frac{n}{x}$ is any constant not zero, and x a variable which approaches zero as a limit, then $\frac{n}{x}$ increases without limit.

That is, if $x \doteq 0$, then $\frac{n}{x} = \infty$.

123. Meanings of $\frac{n}{0}$ and $\frac{0}{0}$.—Zero may be written as the difference between two equal numbers. Thus $0 = k - k$.

Then if a is any constant,

$$\begin{aligned} a \times 0 &= a(k-k) \\ &= ak - ak \\ &= 0. \end{aligned}$$

Hence the product of any constant multiplied by zero equals zero.

Now the fraction $\frac{n}{0}$, where n is any constant number not zero, represents a number which multiplied by 0 gives n. But, by the above principle, no constant multiplied by 0 gives n. Hence,

The symbol $\frac{n}{0}$ has no meaning.

NOTE.—One sometimes sees the expression $\frac{n}{0} = \infty$. But in this expression the symbol 0 is not to be regarded as absolute zero. The expression $\frac{n}{0} = \infty$ is a loose way sometimes used of saying that $\frac{n}{x} = \infty$ when $x \doteq 0$.

The symbol $\frac{0}{0}$ represents a number which multiplied by 0 gives 0. But, by the above principle, any constant a multiplied by 0 gives 0. Hence,

The symbol $\frac{0}{0}$ represents any constant number whatever.

124. Division by Zero. Since $\frac{0}{0}$ may equal any constant whatever, and $\frac{n}{0}$ has no meaning, evidently,

It is never allowable to divide by zero.

If it were allowable to divide by zero, one could prove most any absurd relation between numbers, as the following example illustrates. Point out the fallacy in the reasoning.

Let $a = b.$

Multiplying by a, $a^2 = ab$

Subtracting b^2, $a^2 - b^2 = ab - b^2.$

Factoring, $(a + b)(a - b) = b(a - b).$

Dividing by $a - b$, $a + b = b.$

Substituting a for b, $2a = a.$

Dividing by a, $2 = 1.$

125. Indeterminate Fractions. — For certain values of the literal number involved, a fraction will sometimes take the **indeterminate form** $\frac{0}{0}$.

For example, when $a = 2$, $\frac{a^2 - 4}{a - 2} = \frac{0}{0}$. This fraction, however, is equal to $\frac{(a + 2)(a - 2)}{a - 2}$. Now, as long as a is not equal to 2, we may divide both terms by $a - 2$. Hence, when a is not equal to 2, $\frac{(a + 2)(a - 2)}{a - 2} = a + 2$. This is true, however little a may differ from 2. Hence, if without giving a the value 2, we make it approach 2 as a limit, we may divide both terms of the fraction by $a - 2$. Now, as a approaches 2, evidently $a + 2$ approaches 4. Hence, we say $\frac{a^2 - 4}{a - 2} \doteq 4$ as $a \doteq 2$. Hence, 4 is called the *value* of $\frac{a^2 - 4}{a - 2}$ when $a = 2$.

The *value* of such a fraction, for any particular value of the literal number involved, is defined as the *limit* which the fraction approaches when the literal number approaches this particular value as its limit.

EXAMPLE. — Find the value of $\frac{x^3 - 27}{x - 3}$ when $x = 3$.

When $x = 3$, $\frac{x^3 - 27}{x - 3} = \frac{0}{0}$, the indeterminate form.

For any value of x except 3, $\frac{x^3 - 27}{x - 3} = x^2 + 3x + 9$.

Now, as $x \doteq 3$, $x^2 + 3x + 9 \doteq 27$.

Hence, $\frac{x^3 - 27}{x - 3} = 27$ when $x = 3$.

EXERCISES

Find the value of:

1. $\frac{1 - t}{1 - t^2}$ when $t = 1$.

2. $\frac{n - 2}{n^3 - 8}$ when $n = 2$.

3. $\frac{x^2 - 4}{x^2 - 5x + 6}$ when $x = 2$.

4. $\frac{a^2 + 5a}{a^2 + 3a - 10}$ when $a = -5$

Solve by determinants and interpret solutions:[*]

5. $\begin{cases} x - y = 2, \\ 3x = 3y + 16. \end{cases}$

6. $\begin{cases} 2m + y = 8, \\ 2m = 1 - y. \end{cases}$

7. $\begin{cases} 9A - 2B = 2, \\ 3A = 5 + \frac{2}{3}B. \end{cases}$

8. $\begin{cases} m + n = 6, \\ 2m = 12 - 2n. \end{cases}$

9. $\begin{cases} 5v - t = 3, \\ 2t = 10v - 6. \end{cases}$

10. $\begin{cases} W + 2V = 20. \\ 3W + 6V = 60 \end{cases}$

SUPPLEMENTARY EXERCISES

1. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{5a - 2b}{5c - 2d} = \frac{\sqrt[3]{a^3 - 4b^3}}{\sqrt[3]{c^3 - 4d^3}}$.

2. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a^2 - ab + b^2}{a^2 + ab + b^2} = \frac{c^2 - cd + d^2}{c^2 + cd + d^2}$.

3. The surface of a sphere is $4\pi R^2$, where R is the radius. If S denotes the surface of a sphere and D its diameter, show that

$$\frac{S_1}{S_2} = \frac{R_1^2}{R_2^2} = \frac{D_1^2}{D_2^2}.$$

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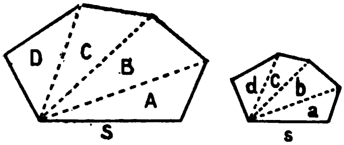
[*] See § 59. If § 59 was omitted, omit these exercises.



4. Two similar polygons whose corresponding sides are S and s, are divided into similar triangles whose areas are A and a, B and b, C and c, D and d, respectively. It is known that

$$\frac{A}{a}=\frac{S^2}{s^2},\ \frac{B}{b}=\frac{S^2}{s^2},\ \frac{C}{c}=\frac{S^2}{s^2} \text{ and } \frac{D}{d}=\frac{S^2}{s^2}.$$

Show that
$$\frac{A+B+C+D}{a+b+c+d}=\frac{S^2}{s^2}.$$

5. Find the mean proportional between $(m^3-n^3)(m^2+mn+n^2)$ and $m-n$.

6. Find the fourth proportional to 1, $\sqrt{2}$, and $\sqrt{6}$.

Representing the constants involved by k, write the equations which express the following physical laws:

7. The force with which a moving body having a given velocity strikes a stationary body varies directly as the mass or weight of the moving body.

8. The force of attraction between two bodies, such as the earth and the sun, varies directly as the product of their masses, and inversely as the square of the distance between them.

9. If an object is whirled in a circle at the end of a string, the tension or stretching force exerted upon the string by the object varies directly as the product of the weight of the object and the square of its velocity, and inversely as the length of the string.

10. The amount of heat received by a body by radiation from a stove varies inversely as the square of the distance of the body from the stove.

11. The resistance offered by a wire to a current of electricity varies directly as the length of the wire and inversely as the area of its cross section.

Find the limit of each of the following when $n = \infty$:

12. $\frac{1}{n}$. 13. $\frac{5}{n}$. 14. $\frac{1}{n+2}$. 15. $\frac{n}{100}$.

Find the limit of each of the following when $n \doteq 0$:

16. $\frac{1}{n}$. 17. $\frac{n^2}{n^2}$. 18. $\frac{n}{n+1}$. 19. $\frac{n}{n^2+4n}$.

Note. — It is easily seen that the symbol $\frac{\infty}{\infty}$ is indeterminate, since the symbol ∞ does not denote a definite value. A fraction which takes this form for some special value of the literal number involved may sometimes be changed to an equivalent fraction which is not of the indeterminate form, and then evaluated, as shown in the following.

Find the limit of each of the following when $n = \infty$:

20. $\frac{n+1}{n+8}$.

Solution. — Dividing both terms by n, $\frac{n+1}{n+8} = \frac{1+\frac{1}{n}}{1+\frac{8}{n}}$.

But when $n = \infty$, $\frac{1}{n} \doteq 0$ and $\frac{8}{n} \doteq 0$.

Hence, when $n = \infty$, $\frac{1+\frac{1}{n}}{1+\frac{8}{n}} \doteq \frac{1}{1}$ or 1.

21. $\frac{n}{n+1}$. 23. $\frac{n+1}{2n}$. 25. $\frac{n^2+4}{n^2+n}$.

22. $\frac{2n}{n-2}$. 24. $\frac{n+2}{3n-2}$. 26. $\frac{2n^2+5n}{4n^2-1}$.

27. $\frac{n(n+1)(n+2)}{n^3}$. 28. $\frac{6n^4-n^2}{(n^2-1)(n^2+3)}$.

CHAPTER XII

PROGRESSIONS

126 Series. — Often, in mathematical work, groups of three or more numbers are encountered that are arranged in a definite order or succession, and that are such that each number of the group may be obtained from one or more of the preceding numbers by some fixed law. Such a group or succession of numbers is called a **series**. The numbers are called the **terms** of the series, and are named from the left to the right, first term, second term, etc.

For example, the odd numbers, 1, 3, 5, 7, ⋯, form a series in which each term after the first is obtained by adding 2 to the preceding term.

Also, 2, 5, 11, 23, ⋯, form a series in which each term after the first is obtained by multiplying the preceding term by 2 and then adding 1 to the product.

And $1, -x, x^2, -x^3, \cdots$, form a series in which each term after the first is obtained by multiplying the preceding term by $-x$.

The law by which the terms of a series are obtained usually may be determined easily by examining the first few terms.

EXERCISES

Give the next 4 terms of each of the following:

1. $1, 2, 3, 4, \cdots$.
2. $2, 4, 6, 8, \cdots$.
3. $20, 17, 14, \cdots$.
4. $1, 2, 4, 8, \cdots$.
5. $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \cdots$.
6. $1, 4, 9, 16, \cdots$.
7. $3, -2, -7, \cdots$.
8. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots$.
9. $1, 2, 5, 14, \cdots$.
10. $1, 2n, 4n^2, 8n^3, \cdots$.
11. $\sqrt{2}, \sqrt{5}, \sqrt{8}, \sqrt{11}, \cdots$.
12. $256t^{16}, -128at^{14}, 64a^2t^{12}, \cdots$.

127. Progressions. — Of the many kinds of series, we shall discuss in this chapter only two of the simplest. They are called **progressions.**

128. Arithmetical Progressions. — An **arithmetical progression** is a series in which the difference between any term and the preceding term is the same for all terms.

This difference is called the **common difference.**

The name arithmetical progression is abbreviated usually to A. P.

Thus, the series 1, 4, 7, 10, ⋯, is an A. P. in which the common difference is 3.

The series 10, 5, 0, − 5, − 10, ⋯, is an A. P. in which the common difference is − 5.

129. The *n*th Term of an Arithmetical Progression. — If the first term of any arithmetical progression is denoted by a, and the common difference by d, the terms of the progression will be represented by

$$a,\ a+d,\ a+2d,\ a+3d,\ a+4d,\ a+5d, \text{ etc.}$$

It is evident from these expressions that the coefficient of d in any term is 1 less than the number of the term. Thus, the twelfth term is $a+11d$. Hence, the general or nth term is $a+(n-1)d$. Therefore, if l represents the value of the nth term in an arithmetical progression whose first term is a and common difference d,

$$l = a + (n-1)d. \qquad (A)$$

This formula will enable one to find the value of any one of the four numbers, l, a, n, d, if the values of the other three are known.

Example 1. — Find the twentieth term of the series 4, 7, 10, 13, ⋯.

Here $a = 4$, $d = 3$, and $n = 20$.

Substituting in formula A,

$$l = 4 + (20-1)3 = 61.$$

Example 2. — In the series 5, 9, 13, 17, ⋯, what term of the series is 65?

Here, $a = 5$, $d = 4$, $l = 65$, n is unknown.

Substituting in formula A,

$$65 = 5 + (n - 1)4.$$

Solving, $$n = 16.$$

Hence, 65 is the sixteenth term of the series.

Example 3. — The fourth and fifteenth terms of an A. P. are 9 and 31, respectively. Write the first six terms.

The first term and common difference must first be found. Since the fourth term is $a + 3d$, and the fifteenth term is $a + 14d$, we get the system

$$\begin{cases} a + 3d = 9, \\ a + 14d = 31. \end{cases}$$

Solving this system, $a = 3$, and $d = 2$.

Hence, the series is 3, 5, 7, 9, 11, 13, ⋯.

EXERCISES

1. Find the thirtieth term of the series 1, 6, 11, 16, ⋯.

2. Find the sixteenth term of the series 1, 5, 9, 13, ⋯.

3. Find the twenty-fourth term of the series -8, -5, -2, 1, ⋯.

4. Find the twentieth term of the series 12, 6, 0, -6, ⋯.

5. Find the eighteenth term of the series $-\frac{3}{2}$, $-\frac{1}{2}$, $\frac{1}{2}$, $\frac{3}{2}$, ⋯.

6. Find the forty-second term of the series 11, 17, 23, ⋯.

7. Which term of the series 1, 4, 7, 10, ⋯, is 46?

8. Which term of the series 2, 7, 12, 17, ⋯, is 152?

9. Which term of the series 12, 8, 4, 0, ⋯, is -132?

10. In an A. P. whose common difference is 3, the sixteenth term is 48. Find the first term.

11. In an A. P. whose common difference is -2, the twenty-eighth term is -49. Find the first term.

12. In an A. P. whose first term is 7, the twenty-second term is 175. Find the common difference.

13. In an A. P. whose first term is 12, the thirtieth term is 99. Find the sixteenth term.

14. The sixth term of an A. P. is 17, and the fifteenth term is 44. Find the first term and the common difference.

15. The third term of an A. P. is 0, and the ninth term is 22. Find the first term and the common difference.

16. The fifth term of an A. P. is 21, and the eighth term is 33. Find the twelfth term. The twentieth term.

17. The second term of an A. P. is 7, and the eleventh term is $20\frac{1}{2}$. Find the seventh term. The fifteenth term.

18. Beginning with 2, find the 100th even number.

19. Beginning with 1, find the 100th odd number.

20. Beginning with 5×4, or 20, find the 120th multiple of 4.

21. Find the twenty-fifth term of the series $n, 4n, 7n, \cdots$.

22. Find the sixteenth term of the series $1, 1 + 3x, 1 + 6x, \cdots$.

23. Find the $(n - 1)$st term of the series $a, a + d, a + 2d, \cdots$.

24. Find the $(n - 3)$d term of the series $a, a + d, a + 2d, \cdots$.

25. A body falls 16 feet the first second, 48 feet the second, 80 feet the third, etc. How far does it fall during the 20th second? During the 60th second?

26. A body falls $\frac{1}{2}a$ feet the first second, $\frac{3}{2}a$ feet the next second, $\frac{5}{2}a$ feet the next second, etc. How far does it fall during the tth second?

27. A projectile is thrown vertically upward with an initial velocity of 448 feet per second. At the end of 1 second its velocity is 416 feet per second, at the end of 2 seconds its velocity is 384 feet per second, etc. In how many seconds will it stop rising and begin to descend?

130. Arithmetical Means. — In an arithmetical progression all of the terms between any two terms are called the **arithmetical means** of those two terms.

For example, since 2, 4, 6, 8, and 10 form an A. P., 4, 6, and 8 are arithmetical means of 2 and 10.

Any number of arithmetical means may be inserted between two given numbers.

EXAMPLE. — Insert seven arithmetical means, between 10 and 30.

Since there are to be seven arithmetical means, 30 must be the ninth term of an A. P. of which 10 is the first term.

Hence, we have $a = 10$, $l = 30$, $n = 9$.

Substituting in formula A,

$$30 = 10 + 8d.$$

Solving, $$d = 2\tfrac{1}{2}.$$

Hence, the means are $12\frac{1}{2}$, 15, $17\frac{1}{2}$, 20, $22\frac{1}{2}$, 25, $27\frac{1}{2}$.

A very important special case is that of one **arithmetical mean** between two numbers. It is called also the **average** of the two numbers.

If A is the arithmetical mean between a and b, then by definition of an A. P.,

$$A - a = b - A.$$

Solving for A, $$A = \frac{a+b}{2}.$$

Hence the arithmetical mean between two numbers equals one half of their sum.

EXERCISES

1. Insert four arithmetical means between 5 and 25.
2. Insert three arithmetical means between 12 and -2.
3. Insert five arithmetical means between 3 and 27.
4. Insert six arithmetical means between -6 and 29.
5. Insert six arithmetical means between $\frac{1}{2}$ and -10.
6. Insert twelve arithmetical means between 1 and 5.
7. Insert twenty arithmetical means between 3 and 48.
8. Insert two arithmetical means between $\frac{1}{2}$ and $\frac{1}{3}$.
9. Insert five arithmetical means between n and $\frac{1}{6}n$

10. Find the arithmetical mean of 5 and 19. Of 8 and 72. Of 16 and 34. Of $\frac{1}{2}$ and $3\frac{1}{2}$. Of -6 and 2. Of $\frac{1}{8}$ and $1\frac{1}{2}$. Of $-\frac{2}{3}$ and $\frac{5}{6}$. Of $2\sqrt{3}$ and $\sqrt{3}$. Of 1 and $2x-1$.

11. Find the arithmetical mean of a^2+ab+b^2 and a^2-ab+b^2.

12. The velocity of a falling body increases uniformly throughout its descent. Its velocity at the beginning of the second second is 32 feet per second, and at the end of the second second 64 feet per second. Find the average or mean velocity during the second second.

13. The velocities of a falling body at the beginning and end of the fifth second are 160 feet per second and 192 feet per second, respectively. Find the average velocity during the fifth second.

14. In going a distance of 20 miles an aviator increased his speed uniformly from 24 miles per hour to 40 miles per hour. What was his average speed for the trip? How long did it take to make the trip?

15. A certain distance is divided into 6 parts, which are in arithmetical progression. The shortest part is 2 inches and the longest 56 inches. Find the lengths of the other parts.

16. A set of 8 weights are in arithmetical progression. The smallest is 1 ounce and the largest 15 ounces. Find the other weights.

17. If the temperature at 6 A.M. is $-4°$, and rises uniformly 2° per hour until 2 P.M., what is the average temperature for the entire period?

131. Sum of n Terms of an Arithmetical Progression. — The sum of n terms of an arithmetical progression may be obtained as follows:

If the sum be represented by S, then

$$S=a+(a+d)+(a+2d)+(a+3d)+\cdots+(l-d)+l$$

If the terms are written in the reverse order,

$$S = l + (l - d) + (l - 2d) + (l - 3d) + \cdots + (a + d) + a.$$

Adding the corresponding terms of these two equations, the d's are eliminated, and we get

$$\begin{aligned} 2S &= (a + l) + (a + l) + (a + l) + \cdots \text{ to } n \text{ terms} \\ &= n(a + l). \end{aligned}$$

Hence,

$$\boldsymbol{S = \tfrac{1}{2} n(a + l).} \qquad (B)$$

Substituting in this formula the value of l given in formula A, § 129, we get another formula:

$$\boldsymbol{S = \tfrac{1}{2} n\{2a + (n - 1)d\}.} \qquad (C)$$

If the value of d is not known and the values of n, a, and l are known, S may be found by formula B. If the value of l is not known, and the values of n, a, and d are known, S may be found by formula C.

If any three of the five numbers a, d, n, l, S, of an A. P. are known, the values of the other two may be found by use of formulas A, B, and C.

EXAMPLE 1. — Find the sum of 25 terms of the series -9, -5, -1, 3, 7, $\cdots$.

Here $a = -9$, $d = 4$, $n = 25$. Hence, substituting in formula C,

$$S = \tfrac{1}{2} \times 25\{2(-9) + (25 - 1)4\} = 975.$$

EXAMPLE 2. — The first term of an A. P. is 2, and the sum of 20 terms is 135. Find the common difference.

Here $a = 2$, $n = 20$, $S = 135$. Hence, from formula C,

$$135 = 10(4 + 19d).$$

Solving,

$$d = \tfrac{1}{2}.$$

EXAMPLE 3. — How many terms of the series 3, 1, -1, -3, -5, $\cdots$, must be taken to make -140?

Here $a = 3$, $d = -2$, $S = -140$. Hence, from formula C,

$$-140 = \tfrac{1}{2} n\{6 + (n - 1)(-2)\}.$$

Simplifying, $n^2 - 4n - 140 = 0.$

Solving, $n = 14 \text{ or } -10.$

The root -10 has no meaning, and hence must be discarded. Therefore 14 terms must be taken.

EXAMPLE 4. — The common difference in an A. P. is $1\frac{1}{2}$, and the twelfth term is $12\frac{1}{2}$. Find the sum of the first 7 terms.

From formula A, $a = -4.$

Hence, from formula C, $S = 3\frac{1}{2}.$

EXERCISES

1. Find the sum of 25 terms of the series 3, 8, 13, ⋯.
2. Find the sum of 16 terms of the series 8, 12, 16, ⋯.
3. Find the sum of 22 terms of the series 42, 39, 36, ⋯.
4. Find the sum of 30 terms of the series 12, 9, 6, ⋯.
5. Find the sum of 40 terms of the series $\frac{1}{2}$, $\frac{5}{8}$, $\frac{3}{4}$, ⋯.
6. Find the sum of 36 terms of the series $12\frac{1}{2}$, 8, $3\frac{1}{2}$,
7. Find the sum of the first 100 whole numbers.
8. Find the sum of the first 100 even numbers.
9. Find the sum of the first 100 odd numbers.
10. Find the sum of all odd numbers of two digits.
11. Find the sum of all even numbers between 100 and 200.
12. Find the sum of all numbers between 50 and 250 that are divisible by 4.
13. Show that the sum of the first n odd numbers is n^2.
14. Find the sum of the first n even numbers.
15. How many of the series of whole numbers, beginning with 1, must be added to give 3240?
16. How many terms of the series 15, 12, 9, ⋯, must be added to give 45?
17. How many terms of the series $6\frac{1}{2}$, $4\frac{1}{2}$, $2\frac{1}{2}$, ⋯, must be added to give -396?

18. How many terms of the series $a, 3a, 5a, \cdots$, must be added to give $400a$?

19. If $a = 4$, $d = 3$, $l = 40$, find n and S.

20. If $a = 2$, $d = 5$, $S = 1428$, find n and l.

21. If $d = -4$, $n = 18$, $S = 144$, find a and l.

22. The first term of an A.P. is -8, and the sum of 10 terms is 100. Write the first 5 terms of the series.

23. If a common clock strikes only the hours, how many strokes does it make in a day?

24. If the hours of the day are numbered from 1 to 24, as in Italy, how many strokes would a clock that strikes only the hours make in a day?

25. In a potato race 50 potatoes are placed in a straight line, 1 yard apart, the first one being 1 yard from the basket. How far must a contestant run to bring them to the basket one at a time?

26. If an elastic ball falls from a height of 30 inches, and at each rebound rises to a point 6 inches below the point to which it rose the time before, find the entire distance that it travels before coming to rest.

27. A car starting down an incline moves 1 foot the first second, 4 feet the second second, 7 feet the third second, etc. How far will it move in 20 seconds? How long will it take the car to run 1335 feet, the entire length of the incline?

28. An object falling freely through space falls approximately 16 feet the first second, 48 feet the next, 80 feet the next, etc. How far will it fall in 6 seconds? In 10 seconds? In 60 seconds?

29. A shot is fired vertically upward, going 1024 feet the first second, and lessening its speed 32 feet per second thereafter. How high does the shot go?

30. If an object falls 16 feet the first second, 48 feet the next, 80 feet the next, etc., and an aviator in time of war drops a bomb from a height of 1600 feet, how long does it take it to reach the ground?

31. A man invests his savings in the shares of a building and loan association, depositing with the association \$1000 the first year. At the beginning of the second year he is credited with \$60 interest on the amount deposited the first year, and pays in only \$940, making his total credit \$2000. At the beginning of the third year he is credited with \$120 interest, and pays in only \$880 cash, etc., What is his credit at the end of 10 years, and how much cash has he paid in then?

32. If a man invests \$1000 of his savings at the end of each year for 10 years at 5% simple interest, how much will he have at the end of 10 years?

33. Cannon balls are piled up into a pyramid with a square base having 12 balls on each side. How many balls are there in the bottom layer? In the second layer? In the third layer? In the whole pyramid?

34. A roll of paper is wrapped about a core whose diameter is 1 inch. The thickness of the paper is $\frac{1}{64}$ inch. The outer diameter of the roll is 10 inches. How long is the strip of paper in the roll?

35. Find three numbers in A. P., such that their sum is 18 and the product of the first and third 14 greater than the second.

36. The sum of three numbers in A. P. is 15, and the sum of their squares is 107. Find the numbers.

132. Geometrical Progressions. — A *geometrical progression* is a series in which the ratio of any term to the preceding term is the same for all terms of the series.

This ratio is called the **common ratio.**

The name geometrical progression is abbreviated usually to G. P.

Thus, the series 1, 2, 4, 8, ⋯, is a G. P., in which the common ratio is 2.

The series $2, -\frac{1}{2}n, \frac{1}{8}n^2, -\frac{1}{32}n^3, \cdots$, is a G. P. in which the common ratio is $-\frac{1}{4}n$.

In either series, if any term is multiplied by the common ratio, the product will be the next term.

133. The nth term of a Geometrical Progression. — If the first term of any geometrical progression is denoted by a, and the common ratio by r, the terms of the progression will be represented by

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5, \text{ etc.}$$

It is seen from these terms that the power of r in any term is 1 less than the number of the term. Thus, the tenth term is ar^9. Hence, the general or nth term is ar^{n-1}. Therefore, if l denotes the value of the nth term in the geometrical progression whose first term is a and common ratio r,

$$\mathbf{l = ar^{n-1}}. \tag{A}$$

This formula will enable one to find the value of any one of the four numbers l, a, r, n, if the values of the other three are known.

Example 1. — **Find the ninth term of the series 2, 6, 18, ⋯.**

Here $a = 2$, $r = 3$, $n = 9$.

Substituting in formula A,

$$l = 2 \times 3^8 = 13{,}122.$$

Example 2. — **The tenth term of a G. P. is $\frac{1}{512}$, and the first term is 1. Find the common ratio.**

Here $a = 1$, $n = 10$, $l = \frac{1}{512}$.

Substituting in formula A,

$$\tfrac{1}{512} = 1 \times r^9.$$

Hence, $\quad r = \frac{1}{2}$.

Example 3. — **The fifth and eighth terms of a G.P. are $\frac{16}{27}$ and $-\frac{128}{729}$, respectively. Write the first 8 terms.**

Since the fifth term is ar^4, and the eighth term is ar^7,

$$\begin{cases} ar^4 = \frac{16}{27}, & (1)\\ ar^7 = -\frac{128}{729}. & (2)\end{cases}$$

Dividing (2) by (1), $\quad r^3 = -\frac{8}{27}$.

Hence, $\quad r = -\frac{2}{3}$.

Substituting $-\frac{2}{3}$ for r in (1), $\quad \frac{16}{81}a = \frac{16}{27}$.

Hence, $\quad a = 3$.

Therefore the series is $3, -2, \frac{4}{3}, -\frac{8}{9}, \frac{16}{27}, -\frac{32}{81}, \frac{64}{243}, -\frac{128}{729}, \cdots$,

EXERCISES

1. Find the eighth term of the series 1, 2, 4, $\cdots$.

2. Find the twelfth term of the series $\frac{1}{1000}$, $\frac{1}{100}$, $\frac{1}{10}$, $\cdots$

3. Find the tenth term of the series 2430, 810, 270, $\cdots$.

4. Find the eighth term of the series $\frac{2}{3}$, 2, 6, $\cdots$.

5. Find the ninth term of the series 4, -8, 16, $\cdots$.

6. Find the sixth term of the series 54, -18, 6, $\cdots$.

7. Find the tenth term of the series $\frac{1}{16}$, $\frac{1}{4}$, 1, $\cdots$.

8. The first term of a G. P. is 4, and the sixth term is 128. Find the common ratio.

9. The common ratio of a G. P. is 3, and the fifth term is 486. Find the first term.

10. The second term of a G. P. is $\frac{1}{4}$, and the eighth term is $\frac{1}{256}$. Find the tenth term.

11. The first term of a G. P. is 3, and the third term is 6. Find the sixth term.

12. The second term of a G. P. is 3, and the fifth term is $\frac{81}{8}$. Find the fourth term.

13. The eighth term of a G. P. is 1152, and the tenth term is 4608. Find the first five terms.

14. The third term of a G. P. is $\frac{1}{8}$, and the eighth term is 128. Find the first six terms.

15. Find the sixth term of the series $2\sqrt{2}$, 4, $4\sqrt{2}$, $\cdots$.

16. Find the fifth term of the series $\frac{2}{3\sqrt{2}}$, 1, $\frac{3\sqrt{2}}{2}$, $\cdots$.

17. The first and second terms of a G. P. are m and n. Find the next two terms.

18. In making an inventory at the close of each year, a manufacturer deducted 10% from the value of his machinery at the previous inventory, because of deterioration. The machinery cost $20,000. What was the value at the end of the fifth year?

SUGGESTION. — The value at the end of the first year was $.90 \times \$20,000$, at the end of the second year, $.90 \times .90 \times \$20,000$, or $.90^2 \times \$20,000$; etc. What was the common ratio?

19. A boy puts $100 in a savings bank, which pays 3 % compound interest, compounded annually. What does it amount to at the end of six years?

Suggestion. — The value at the end of the first year is $1.03 \times \$100$; at the end of the second year, $1.03^2 \times \$100$; etc.

20. To how much will $500 amount at the end of five years, if put in a savings bank which pays 3 % compound interest, compounded annually?

21. The population of a city is 100,000, and it increases 10 % every year. What will the population be in 10 years?

22. If a rubber ball rebounds $\frac{1}{4}$ of the distance it falls each time, and falls the first time from a height of 60 feet, how high will it rise after the eighth fall?

134. Geometrical Means. — In a geometrical progression all of the terms between any two terms are called the **geometrical means** of those two terms.

For example, since 5, 10, 20, 40, 80, and 160 form a G. P., 10, 20, 40, and 80 are geometrical means of 5 and 160.

Any number of geometrical means may be inserted between two given numbers.

Example. — Insert two geometrical means between 8 and 27.

Since there are to be two geometrical means, 27 must be the fourth term of a G. P. of which 8 is the first term.

Hence, we have $a = 8$, $l = 27$, and $n = 4$.

Substituting in formula A,

$$27 = 8r^3.$$

Solving, $r = 1\frac{1}{2}$.

Hence, the means are 12 and 18.

An important special case is that of one **geometrical mean** between two numbers. It is called also the **mean proportional** between the two numbers.

If G is the geometrical mean between a and b, then by definition of a G.P.,

$$\frac{b}{G}=\frac{G}{a}.$$

Solving for G, $G=\sqrt{ab}$.

Hence, the geometrical mean between two numbers equals the square root of their product.

EXERCISES

1. Insert two geometrical means between 7 and 56.

2. Insert two geometrical means between 2 and 54.

3. Insert three geometrical means between 16 and 54.

4. Insert four geometrical means between 1 and 1024.

5. Insert three geometrical means between 80 and 5.

6. Insert four geometrical means between 24 and $3\frac{13}{81}$.

7. Insert three geometrical means between -9 and -144.

8. Insert two geometrical means between 1 and 2.

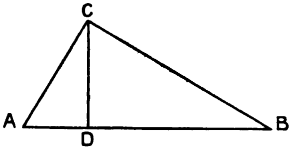
9. Insert two geometrical means between m and n.

10. Find the gometrical mean of 1 and 4. Of 9 and 16. Of 5 and 20. Of 3 and 4. Of $\sqrt{2}$ and $\sqrt{8}$.

11. Find the geometrical mean of $a-b$ and a^3-a^2b.

12. Triangle ABC is a right triangle, with CD perpendicular to the hypotenuse AB. It is shown in geometry that CD is the geometrical mean of AD and DB.

If $AD=9$ and $DB=25$, find CD.
If $AD=4$ and $DB=12$, find CD.

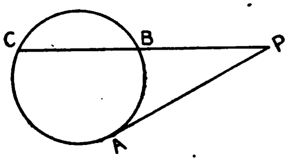


13. In the figure of Ex. 12, if $CD=8$ and $AB=20$, find AD and DB.

14. In the right triangle in Ex. 12, AC is the geometrical mean of AD and AB. If $AD=5$ and $AB=20$, find AC.

Also CB is the geometrical mean of DB and AB. If $DB=12$ and $AB=27$, find CB.

15. If PA touches a circle at A, and PB cuts the circle at B and C, PA is a geometrical mean of PB and PC.



If $PB = 16$ and $PC = 49$, find PA.

If $BC = 12$ and $PA = 16$, find PB and PC.

135. Sum of *n* Terms of a Geometrical Progression. — The sum of n terms of a geometrical progression is found as follows:

If S represents the sum of n terms of a geometrical progression whose first term is a and common ratio r, then

$$S = a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1}. \qquad (1)$$

Hence, $$rS = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n. \qquad (2)$$

Now, if (2) be subtracted from (1), all terms cancel except a of (1) and ar^n of (2). Hence, subtracting (2) from (1),

$$S - rS = a - ar^n.$$

Factoring, $$S(1 - r) = a(1 - r^n).$$

Hence, dividing by $1 - r$, we get

$$S = \frac{a(1 - r^n)}{1 - r}. \qquad (B)$$

Since $l = ar^{n-1}$, by substituting in (B) we get

$$S = \frac{a - rl}{1 - r}. \qquad (C)$$

If any three of the numbers a, r, n, l, S, of a G. P. are known, the other two may be found by use of formulas A, B, and C.

Example 1. — Find the sum of the first 6 terms of the series 4, 6, 9, ⋯.

Here $$a = 4,\ r = \tfrac{3}{2},\ n = 6.$$

Hence, $$S = \frac{4\{1 - (\frac{3}{2})^6\}}{1 - \frac{3}{2}} = 83\tfrac{1}{8}.$$

Example 2. — The first term of a G. P. is 3, the sixth term is 9375, and the sum of the first 6 terms is 11,718. Find the common ratio.

By formula C, $$11{,}718 = \frac{3 - 9375\,r}{1 - r}.$$

Solving, $$r = 5.$$

EXERCISES

1. Find the sum of 10 terms of $6, 12, 24, \cdots$.

2. Find the sum of 6 terms of $1, 3, 9, \cdots$.

3. Find the sum of 7 terms of $2, -4, 8, \cdots$.

4. Find the sum of 8 terms of $\frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \cdots$.

5. Find the sum of 6 terms of $48, 12, 3, \cdots$.

6. Find the sum of 5 terms of $5, -30, 180, \cdots$.

7. Find the sum of 10 terms of $1, x, x^2, \cdots$.

8. Find the sum of 9 terms of $\frac{2}{3}, 1, \frac{3}{2}, \cdots$.

9. Find the sum of 6 terms of $6n, -12n^2, 24n^3, \cdots$.

10. Find the sum of 5 terms of $2\sqrt{3}, 6, 6\sqrt{3}, \cdots$.

11. Find the sum of 10 terms of $\frac{4}{3}, \frac{2}{3}, \frac{1}{3}, \cdots$.

12. In a G. P. whose first term is 7, the sum of 6 terms is 2548. Find the common ratio.

13. In a G. P. whose first term is $\frac{1}{4}$, the sum of 5 terms is $\frac{211}{324}$. Find the common ratio. Find the fifth term.

14. In the series $1, 2, 4, \cdots$, how many terms must be added to make 255?

15. In the series $100, 40, 16, \cdots$, how many terms must be added to make $\frac{20748}{125}$?

16. A ball falls from a height of 100 feet, and rebounds after each fall one fifth of the distance it fell. Through what distance will it have traveled at the end of the fifth fall?

17. The bell jar of an air pump holds 432 cubic inches of air. At each stroke of the pump $\frac{1}{12}$ of the air left in the jar is removed. How much air is removed in 5 strokes?

18. If a man saves each year twice as much as the preceding year, and saves $150 the first year, how long will it take for him to save $2250?

19. A vessel containing vinegar is emptied of one-half its contents, and then filled with water. Half of this mixture is then drawn off and the vessel refilled with water. This is repeated five times. What part of the remaining mixture is water?

20. To what sum will \$1 amount in 4 years at 3 % compound interest?

SUGGESTION. — $a = \$1$, $r = \$1.03$, $n = 5$.

21. To what sum will \$1000 amount in 5 years at 3 % compound interest?

22. A man puts \$100 in a savings bank which pays 3 % compound interest, compounded annually. To what sum will it amount in 6 years?

23. The **present worth** of a debt due at some future time is a sum such that, if invested at compound interest, the amount at the end of the time will equal the debt.

Find the present worth of \$5000 due in 5 years, the money being worth 6 % interest compounded annually.

24. A man, in making his will, left \$25,000 to his daughter. It was made payable on her twenty-fifth birthday, but in case she married before that time she was to receive the present worth of the sum at the time of her marriage, money being considered worth 6 % compounded annually. She married on her twentieth birthday. How much money did she receive?

136. Infinite Geometrical Progressions. — In a G. P. such as 1, 2, 4, 8, ⋯, in which the common ratio is greater than 1, and positive, each term is greater than the preceding term. Evidently, if the number of terms is increased without limit, their sum will become infinite.

But in a G. P. such as $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$, in which the common ratio is less than 1, and positive, each term is less than the preceding term, and as the number of terms becomes very great each term becomes very small. In such a series, as the number of terms taken is increased without limit, the sum of the terms does not increase without limit, but approaches some definite number as a limit.

For example, in the series 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ⋯, the sum of 4 terms is $1\frac{7}{8}$, the sum of 8 terms is $1\frac{127}{128}$, the sum of 16 terms is $1\frac{32767}{32768}$, etc. Evidently, as the number of terms is gradually increased the sum approaches nearer and nearer to 2, but always remains less than 2. When the number of terms taken is increased without limit, the sum approaches 2 as a limit.

In general, the formula $S=\frac{a(1-r^n)}{1-r}$ may be written in the form $S=\frac{a}{1-r}-\frac{ar^n}{1-r}$. Now, if r is less than 1, r^n is still smaller, and when $n=\infty$, $r^n \doteq 0$. When $r^n \doteq 0$, $\frac{ar^n}{1-r} \doteq 0$. Hence, when n is increased without limit, the sum $\frac{a}{1-r}-\frac{ar^n}{1-r}$ approaches $\frac{a}{1-r}$ as a limit.

Consequently, the sum of an infinite number of terms of a G. P. in which the common ratio is less than 1 *is defined as the limit* $\frac{a}{1-r}$.

That is, when n is infinite and r less than 1,

$$S=\frac{a}{1-r}. \qquad (D)$$

Example. — Find the sum of the infinite series of terms 9, 6, 4, ⋯.

Here $r=\frac{2}{3}$. Hence from formula D,

$$S=\frac{9}{1-\frac{2}{3}}=27.$$

By aid of formula D the common fraction to which any repeating decimal is equal may be found.

Example. — Find the value of 12.636363 ⋯.

The repeating part .636363 ⋯ may be written in the form $\frac{63}{100}+\frac{63}{10000}+\frac{63}{1000000}+\cdots$. This is an infinite G. P. in which $a=\frac{63}{100}$ and $r=\frac{1}{100}$.

Hence, $.636363\cdots=\frac{\frac{63}{100}}{1-\frac{1}{100}}=\frac{63}{99}=\frac{7}{11}.$

Hence, $12.636363\cdots=12\frac{7}{11}.$

EXERCISES

Find the sum of an infinite number of terms of:

1. $8, 4, 2, \cdots$.

2. $12, 4, 1\frac{1}{3}, \cdots$.

3. $15, 5, 1\frac{2}{3}, \cdots$.

4. $3, \frac{1}{4}, \frac{1}{48}, \cdots$.

5. $\frac{3}{2}, 1, \frac{2}{3}, \cdots$.

6. $\frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \cdots$.

7. $1, \frac{1}{5}, \frac{1}{25}, \cdots$.

8. $2, \sqrt{2}, 1, \cdots$.

9. $1, n, n^2, \cdots$, when n is less than 1.

10. $1, \frac{1}{n^2}, \frac{1}{n^4}, \cdots$, when n is greater than 1.

11. In a G. P. the common ratio is $\frac{1}{4}$. What must be the first term in order that the sum of an infinite number of terms shall be 80?

12. In a G. P. the first term is 5. What must be the common ratio in order that the sum of an infinite number of terms shall be $4\frac{6}{11}$?

13. Find the first five terms of a G. P. the sum of an infinite number of terms of which is 2, and whose second term is $\frac{3}{8}$.

14. Find the eighth term of a G. P. the sum of an infinite number of terms of which is 243, and whose second term is 54.

Find the common fractions to which the following are equal:

15. $0.66666 \cdots$.

16. $0.272727 \cdots$.

17. $0.393939 \cdots$.

18. $0.454545 \cdots$.

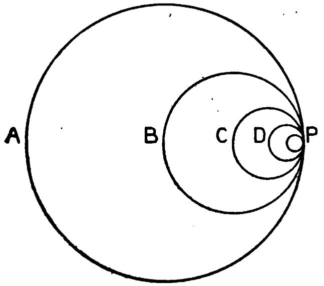
19. $0.150150 \cdots$.

20. $1.333333 \cdots$.

21. $4.363636 \cdots$.

22. $3.721212 \cdots$.

23. $0.046363 \cdots$.



24. The circles A, B, C, etc., are tangent internally at P. The diameter of B is $\frac{1}{2}$ that of A, the diameter of C is $\frac{1}{2}$ that of B, etc. If the diameter of A is 6 inches, and the number of circles thus constructed is continued without limit, find the sum of the circumferences of all of the circles.

25. In the figure of Ex. 24, the area of circle B is $\frac{1}{4}$ that of circle A, the area of circle C is $\frac{1}{4}$ that of circle B, etc. If the diameter of A is 4 inches, find the sum of the areas of all of the circles.

SUPPLEMENTARY EXERCISES

The numbers a, b, c, d, etc., are said to form a **harmonic progression** when $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$, etc., form an arithmetical progression. Problems relating to harmonic progressions may be solved by forming the corresponding arithmetical progressions and using the principles applicable to the latter.

1. Find the twelfth term of the series $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \cdots$.

SOLUTION. — The corresponding A. P. is $1, 3, 5, 7, \cdots$.
The twelfth term of this is $1 + 11 \times 2$, or 23.
Hence, the twelfth term of the H. P. is $\frac{1}{23}$.

2. Find the seventh term of the series $1, \frac{1}{6}, \frac{1}{11}, \cdots$.

3. Find the twentieth term of the series $-\frac{1}{8}, -\frac{1}{5}, -\frac{1}{2}, \cdots$.

4. Find the fifteenth term of the series $-\frac{2}{3}, -2, 2, \frac{2}{3}, \cdots$.

5. Find the twelfth term of the series $2, \frac{1}{4}, \frac{2}{15}, \cdots$.

6. Find the tenth term of the H. P. in which the third term is $\frac{1}{3}$ and the seventh term is $\frac{1}{33}$.

7. Find the first five terms of the H. P. in which the third term is 2 and the thirty-second term is $\frac{2}{59}$.

8. In any H. P. all of the terms between any two given terms are called the *harmonic means* of those terms.

Find the harmonic mean of $\frac{1}{6}$ and $\frac{1}{12}$. Of $-\frac{3}{2}$ and $\frac{3}{4}$.

9. Insert 4 harmonic means between $-\frac{6}{5}$ and $\frac{6}{5}$.

10. Insert 5 harmonic means between $\frac{1}{4}$ and 3.

11. If H represents the harmonic mean between a and b, show that $H = \frac{2ab}{a+b}$.

12. If A, G, and H represent the arithmetical, geometrical, and harmonic means, respectively, of two numbers, then show that G is a mean proportional between A and H.

SUGGESTION. — If the two numbers are a and b, then

$$A = \frac{a+b}{2}, \quad G = \sqrt{ab}, \quad H = \frac{2ab}{a+b}.$$

Now show that $AH = G^2$, by showing that they equal the same thing.

13. The arithmetical mean of two numbers is 8, and their harmonic mean is 18. Find their geometrical mean.

14. The geometrical mean of two numbers is 4, and their harmonic mean is $\frac{1}{5}$. Find the numbers.

CHAPTER XIII

EXPONENTS

137. Positive Integral Exponents. — A positive integral **exponent**, as defined and used in the FIRST COURSE and in this volume, indicates how many times the base is to be taken as a factor.

The following fundamental laws of positive integral exponents, which have been applied at various points of the work, were proved in Chapter XVI of the FIRST COURSE by use of the above definition. Let the pupil establish each of them in review.

I. LAW FOR MULTIPLICATION:

$$a^m \times a^n = a^{m+n}.$$

That is, *the exponent of the product of two powers of the same number equals the sum of the exponents of the two factors.*

SUGGESTION. — Show by definition of an exponent that $a^m \times a^n$ contains a as a factor $m + n$ times.

This law may be extended to apply to the product of more than two factors. Thus, in general,

$$a^m \times a^n \times a^p \cdots = a^{m+n+p+\cdots}.$$

II. LAW FOR DIVISION:

$$a^m \div a^n = a^{m-n}.$$

That is, *the exponent of the quotient of two powers of the same number equals the exponent of the dividend minus the exponent of the divisor.*

SUGGESTION. — By using the principle that *quotient* × *divisor* = *dividend*, show that this law follows from Law I.

III. LAW FOR A POWER OF A POWER:

$$(a^n)^m = a^{mn}.$$

That is, *the mth power of the nth power of any number equals the mnth power of the number.*

SUGGESTION. — Show that this follows from Law I. Thus,

$$(a^n)^m = a^n \times a^n \times a^n \cdots \text{ to } m \text{ factors, etc.}$$

This law may be extended thus:

$$((a^n)^m)^p = a^{pmn}; \ (((a^n)^m)^p)^q = a^{qpmn}; \text{ etc.}$$

IV. LAW FOR A POWER OF A PRODUCT:

$$(ab)^n = a^n b^n.$$

That is, *the nth power of the product of two numbers equals the product of the nth powers of the numbers.*

SUGGESTION — Show that $(ab)^n$ contains a and b each as a factor n times. Thus, $(ab)^n = (ab)(ab)(ab) \cdots$ to n factors, etc.

This law may be extended to apply to a power of a product of three or more factors. Thus, in general,

$$(abc \cdots)^n = a^n b^n c^n \cdots.$$

V. LAW FOR POWER OF A FRACTION:

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

That is, *the nth power of a fraction equals the nth power of the numerator divided by the nth power of the denominator.*

SUGGESTION. — Proceed as in establishing Law IV.

VI. LAW FOR ROOT OF A POWER, OR POWER OF A ROOT:

$$\sqrt[n]{a^m} = a^{\frac{m}{n}}.$$

That is, *the exponent of a root of a power of a number equals the exponent of the power divided by the index of the root.*

SUGGESTION. — $(a^{\frac{m}{n}})^n = a^{\frac{m}{n} \times n} = a^m$, by Law III.

Since by this law $\sqrt[n]{a}=a^{\frac{1}{n}}$, $(\sqrt[n]{a})^m=(a^{\frac{1}{n}})^m$. Hence by Law III,

$$(\sqrt[n]{a})^m=a^{\frac{m}{n}}.$$

That is, *the exponent of a power of a root of a number equals the exponent of the power divided by the index of the root.*

By use of one or more of the above laws of exponents, any power or root of any monomial, or the product or quotient of any monomials, may be obtained.

EXERCISES

By the above laws find the values of:

1. $n^3 \times n^7$.
2. $V^8 \times V^4$.
3. $k \times k^5 \times k^3$.
4. $R^6 \times R \times R^8$.
5. $2^8 \times 2^2 \times 2$.
6. $5 \times 5^2 \times 5^4$.
7. $3^4 \times 3 \times 3^4$.
8. $(-1)^2(-1)^3$.
9. $(-2)^4(-2)^2$.
10. $(-a)^2(-a)^6$.
11. $x^n \times x^n$.
12. $t^{n+1} \times t$.
13. $P^{m+3} \times P^{m-8}$.
14. $a^9 \div a^2$.
15. $y^{12} \div y^8$.
16. $D^9 \div D$.
17. $\pi r^2 \div \pi r$.
18. $mv^n \div v^{n-2}$.
19. $at^2 \div \frac{1}{2}\,at$.
20. $2^8 \div 2^5$.
21. $3^{12} \div 3^{10}$.
22. $(-1)^5 \div (-1)^3$.
23. $(-n)^9 \div (-n)^5$.
24. $(x^4)^5$.
25. $(N^6)^3$.
26. $(s^{12})^4$.
27. $(k^{20})^5$.
28. $(2^3)^2$.
29. $(3^2)^4$.
30. $[(-1)^3]^6$.
31. $[(-b)^5]^4$.
32. $(x^{n-1})^n$.
33. $(mn)^4$.
34. $(a^2b^8)^6$.
35. $(x^5y)^3$.
36. $(v^9t^4)^4$.
37. $(2 \times 5)^3$.
38. $(7 \times 3)^2$.
39. $(a^nb^{2n})^5$.
40. $(\frac{3}{4})^3$.
41. $(\frac{2}{3})^4$.
42. $(\frac{4}{5})^4$.
43. $(-\frac{1}{2})^6$.
44. $(-\frac{2}{3})^3$.
45. $\left(\frac{a}{b}\right)^4$.
46. $\left(\frac{2\,m^2}{3\,n^5}\right)^3$.
47. $\left(\frac{v^3t^5}{2\,a^4s^2}\right)^5$.
48. $\sqrt[4]{a^{12}}$.
49. $\sqrt[6]{n^{18}}$.
50. $\sqrt{x^4y^8}$.
51. $\sqrt[5]{32\,r^{20}}$.
52. $\sqrt[n]{a^{2n}b^{3n}}$.
53. $(\sqrt[3]{a})^6$.
54. $(\sqrt[4]{t})^8$.
55. $(\sqrt[3]{m^2})^9$.
56. $(2\,a^2\sqrt[5]{b^2})^{10}$.

138. Zero Exponents. — The application of Law II, § 137, when the exponents of the dividend and divisor are equal, gives rise to a **zero exponent**.

Thus, $a^4 \div a^4 = a^{4-4} = a^0$; $r^9 \div r^9 = r^{9-9} = r^0$.

The meaning of a zero exponent may be shown as follows:

In general, $a^n \div a^n = a^{n-n} = a^0$.

But, $a^n \div a^n = 1$.

Hence, since a^0 and 1 are but two different ways of writing the same quotient,

$$a^0 = 1.$$

That is, *any base with the exponent* 0 *equals* 1.

EXERCISES

Give the values of:

1. 2^0.
2. 0.1^0.
3. 100^0.
4. $(\frac{1}{4})^0$.
5. $(-1)^0$.
6. $3^4 \times 3^0$.
7. $n^5 \div n^0$.
8. $(a^4)^0$.

139. Negative Exponents. — The application of Law II, § 137, when the exponent of the divisor is greater than that of the dividend, gives rise to a **negative exponent**.

Thus, $a^3 \div a^8 = a^{3-8} = a^{-5}$; $v^4 \div v^{10} = v^{-6}$.

It is evident that a negative exponent has not the same meaning as an exponent that is a positive whole number. For example, to say that n^{-4} indicates that n is to be taken as a factor -4 times is absurd. But this new kind of number symbol has a definite meaning which may be shown as follows:

Assuming that Law I, § 137, applies to negative exponents,

$$a^n \times a^{-n} = a^{n-n} = a^0,$$

or, $$a^n \times a^{-n} = 1.$$

Dividing both members of this equation by a^n,

$$a^{-n} = \frac{1}{a^n}.$$

That is, *a negative exponent indicates a power of the base that is to be used as a divisor.*

It follows from the meaning of a negative exponent that

Any factor may be changed from the numerator to the denominator of a fraction, or vice versa, by changing the sign of its exponent.

Thus,
$$\frac{a^2b^{-2}}{c^3d^{-3}} = \frac{a^2 \times \frac{1}{b^2}}{c^3 \times \frac{1}{d^3}} = \frac{\frac{a^2}{b^2}}{\frac{c^3}{d^3}} = \frac{a^2d^3}{c^3b^2}.$$

Evidently, this result might have been obtained in a single step by putting the b into the denominator and the d into the numerator and changing the signs of their exponents.

Similarly,
$$\frac{3^{-4}}{2^{-5}} = \frac{2^5}{3^4} = \frac{32}{81}.$$

EXERCISES

Find the values of:

1. 2^{-2}.
2. 2^{-5}.
3. 3^{-4}.
4. 7^{-1}.
5. 10^{-5}.
6. 1^{-1}.
7. 0.5^{-2}.
8. 0.01^{-3}.
9. $(\frac{1}{2})^{-2}$.
10. $(\frac{2}{3})^{-1}$.
11. $(\frac{1}{5})^{-3}$.
12. $(\frac{3}{8})^{-2}$.
13. $(\frac{2}{5})^{-4}$.
14. $2^{-5} \times (\frac{1}{2})^{-1}$.
15. $10^{-3} \div (\frac{3}{4})^{-2}$.
16. $(\frac{1}{2})^{-6} \times (\frac{2}{3})^{-4}$.

Write with positive exponents:

17. n^{-4}.
18. a^2b^{-3}.
19. πR^2r^{-2}.

20 $\frac{m^{-2}}{n^{-4}}$.

21. $\frac{a^5}{b^{-5}}$.
22. $\frac{a^{-1}b^{-2}}{c^3}$.
23. $\frac{2x^{-2}}{-y^2}$.
24. $\frac{-5P^{-4}}{-3N^{-3}}$.
25. $\frac{x^{-1}y^{-2}z^{-3}}{xy^2z^3}$.
26. $\frac{x^3y^{-3}}{1}$.
27. $\frac{a^{-6}b}{-6}$.
28. $\frac{V^{-2}}{5}$.
29. $\frac{m}{n} + \frac{m^{-1}}{n^{-1}}$.
30. $\frac{a^{-1}+b^{-1}}{a^{-1}-b^{-1}}$.
31. $\frac{x^2-y^{-2}}{x^2+y^{-2}}$.
32. $\frac{(a+b)^{-1}}{(a-b)^{-1}}$.
33. $(N^{-2})^{-3}$.

Write in integral form:

34. $\frac{a^2}{b^2}$. 36. $\frac{1}{-5n}$. 38. $\frac{1}{x}+\frac{1}{y}$. 40. $\frac{m}{n^{-1}}+\frac{n}{m^{-1}}$.

35. $\frac{2x}{3y}$. 37. $\frac{v^{-2}}{t^{-2}}$. 39. $\frac{a}{b}+\frac{b}{a}$. 41. $\frac{1}{a^2+ab+b^2}$.

140. Fractional Exponents. — The application of Law VI, § 137, when the exponent of the base is not divisible without remainder by the index of the root, gives rise to a **fractional exponent.**

Thus, $\sqrt[5]{a^2} = a^{\frac{2}{5}}$; $\sqrt[3]{n^4} = n^{\frac{4}{3}}$; $\sqrt{5} = 5^{\frac{1}{2}}$.

It is clear that a fractional exponent has not the same meaning as an integral exponent. For example, $a^{\frac{3}{5}}$ cannot indicate that a is to be used as a factor $\frac{3}{5}$ of a time. Such a statement is absurd. The meaning of every fractional exponent may be shown as follows:

Assuming that Law I, § 137, applies to fractional exponents,

$$\begin{aligned}(a^{\frac{n}{m}})^m &= a^{\frac{n}{m}} \times a^{\frac{n}{m}} \times a^{\frac{n}{m}} \cdots \text{ to } m \text{ factors}\\ &= a^{\frac{n}{m}+\frac{n}{m}+\frac{n}{m}+\cdots \text{ to } m \text{ terms}}\\ &= a^n.\end{aligned}$$

Hence, by definition of a root,

$$a^{\frac{n}{m}} = \sqrt[m]{a^n}.$$

That is, *a fractional exponent indicates a root of a power of the base, the numerator indicating the power of the base, and the denominator indicating the root.*

For example, $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$; $(9n^4)^{\frac{1}{2}} = \sqrt{9n^4} = 3n^2$.

An expression involving negative fractional exponents may be written with positive integral exponents by first interpreting the negative signs, then the fractional forms, of the exponents.

For example, $n^{-\frac{3}{4}} = \frac{1}{n^{\frac{3}{4}}} = \frac{1}{\sqrt[4]{n^3}}$.

And $\left(\frac{8}{27}\right)^{-\frac{2}{3}} = \frac{1}{\sqrt[3]{(\frac{8}{27})^2}} = \frac{1}{\frac{4}{9}} = \frac{9}{4}$.

EXERCISES

Find the values of:

1. $4^{\frac{1}{2}}$.
2. $9^{\frac{1}{2}}$.
3. $27^{\frac{2}{3}}$
4. $16^{\frac{3}{4}}$.
5. $25^{\frac{3}{2}}$.
6. $4^{\frac{3}{2}}$.
7. $81^{\frac{3}{4}}$.
8. $125^{\frac{2}{3}}$.
9. $32^{\frac{3}{5}}$.
10. $64^{\frac{5}{6}}$.
11. $9^{\frac{5}{2}}$.
12. $64^{\frac{4}{3}}$.
13. $4^{-\frac{1}{2}}$.
14. $16^{-\frac{3}{4}}$.
15. $27^{-\frac{2}{3}}$.
16. $256^{-\frac{3}{4}}$.
17. $(\frac{8}{27})^{-\frac{2}{3}}$.
18. $(\frac{64}{125})^{\frac{4}{3}}$.
19. $(\frac{81}{256})^{-\frac{1}{2}}$.
20. $(\frac{16}{81})^{-\frac{3}{4}}$.
21. $8^{\frac{2}{3}} \times 9^{-\frac{3}{2}}$.
22. $125^{-\frac{5}{3}} \times 32^{\frac{3}{5}}$.
23. $25^{-\frac{1}{2}} \div 81^{-\frac{1}{2}}$.
24. $(\frac{1}{27})^{-\frac{1}{3}} \div (\frac{1}{16})^{\frac{3}{4}}$.
25. $8^{-\frac{2}{3}}(25^{\frac{1}{2}} - 81^{\frac{1}{4}})$.

Express with positive integral exponents:

26. $a^{\frac{2}{3}}b^{\frac{1}{3}}$.
27. $p^{-\frac{1}{2}}q^{\frac{1}{3}}$.
28. $M^{-\frac{2}{3}}V^{-\frac{3}{4}}$.
29. $4x^{-\frac{5}{2}}$.
30. $-9t^{\frac{3}{4}}$.
31. $2m^{\frac{3}{2}}n^{-\frac{2}{3}}$.
32. $9V^{-\frac{4}{5}}t^{\frac{2}{5}}$.
33. $\dfrac{x^{-\frac{1}{2}}y^{-\frac{1}{2}}}{z^{-1}}$.
34. $\dfrac{m^{-2}}{p^{-\frac{2}{3}}q^{-\frac{3}{2}}}$.
35. $\dfrac{(a-b)^{-\frac{3}{4}}}{(a+b)^{-\frac{3}{4}}}$.
36. $\dfrac{(h-k)^{\frac{1}{3}}}{(h-k)^{-\frac{2}{3}}}$.
37. $\dfrac{m^{\frac{1}{2}}+n^{\frac{1}{2}}}{m^{-\frac{1}{2}}+n^{-\frac{1}{2}}}$.

Express without radical signs:

38. $\sqrt[5]{v^2}$.
39. $\sqrt[3]{n^7}$.
40. $a\sqrt{a}$.
41. $\sqrt{w}\sqrt[3]{t}$.
42. $\sqrt[4]{a^2b^3mn}$.
43. $\sqrt[6]{27x^3y^6}$.
44. $\sqrt[4]{b^3}\sqrt{c}$.
45. $\sqrt[mn]{a^n}$.
46. $\sqrt[2n]{x^{mn}}$.
47. $\sqrt{a-b}$.
48. $\sqrt[3]{1+x^2}$.
49. $\sqrt[m]{x^n}\sqrt[n]{x^m}$.

Write as integral expressions without radical signs:

50. $\dfrac{1}{\sqrt{1-x}}$.
51. $\dfrac{1}{\sqrt[3]{(1+x^2)^2}}$.
52. $\dfrac{\sqrt{(a+b)^3}}{\sqrt{(a-b)^{-3}}}$.

141. Laws Applied to Negative and Fractional Exponents. — In determining the meanings of negative and fractional exponents, it was assumed that Law I of positive integral exponents, § 137, is applicable to these new kinds of exponents also. It may be proved that all of the fundamental laws of positive integral exponents given in § 137 hold also for negative and fractional exponents. The proofs are omitted from this course.

Example 1. — Simplify $n^{\frac{2}{3}} \times n^{\frac{3}{4}} \times n^{\frac{1}{2}}$.

$$n^{\frac{2}{3}} \times n^{\frac{3}{4}} \times n^{\frac{1}{2}} = n^{\frac{8}{12}} \times n^{\frac{9}{12}} \times n^{\frac{6}{12}} = n^{\frac{8}{12}+\frac{9}{12}+\frac{6}{12}} = n^{\frac{23}{12}}.$$

Example 2. — Simplify $a^{-\frac{2}{5}} \div a^{-\frac{3}{10}}$.

$$a^{-\frac{2}{5}} \div a^{-\frac{3}{10}} = a^{-\frac{4}{10}} \div a^{-\frac{3}{10}} = a^{-\frac{1}{10}}.$$

Example 3. — Simplify $(A^{-4}B^{-\frac{3}{4}})^{-\frac{1}{2}}$.

$$(A^{-4}B^{-\frac{3}{4}})^{-\frac{1}{2}} = A^2B^{\frac{3}{8}}.$$

Example 4. — Simplify $(m^{\frac{1}{3}} + n^{-\frac{1}{3}})(m^{\frac{2}{3}} - m^{\frac{1}{3}}n^{-\frac{1}{3}} + n^{-\frac{2}{3}})$.

$$\begin{array}{llll}
m^{\frac{2}{3}} & - m^{\frac{1}{3}}n^{-\frac{1}{3}} & + n^{-\frac{2}{3}} & \\
m^{\frac{1}{3}} & + n^{-\frac{1}{3}} & & \\
\hline
m & - m^{\frac{2}{3}}n^{-\frac{1}{3}} & + m^{\frac{1}{3}}n^{-\frac{2}{3}} & \\
 & m^{\frac{2}{3}}n^{-\frac{1}{3}} & - m^{\frac{1}{3}}n^{-\frac{2}{3}} & + n^{-1} \\
\hline
m & & & + n^{-1}
\end{array}$$

Operations with expressions involving radicals may be performed by first expressing the radicals by means of fractional exponents.

Example 5. — Simplify $\dfrac{x-y}{\sqrt[3]{x}-\sqrt[3]{y}}$.

Expressing the radicals by means of fractional exponents, and dividing,

$$\begin{array}{r|llll}
 & x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} & & & \\
x^{\frac{1}{3}} - y^{\frac{1}{3}}) & x & & & -y \\
 & \underline{x - x^{\frac{2}{3}}y} & & & \\
 & & x^{\frac{2}{3}}y & & -y \\
 & & \underline{x^{\frac{2}{3}}y} & \underline{- x^{\frac{1}{3}}y^{\frac{2}{3}}} & \\
 & & & x^{\frac{1}{3}}y^{\frac{2}{3}} & -y \\
 & & & \underline{x^{\frac{1}{3}}y^{\frac{2}{3}}} & \underline{-y}
\end{array}$$

Hence, $$\frac{x-y}{\sqrt[3]{x}-\sqrt[3]{y}} = x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} = \sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}.$$

EXERCISES

Simplify:

1. $A^{-3} \times A^{-2} \times A^{7}$.
2. $4\,n^{-5} \times n^{3} \times n^{3}$.
3. $R^{-1} \times R^{6} \times R^{-5}$.
4. $12\,t^{10} \times t^{-4} \times t^{-2}$.
5. $a^{\frac{1}{2}} \times a^{-\frac{3}{2}} \times a^{2}$.
6. $5\,p^{-\frac{2}{3}} \times 3\,p^{\frac{1}{2}}$.
7. $h^{-\frac{4}{5}}k^{\frac{2}{3}} \times hk^{-\frac{1}{2}}$.
8. $V^{\frac{1}{4}}t^{\frac{1}{5}} \times V^{-\frac{1}{3}}t^{-\frac{1}{4}}$.
9. $y^{2}z^{-\frac{1}{6}} \times y^{-\frac{3}{4}}z^{\frac{3}{4}}$.
10. $x^{-2} \div x^{-4}$.
11. $an^{-1} \div a^{3}n^{4}$.
12. $w^{2}v^{-3} \div w^{-1}v^{-1}$.
13. $p^{-1}q^{-2}r^{-3} \div p^{3}q^{2}r$.
14. $N^{-\frac{4}{5}} \div N^{-\frac{7}{5}}$.
15. $K^{6} \div K^{\frac{11}{2}}$.
16. $b^{\frac{7}{10}} \div a^{-1}b^{-\frac{3}{10}}$.
17. $x^{\frac{1}{2}}y^{\frac{1}{3}}z^{-\frac{1}{4}} \div xy^{-\frac{2}{3}}z$.
18. $6\,P^{-\frac{3}{2}}V^{\frac{6}{7}} \div 3\,P^{-\frac{1}{4}}V$.
19. $(a^{-2}b^{-3})^{-2}$.
20. $(x^{4}y^{-1})^{-5}$.
21. $(\pi R^{2})^{-4}$.
22. $(m^{-\frac{1}{2}}n^{-\frac{3}{2}})^{2}$.
23. $(2\,H^{\frac{3}{4}}n^{-1})^{\frac{2}{3}}$.
24. $(5\,M^{-\frac{1}{6}}A^{\frac{1}{3}})^{-3}$.
25. $(4\,x^{-\frac{2}{3}}y^{-\frac{1}{4}}z^{-1})^{-\frac{3}{2}}$.
26. $(81\,a^{12}b^{-6}c^{-9})^{-\frac{1}{12}}$.
27. $(-27\,p^{-9}d^{-\frac{3}{4}})^{-\frac{2}{3}}$.
28. $(32\,r^{-\frac{5}{3}}s^{\frac{10}{11}})^{-\frac{3}{5}}$.
29. $\sqrt{a} \times \sqrt[3]{a}$.
30. $\sqrt[4]{t^{3}} \times \sqrt[3]{t^{4}}$.
31. $\sqrt{n} \times \sqrt[4]{n} \times \sqrt{n}$.
32. $\sqrt[3]{k^{-5}} \div \sqrt[3]{k^{-8}}$.
33. $\sqrt[5]{a^{-2}x^{-1}} \div \sqrt[5]{a^{-12}x^{-6}}$.
34. $\sqrt{a^{3}} \div \sqrt[3]{a^{2}}$.
35. $\sqrt[3]{wv} \div \sqrt[4]{w^{3}v^{2}}$.
36. $\dfrac{\sqrt[3]{n^{2}}}{\sqrt[4]{mn^{3}}} \div \dfrac{\sqrt[6]{m^{3}n^{2}}}{\sqrt[12]{m^{7}n^{9}}}$.
37. $\dfrac{1}{\sqrt[3]{k^{2}n}} \div \sqrt{kn^{2}}$.
38. $\sqrt[3]{\dfrac{a^{-1}}{b^{-3}}} \div \left(\dfrac{a^{-\frac{2}{3}}b^{\frac{1}{2}}}{b^{-1}a}\right)^{-2}$.
39. $\sqrt[13]{(\sqrt[3]{t^{2}} \times \sqrt{t^{4}})^{-14}}$.
40. $\sqrt[4]{\sqrt[3]{n^{8}}}$.
41. $\sqrt[3]{\sqrt{\sqrt[4]{x^{-6}}}}$.
42. $(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^{\frac{1}{2}} - y^{\frac{1}{2}})$.
43. $(W^{\frac{3}{2}} - V^{-\frac{3}{2}})(W^{\frac{3}{2}} + V^{-\frac{3}{2}})$.

44. $(m-n)\div(m^{\frac{1}{2}}+n^{\frac{1}{2}})$.

45. $(R+r)\div(R^{\frac{1}{3}}+r^{\frac{1}{3}})$.

46. $(x^2-y^2)\div(x^{\frac{2}{3}}-y^{\frac{2}{3}})$.

47. $(x^{\frac{1}{2}}+2)(x^{\frac{1}{2}}-1)$.

48. $(3\,m^{-\frac{2}{3}}-4\,n^{\frac{5}{2}})(3\,m^{-\frac{2}{3}}+4\,n^{\frac{5}{2}})$.

49. $(a-4\,b)\div(a^{\frac{1}{2}}-2\,b^{\frac{1}{2}})$.

50. $(t^{-3}-1)\div(t^{-1}-1)$.

51. $(1+y^{-1})\div(1+y^{-\frac{1}{3}})$.

52. $(n^{-3}+27)\div(n^{-1}+3)$.

53. $(k^{-4}-1)\div(k^{-1}-1)$.

54. $(v^2+t^2)\div(v^{\frac{2}{3}}+t^{\frac{2}{3}})$.

55. $(M^{\frac{2}{3}}+N^{-\frac{3}{4}})^2$.

56. $(a^{\frac{1}{2}}-b^{-\frac{1}{2}})^2$.

57. $(1-x^{-\frac{1}{4}})^2$.

58. $(E^{-\frac{1}{4}}+F^{-\frac{1}{4}})^2$.

59. $(e^x+e^{-x}-2)^2$.

60. $(1-x^{-1}+x^{-3})^2$.

61. $(n^{\frac{1}{4}}+1+n^{-\frac{1}{4}})^2$.

62. $(h-k)\div(\sqrt[4]{h}-\sqrt[4]{k})$.

63. $(A^{-\frac{1}{2}n}-1)^2$.

64. $(a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}})(a^{\frac{1}{3}}-b^{\frac{1}{3}})$.

65. $(x^{\frac{4}{3}}+x^{\frac{2}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}})(x^{\frac{4}{3}}-x^{\frac{2}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}})$.

66. $(t^{\frac{5}{4}}-t^{\frac{3}{4}}+t^{\frac{1}{4}}-t^{-\frac{1}{4}})(t^{\frac{3}{4}}+t^{\frac{1}{4}})$.

67. $(8\,n^{-2}-8\,n^2+5\,n^6-3\,n^{-6})\div(5\,n^2-3\,n^{-2})$.

68. $(p+32\,q)\div(p^{\frac{1}{5}}+2\,q^{\frac{1}{5}})+2\,p^{\frac{3}{5}}q^{\frac{1}{5}}+8\,p^{\frac{1}{5}}q^{\frac{3}{5}}$.

69. $(\sqrt[3]{m^4}+\sqrt[3]{m^2n^2}+\sqrt[3]{n^4})(\sqrt[3]{m^2}-\sqrt[3]{n^2})$.

70. $(\sqrt[3]{E^2}-\sqrt[3]{EF}+\sqrt[3]{F^2})(\sqrt[3]{E}+\sqrt[3]{F})$.

71. $(\sqrt{x}+\sqrt[4]{x}+1)\left(\dfrac{1}{\sqrt{x}}-\dfrac{1}{\sqrt[4]{x}}+1\right)$.

72. $\left(\dfrac{4}{\sqrt[3]{n^4}}-\dfrac{12}{\sqrt[3]{n^2}}+9\right)\div\left(\dfrac{2}{\sqrt[3]{n^2}}-3\right)$.

SUPPLEMENTARY EXERCISES

Simplify:

1. $(a^{-3n}-1)\div(a^{-n}-1)$.

2. $(x^{\frac{2n}{3}}-x^{\frac{n}{3}}y^{\frac{n}{3}}+y^{\frac{2n}{3}})(x^{\frac{n}{3}}+y^{\frac{n}{3}})$.

3. $(t^{-2n}-t^{-n}+1-t^n+t^{2n})^2$.

4. $(m+n+2\sqrt[3]{mn^2}+2\sqrt[3]{m^2n})\div(\sqrt[3]{m^2}+\sqrt[3]{n^2}+\sqrt[3]{mn})$.

Factor:

5. $16 - n^{-2}$.

6. $R^2 + 2 + R^{-2}$.

7. $27 - N^{-3}$.

8. $k^{-2} - k^{-1} - 6$.

Find the square root of:

9. $9x - 12x^{\frac{1}{2}} + 10 - 4x^{-\frac{1}{2}} + x^{-1}$.

10. $m^2 - 6m^{\frac{3}{2}}n^{\frac{1}{3}} + 15mn^{\frac{2}{3}} - 18m^{\frac{1}{2}}n + 9n^{\frac{4}{3}}$.

11. $A^2 + 8A + 12 - 16A^{-1} + 4A^{-2}$.

12. $2 + h^2k^{-2} + 2hk + k^4 + 2h^{-1}k^3 + h^{-2}k^2$.

13. $r^{\frac{4}{5}} + 4r^{\frac{3}{5}} + 4r^{\frac{2}{5}} + 2r^{\frac{1}{5}} + 4 + r^{-\frac{2}{5}}$.

14. $a^{-\frac{4}{3}} - 2a^{-1}b^{\frac{1}{3}} - 2a^{-\frac{1}{3}}b + 3a^{-\frac{2}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}}$.

CHAPTER XIV

THE BINOMIAL THEOREM

142. The Binomial Theorem. — By actual multiplication it may be shown that:

$$(a+b)^2 = a^2 + 2\,ab + b^2;$$
$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3;$$
$$(a+b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4;$$
$$(a+b)^5 = a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5;$$
$$(a+b)^6 = a^6 + 6\,a^5b + 15\,a^4b^2 + 20\,a^3b^3 + 15\,a^2b^4 + 6\,ab^5 + b^6; \text{ etc.}$$

If $a - b$ is taken instead of $a + b$, the even-numbered terms in each of the above expansions will be negative and the odd-numbered terms will be positive. Thus,

$$(a-b)^5 = a^5 - 5\,a^4b + 10\,a^3b^2 - 10\,a^2b^3 + 5\,ab^4 - b^5.$$

A comparison of these few values of different powers of $a + b$ and $a - b$ will show that each of them may be written down, without performing the multiplication, by use of the following laws:

(1) *The first term in each case is* ***a*** *with an exponent equal to that of the binomial, and the last term is* ***b*** *with the same exponent.*

(2) *The exponents of* ***a*** *decrease by* 1 *in each term after the first; and* ***b*** *appears to the first power in the second term, and its exponents increase by* 1 *in each term after the second. The sum of the exponents of* ***a*** *and* ***b*** *in every term is equal to the exponent of the binomial.*

(3) *The numerical coefficient in the second term equals the exponent of the first term. And the coefficient of any term after the second may be obtained from the preceding term by multiplying the coeffi-*

cient of that term by the exponent of **a** *and dividing the product by* 1 *more than the exponent of* **b** *in that term.*

(4) *If* **a** *and* **b** *are both positive, the signs of all terms are positive; if* **b** *is negative, the signs of all even-numbered terms are negative and all odd-numbered terms positive.*

(5) *The number of terms is greater by* 1 *than the exponent of the binomial.*

These laws taken collectively, by means of which any power of any binomial may be written down directly, constitute what is known as the **Binomial Theorem.***

NOTE. — If the coefficients of each of the various powers of $a+b$, including the zero and first powers, are written down in order in a row, these rows of numbers form a triangular arrangement, as shown in the margin. This arrangement is known as *Pascal's Triangle.* Any number may be obtained by adding the two numbers, one to the right and the other to the left of it, in the row above. The triangle may thus be continued indefinitely.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
• • • • • •

In writing down the expansion of any power of a binomial by the Binomial Theorem, if the terms of the binomial contain coefficients or exponents, it is wise to inclose each of these terms in parentheses, and then to simplify in a second step the expression obtained. In this way confusion in finding the exponents and coefficients of the various terms of the expansion will be avoided.

EXAMPLE. — Find by the Binomial Theorem $(2x^2-3y^3)^4$.

$$(2x^2-3y^3)^4 = (2x^2)^4 - 4(2x^2)^3(3y^3) + 6(2x^2)^2(3y^3)^2 - 4(2x^2)(3y^3)^3$$
$$+ (3y^3)^4 = 16x^8 - 96x^6y^3 + 216x^4y^6 - 216x^2y^9 + 81y^{12}.$$

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* The Binomial Theorem has a long and interesting history. The ancient Hindus were familiar with special cases of the theorem, and used it in extracting square roots and cube roots. Sir Isaac Newton, the great mathematician and scientist, was the first to recognize the general truth of the theorem, about the year 1665; and he used it in making some of his greatest mathematical discoveries. Napier was led by use of the Binomial Theorem to the discovery of logarithms. (See Chapter XV.)

EXERCISES

Find by the Binomial Theorem:

1. $(a+b)^7$.
2. $(a-b)^7$.
3. $(m+n)^8$.
4. $(m-n)^8$.
5. $(x+y)^9$.
6. $(v-t)^{10}$.
7. $(h+k)^{12}$.
8. $(P-Q)^{13}$.
9. $(1+x)^4$.
10. $(1-n)^5$.
11. $(a+1)^6$.
12. $(2x+y)^5$.
13. $(a^2-b^2)^8$.
14. $(M-3N^2)^6$.
15. $(2V+3t)^6$.
16. $(1-2x^3)^4$.
17. $(2p^2+q)^7$.
18. $(1-3V^3)^5$.
19. $(y^3+z^3)^{10}$.
20. $(1+2b^4)^8$.
21. $(1-\frac{1}{2}n)^6$.
22. $(\frac{1}{2}E^2+\frac{1}{3}F^2)^6$.
23. $(\frac{2}{3}a^4-1)^5$.
24. $(6R^2+\frac{1}{2})^4$.

Find the first four terms of:

25. $(a+b)^{18}$.
26. $(a-b)^{20}$.
27. $(1+N)^{40}$.
28. $(t-1)^{30}$.
29. $(x+2)^{20}$.
30. $(M^2-2N)^{24}$.
31. $(w^2-3v^2)^{22}$.
32. $(\frac{1}{2}x+2y)^{30}$.
33. $\left(n-\frac{2}{n}\right)^{40}$.
34. $\left(\frac{2a}{b}+\frac{b^2}{4a^2}\right)^{16}$.
35. Write the first five terms of $(a-b)^n$.

Compute the following correct to two decimal places:

36. $(1.1)^{12}$.

SUGGESTION. — $(1.1)^{12}=(1+.1)^{12}=$ etc.

37. $(1.2)^{10}$. 38. $(1.04)^8$. 39. $(.31)^{10}$. 40. $(2.5)^{12}$.

41. Form the first ten lines of Pascal's Triangle.

143. Proof for Positive Integral Exponents. — That the Binomial Theorem holds true for all positive integral powers of any binomial may be proved as follows:

Assuming that the theorem holds true for the nth power of $a+b$, where n is any positive integer, we may write $(a+b)^n=$

$$a^n+na^{n-1}b+\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3+\cdots+b^n.$$

Multiplying both members of this equation by $a+b$,

$$(a+b)^{n+1}=a^{n+1}+(n+1)a^nb+\frac{(n+1)(n)}{1\cdot 2}a^{n-1}b^2$$
$$+\frac{(n+1)(n)(n-1)}{1\cdot 2\cdot 3}a^{n-2}b^3+\cdots+b^{n+1}.$$

Now it is seen that this value of $(a+b)^{n+1}$ is of the same form as the value of $(a+b)^n$, and that the result is the same as if it had been obtained directly by use of the Binomial Theorem.

Hence, if the theorem holds true for the nth power of $a+b$, it holds true for the $(n+1)$th power.

But it is proved by actual multiplication that the theorem holds true for the sixth power. Hence, since it holds for the sixth power, it holds for the seventh power. Now since it holds for the seventh power, it holds also for the eighth power; since it holds for the eighth power, it holds also for the ninth power; etc.

Hence, it holds true for all positive integral powers of $a+b$.

By similar proof it may be shown to hold for all positive integral powers of $a-b$.

NOTE. — It is unsafe to reason that just because a law holds true for a few, or even many special cases, it always holds true. Certain laws have been shown to hold true for many special cases, yet finally fail. Thus, the ancient Chinese thought that if $2^{n-1}-1$ were exactly divisible by n, then n must be a prime number. This holds true for all values of n up to 341. But for this value it fails. Hence the necessity is seen of finding a general proof of every law, such as that in this section. Mathematics undertakes to find general proofs of all of its principles.

144. Negative or Fractional Powers of Binomials. — The Binomial Theorem holds true also for negative and fractional powers of a binomial $a+b$, provided that the absolute value of a is greater than that of b, except that in these cases the number of terms in the expansion is unlimited. The proof is too difficult for this course.

EXAMPLE 1. — Expand to 5 terms $(1+x^2)^{-2}$.

$$(1+x^2)^{-2}=(1)^{-2}+(-2)(1)^{-3}(x^2)+(3)(1)^{-4}(x^2)^2+(-4)(1)^{-5}(x^2)^3$$
$$+(5)(1)^{-6}(x^2)^4+\cdots$$
$$=1-2x^2+3x^4-4x^6+5x^8-\cdots.$$

Example 2. — Expand to 5 terms $(1-2n)^{\frac{1}{2}}$.

$$(1-2n)^{\frac{1}{2}} = (1)^{\frac{1}{2}} - (\tfrac{1}{2})(1)^{-\frac{1}{2}}(2n) + (-\tfrac{1}{8})(1)^{-\frac{3}{2}}(2n)^2 - (\tfrac{1}{16})(1)^{-\frac{5}{2}}(2n)^3 + (-\tfrac{5}{128})(1)^{-\frac{7}{2}}(2n)^4 - \cdots$$
$$= 1 - n - \tfrac{1}{2}n^2 - \tfrac{1}{2}n^3 - \tfrac{5}{8}n^4 - \cdots.$$

EXERCISES

Expand to 5 terms:

1. $(1+n)^{-1}$.
2. $(1-t)^{-3}$.
3. $(1+x)^{\frac{1}{2}}$.
4. $(1-R)^{\frac{1}{3}}$.
5. $(1+a)^{\frac{1}{4}}$.
6. $(1-x^3)^{-1}$.
7. $(1+k^2)^{-2}$.
8. $(1+P^2)^{-\frac{1}{2}}$.
9. $(1+4a^2)^{-\frac{1}{2}}$.
10. $(1-9y^3)^{-\frac{1}{3}}$.
11. $(a^2-b^2)^{-\frac{1}{4}}$.
12. $(x^{\frac{1}{2}}+y^{\frac{1}{2}})^{\frac{1}{2}}$.
13. $(m^{-\frac{1}{2}}-n^{-\frac{1}{2}})^{-\frac{1}{2}}$.
14. $(2v^{\frac{2}{3}}+t^{\frac{3}{2}})^{\frac{2}{3}}$.
15. $(3p^{-\frac{3}{4}}-2q^{-1})^{-\frac{1}{3}}$.

145. Extraction of Roots by the Binomial Theorem. — The Binomial Theorem may be used to extract roots of arithmetical numbers. The process is best shown by an example.

Example. — Find to 3 decimal places the value of $\sqrt[5]{29}$.

$$\sqrt[5]{29} = 29^{\frac{1}{5}} = (2^5 - 3)^{\frac{1}{5}}$$
$$= (2^5)^{\frac{1}{5}} - (\tfrac{1}{5})(2^5)^{-\frac{4}{5}}(3) + \frac{(\frac{1}{5})(-\frac{4}{5})}{1\cdot 2}(2^5)^{-\frac{9}{5}}(3)^2$$
$$- \frac{(\frac{1}{5})(-\frac{4}{5})(-\frac{9}{5})}{1\cdot 2\cdot 3}(2^5)^{-\frac{14}{5}}(3)^3 + \cdots$$
$$= 2 - \tfrac{3}{80} - \tfrac{9}{6400} - \tfrac{81}{1024000} - \cdots$$
$$= 2 - 0.0375 - 0.0014 - 0.0001 - \cdots$$
$$= 1.961, \text{ approximately.}$$

A similar process may be used in any case. Hence the rule:

To find a root of a number, separate the number into two parts, the first of which is the nearest possible perfect power of which the required root can be found; then expand the resulting binomial by the Binomial Theorem, and combine the values of the terms thus obtained.

Thus, $\sqrt[3]{65} = \sqrt[3]{64+1} = (4^3+1)^{\frac{1}{3}}$; $\sqrt[4]{623} = \sqrt[4]{625-2} = (5^4-2)^{\frac{1}{4}}$.

It is evident that the first few terms of the expansion will give a close approximation to the value of the root, if the successive terms decrease in value rapidly; that is, if the second term of the binomial is much smaller than the first.

NOTE. — A shorter method of finding the approximate value of any root of a number is by the use of logarithms, as shown in the next chapter.

EXERCISES

Find to 3 places of decimals the value of:

1. $\sqrt[4]{15}$.
2. $\sqrt[3]{28}$.
3. $\sqrt[3]{66}$.
4. $\sqrt[3]{25}$.
5. $\sqrt[3]{130}$.
6. $\sqrt[3]{219}$.
7. $\sqrt[3]{9}$.
8. $\sqrt[4]{261}$.
9. $\sqrt[5]{240}$.
10. $\sqrt[6]{67}$.
11. $\sqrt[5]{3120}$.
12. $\sqrt[6]{730}$.
13. $\sqrt[4]{85}$.
14. $\sqrt[5]{29}$.
15. $\sqrt[6]{70}$.

146. The *r*th Term of the Binomial Expansion. — In the expansion $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots + b^n$, it is seen that the numerator of the coefficient of the 3rd term is the product of 2 factors, $n(n-1)$; the numerator of the coefficient of the 4th term is the product of 3 factors, $n(n-1)(n-2)$; etc. In general, the numerator of the coefficient of the rth term is the product of $r-1$ factors, $n(n-1)(n-2)\cdots$.

The denominator of the coefficient of the third term is $1\cdot 2$; of the fourth term, $1\cdot 2\cdot 3$; and in general, of the rth term, $1\cdot 2\cdot 3\cdot 4 \cdots$ to $r-1$ factors.

The exponent of a in the third term is $n-2$; in the fourth term $n-3$; and in general, in the rth term, $n-(r-1)$.

The exponent of b in the third term is 2; in the fourth term 3; and in general, in the rth term, $r-1$.

Hence, in the expansion of $(a+b)^n$ the

$$\textbf{\textit{r}th term} = \frac{n(n-1)(n-2)\cdots \text{ to } r-1 \text{ factors}}{1\cdot 2\cdot 3\cdots \text{ to } r-1 \text{ factors}}\,a^{n-(r-1)}b^{r-1}.$$

In the expansion of $(a-b)^n$, the rth term is the same expression, except that it is preceded by $+$ only when r is *odd*, and by $-$ when r is *even*.

By substituting the values of a, b, n, and r in this formula for the rth term, we may write down directly any desired term of the expansion of any power of any binomial, without writing the preceding terms.

Example. — Find the eighth term in the expansion of $(1-t^3)^{-2}$.

Here $a=1$, $b=t^3$, $n=-2$, and $r=8$.

Hence, the eighth term

$$=-\frac{(-2)(-3)(-4)(-5)(-6)(-7)(-8)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}(1)^{-9}(t^3)^7$$

$$=8\,t^{21}.$$

EXERCISES

Find:

1. The fourth term in the expansion of $(a^2+b^2)^6$.
2. The fifth term in the expansion of $(1+x)^{-4}$.
3. The eighth term in the expansion of $(1-t^2)^{20}$.
4. The sixth term in the expansion of $(m-n)^{40}$.
5. The fourth term in the expansion of $(x^3+y^3)^{12}$.
6. The tenth term in the expansion of $(A^4-B^2)^{-1}$.
7. The seventh term in the expansion of $(1+2\,x^3)^{\frac{1}{2}}$.
8. The twelfth term in the expansion of $(1-V^{-1})^{-\frac{1}{2}}$.
9. The eighth term in the expansion of $(x^{\frac{1}{2}}+y^{\frac{1}{2}})^{\frac{1}{4}}$.
10. The ninth term in the expansion of $(R^{-1}-r^2)^{-\frac{1}{3}}$.
11. The fourth term in the expansion of $(2\,a+b^2)^{\frac{1}{2}}$.
12. The sixteenth term in the expansion of $(1-a^{-\frac{1}{2}})^{-\frac{1}{4}}$.
13. The sixth term in the expansion of $(2-k)^{-3}$.
14. The eighth term in the expansion of $(2+\frac{1}{2}\,n^2)^{\frac{1}{2}}$.
15. The eleventh term in the expansion of $(x^{\frac{2}{3}}+x^{-\frac{2}{3}})^{\frac{2}{3}}$.
16. The fifth term in the expansion of $(e^{\frac{1}{2}}-e^{-\frac{1}{2}})^{-2}$.

SUPPLEMENTARY EXERCISES

Expand:

1. $(\sqrt{x}-\sqrt{y})^8$.
2. $(\sqrt[3]{a^2}+\sqrt[3]{b^2})^6$.
3. $(m\sqrt{m}+n\sqrt{n})^{10}$.
4. $(e^2\sqrt[4]{e}-1)^{12}$.
5. $\left(\frac{\sqrt{a}}{b}+\frac{\sqrt{b}}{a}\right)^{10}$.
6. $\left(\frac{3m}{n^2}+n\sqrt{m}\right)^4$.
7. $\left(\frac{\sqrt{x}}{\sqrt[3]{y^2}}-\sqrt{\frac{y}{x}}\right)^6$.
8. $\left(\frac{a\sqrt[3]{a}}{\sqrt[4]{b}}+\frac{\sqrt[5]{b}}{2a}\right)^4$.

By grouping terms, express as binomials and expand:

9. $(1+x-x^2)^4$.

SUGGESTION. — $(1+x-x^2)^4=([1+x]-x^2)^4$
$=[1+x]^4-4[1+x]^3[x^2]+\cdots$.
Now expand $[1+x]^4$, $[1+x]^3$, etc., and combine terms.

10. $(x^2+y^2+z^2)^3$.
11. $(2-M+N)^4$.
12. $(a-b+c-d)^4$.
13. $(1-t+t^2-t^3)^3$.
14. $(2a-b+3c^2)^4$.
15. $(n^{-2}+1+n^2)^4$.

16. Write the formula for the $(r+1)$th term in the expansion of $(a+b)^n$.

17. Write the formula for the $(r-1)$th term in the expansion of $(a+b)^n$.

Find:

18. The sixth term in the expansion of $(v\sqrt{w}-w\sqrt{v})^8$.

19. The fifth term in the expansion of $\left(\frac{\sqrt[5]{a^2}}{b}+\frac{\sqrt[5]{b^2}}{a}\right)^{10}$.

20. Find to four decimal places the value of $\sqrt[3]{220}$.

CHAPTER XV

LOGARITHMS

147. Logarithms. — Long and complicated computations, such as are encountered often in practical work, may be shortened and simplified by use of **logarithms.**

Thus, to compute the area of a triangle by the formula *Area* $= \sqrt{s(s-a)(s-b)(s-c)}$ one must find the product of four factors and then take the square root of this product. By the use of logarithms, as we shall see later, the multiplication is replaced by simple column addition, and the extraction of the root by division by 2.

The **logarithm** of a number is the *exponent* which indicates the power to which a given base must be raised to produce that number. That is, if $b^e = n$, then e is the logarithm of n to the base b, and is written

$b^e = n$

$$e = \log_b n.$$

Thus, since $2^3 = 8$, $\log_2 8 = 3$. Since $5^4 = 625$, $\log_5 625 = 4$. Since $4^{-2} = \frac{1}{16}$, $\log_4 \frac{1}{16} = -2$.

By the definition of a logarithm, the equations $e = \log_b n$ and $b^e = n$ express the same relation.

EXERCISES

Express the following relations in terms of logarithms:

1. $2^5 = 32$.

SOLUTION. — $\log_2 32 = 5$.

2. $2^6 = 64$.
3. $5^2 = 25$.
4. $3^4 = 81$.
5. $4^3 = 64$.
6. $5^3 = 125$.
7. $7^3 = 343$.
8. $10^4 = 10000$.
9. $5^6 = 15625$.
10. $6^{-2} = \frac{1}{36}$.
11. $4^{-3} = \frac{1}{64}$.
12. $16^{\frac{1}{4}} = 2$.
13. $8^{-\frac{2}{3}} = \frac{1}{4}$.

Express the following relations by means of exponents:

14. $\log_3 9 = 2$.

SOLUTION — $3^2 = 9$.

15. $\log_2 16 = 4$.
16. $\log_4 16 = 2$.
17. $\log_3 27 = 3$.
18. $\log_{10} 1000 = 3$.
19. $\log_6 216 = 3$.
20. $\log_4 256 = 4$.
21. $\log_8 4 = \frac{2}{3}$.
22. $\log_{27} 81 = \frac{4}{3}$.
23. $\log_7 \frac{1}{49} = -2$.
24. $\log_3 \frac{1}{243} = -5$.
25. $\log_{10} 0.001 = -3$.
26. $\log_{100} 0.001 = -\frac{3}{2}$.

Find the values of the following logarithms:

27. $\log_2 16$.
28. $\log_3 27$.
29. $\log_{10} 100$.
30. $\log_{10} 0.0001$.
31. $\log_4 64$.
32. $\log_2 0.5$.
33. $\log_2 0.25$.
34. $\log_8 \frac{1}{64}$.
35. $\log_6 216$.
36. $\log_5 \frac{1}{125}$.
37. $\log_{10} 1$.
38. $\log_4 1$.

To the base 4 what numbers have the following logarithms?

39. 1.
40. 3.
41. 5.
42. -2.
43. -4.
44. $\frac{1}{2}$.
45. $-\frac{1}{2}$.
46. $-\frac{3}{2}$.

148. Common Logarithms. — The logarithm of a number to the base 10 is called a **common logarithm**.

Thus, $\log_{10} 6$, $\log_{10} \frac{1}{16}$, $\log_{10} 0.001$ are *common logarithms*.

The logarithms of all arithmetical numbers to any one base constitute a **system of logarithms**. The common logarithms of all arithmetical numbers constitute what is called the **common system** of logarithms.*

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* The common system of logarithms was first worked out by an Englishman named Henry Briggs, in the early part of the seventeenth century. Another important system is the **Napierian** system, named after John Napier, an Englishman who was a contemporary of Briggs. Napier was the real inventor of logarithms, and Briggs got his inspiration from Napier. The base of the Napierian system is the sum of an infinite series $1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots$. The sum of this series is approximately 2.7182818, and is represented in mathematical writings by the letter e.

While the common system is used in practical computations, the Napierian system is used in theoretical investigations.

The common system is superior to any other system of logarithms for practical use because its base 10 is also the base of our decimal system of numbers.

The rest of this chapter will be devoted to common logarithms, and in the following sections the base will be understood to be 10 and will not be written.

149. Characteristic and Mantissa. — From the definition of a logarithm,

$$\begin{array}{llll} & \text{since } 10^0 = 1, & \log 1 & = 0; \\ & \text{since } 10^1 = 10, & \log 10 & = 1; \\ & \text{since } 10^2 = 100, & \log 100 & = 2; \\ & \text{since } 10^3 = 1000, & \log 1000 & = 3; \text{ etc.} \\ \text{And,} & \text{since } 10^{-1} = 0.1, & \log 0.1 & = -1; \\ & \text{since } 10^{-2} = 0.01, & \log 0.01 & = -2; \\ & \text{since } 10^{-3} = 0.001, & \log 0.001 & = -3; \text{ etc.} \end{array}$$

It is evident from the above that the logarithm of a positive integral power of 10 is a positive integer, and that the logarithm of a negative integral power of 10 is a negative integer.

Furthermore, since 65, say, is greater than 10^1 and less than 10^2, log 65 is greater than 1 and less than 2. Hence, $\log 65 = 1 +$ *some decimal.* Likewise, since 382 is greater than 10^2 and less than 10^3, log 382 is greater than 2 and less than 3. Hence, $\log 382 = 2 +$ *some decimal.* Evidently, in general, the logarithm of any positive number, except a positive or negative integral power of 10, consists of *an integral and a decimal part.*

Thus, $\log 825 = 2.916+$.

The integral part of a logarithm is called its **characteristic,** and the decimal part is called its **mantissa.**

150. Determination of the Characteristic. — A number having one figure in its integral part lies between 10^0 and 10^1. Hence, its logarithm lies between 0 and 1; *i.e.* it equals $0 +$ *some decimal.* A number having two figures in its integral part lies between 10^1 and 10^2. Hence, its logarithm lies between 1 and 2; *i.e.* it equals $1 +$ *some decimal.* Similarly, if a number has three figures in its integral part, its logarithm lies between 2 and 3; *i.e.* it equals $2 +$ *some decimal,* and so on. Therefore,

If a number is greater than 1, *the characteristic of its logarithm is positive, and is less by* 1 *than the number of figures in the integral part of the number.*

Thus, in log 6841.27, the characteristic is 3. In log 362.781, the characteristic is 2.

Again, a number less than 1, and having no zero immediately following the decimal point, lies between 10^0 and 10^{-1}. Hence, the logarithm lies between 0 and -1; *i.e.* it equals $-1 +$ *some decimal.* A number less than 1, and having one zero immediately following the decimal point, lies between 10^{-1} and 10^{-2}. Hence, its logarithm lies between -1 and -2; *i.e.* it equals $-2 +$ *some decimal.* Similarly, if a number less than 1 has two zeros immediately following the decimal point, its logarithm lies between -2 and -3; *i.e.* it equals $-3 +$ *some decimal,* and so on. Therefore,

If a number is less than 1, *the characteristic of its logarithm is negative, and is greater by* 1 *than the number of zeros immediately following the decimal point.*

Thus, in log 0.0683 the characteristic is -2. In log 0.0008 the characteristic is -4. In log 0.3974 the characteristic is -1.

In writing the logarithm of a number the mantissa is always positive, and if the characteristic is negative, the minus sign is written above the characteristic to signify that it applies to the characteristic alone.

Thus $\log 0.00357 = \bar{3}.5527$. This means $-3 + 0.5527$.

EXERCISES

Find the characteristics in the logarithms of:

1. 28.
2. 521.
3. 0.07.
4. 0.00064.
5. 6.871.
6. 98743.
7. 1.426.
8. 3.1416.
9. 0.729.
10. 5280.
11. 12.65.
12. 0.0082.

151. Tables of Mantissas. — Since changing the position of the decimal point in a number is equivalent to multiplying or dividing the number by some integral power of 10, which has the effect of changing its logarithm by some whole number only,

The common logarithms of numbers which do not differ except in the positions of the decimal point have the same mantissa.

For example, $\log 2680 = 3.4281$; $\log 268 = 2.4281$; $\log 26.8 = 1.4281$; $\log 2.68 = 0.4281$; $\log 0.268 = \bar{1}.4281$; $\log 0.0268 = \bar{2}.4281$. The mantissas are the same in all of these logarithms.

Since the mantissa of the logarithm of a number depends only upon the sequence of figures, and not upon the position of the decimal point, it is necessary to know only the mantissas of the logarithms of whole numbers. The common logarithms of sets of consecutive whole numbers have been computed and tabulated. On pages 252 and 253 is a table which contains the mantissas of the logarithms of all whole numbers from 1 to 1000. Logarithms may be computed to any number of decimal places, the number taken depending upon the degree of accuracy required in their use. In the table in this book the mantissas are computed to four decimal places. In this table the first two figures of each number are found in the column headed N, and the third figure in the horizontal line at the top of the table. The mantissas, with decimal points omitted, are found in the columns headed 0, 1, 2, 3, etc.

To find the logarithm of a number, find its characteristic by the rules in § 150, and look in the table for the mantissa. If a number has less than three figures, annex ciphers until it has three figures, in looking for the mantissa.

Thus to find the mantissa of log 38, look for the mantissa of log 380. To find the mantissa of log 3, look for the mantissa of log 300.

152. To find the Logarithm of a Given Number. — (*a*) *When the given number contains not more than three figures,* the mantissa of its logarithm is obtained directly from the table.

Example 1. — Find log 32.7.

By § 150 its characteristic is 1. The required mantissa is the mantissa of log 327.

Look for 32 in the column headed N in the table. Looking along the horizontal line of numbers opposite 32, to the column headed 7, we find .5145, the required mantissa. Hence,

$$\log 32.7 = 1.5145.$$

Example 2. — Find log 0.91.

By § 150 the characteristic is — 1. The required mantissa is the mantissa of log 910. This is opposite 91 in the column headed 0, and is seen to be .9590. Hence,

$$\log 0.91 = \bar{1}.9590.$$

(*b*) *When the given number contains more than three figures,* use is made of the principle that when the difference of two numbers is small compared with either of them, the difference of the numbers is approximately proportional to the difference of their logarithms.

Example 3. — Find log 2.8465.

Shift the decimal point until it follows the third figure.

The required mantissa is that of log 284.65.

Now 284.65 is greater than 284 by .65.

The mantissa of log 284 = .4533.

The mantissa of log 285 = .4548.

Subtracting, .4548 — .4533 = .0015.

Hence, an increase of 1 in 284 causes an increase of .0015 in the corresponding mantissa. Therefore, an increase of .65 will cause an increase of .65 × .0015, or approximately .0010, in the mantissa.

Adding .0010 to the mantissa of log 284 gives .4543.

Hence, attaching the characteristic 0,

$$\log 2.8465 = 0.4543.$$

To find the mantissa of the logarithm of a number of more than three figures, set the decimal point after the third figure. Find in the table the mantissas corresponding to the integral part of the number thus pointed off and to the number 1 greater. Take the difference of these mantissas, and multiply the result by the decimal part of the number as pointed off. Add this product to the smaller mantissa. This gives the mantissa required.

EXERCISES

Find the logarithms of:

1. 215.
2. 673.
3. 940.
4. 717.
5. 4.62.
6. 19.9.
7. 830.
8. 16.
9. 29.
10. 8.5.
11. 100.
12. 900.
13. 1.
14. 2.
15. 1684.
16. 34.27.
17. 100.5.
18. 926.81.
19. 0.8632.
20. 0.00315.
21. 0.06843.
22. 4268.4.
23. 1.096.
24. 0.00012.
25. 99.99.
26. 2031.7.
27. 0.0083326.
28. 0.50416.
29. 68593.
30. 0.074803.

153. To find a Number whose Logarithm is Given. —

(*a*) *When the given mantissa is found in the table*, the sequence of figures of the required number may be obtained by reversing the process (*a*), § 152. The position of the decimal point is determined by reversing the rules in § 150.

EXAMPLE. — Find the number whose logarithm is $\bar{2}.9325$.

Looking in the table, we find the mantissa .9325 opposite 85 and in the column headed 6. Hence,

$$.9325 = \text{mantissa of } \log 856.$$

Since the characteristic is -2, the number must be less than 1, and must contain one zero immediately to the right of the decimal point. Hence,

$$\bar{2}.9325 = \log 0.0856.$$

(*b*) *When the given mantissa is not found in the table*, the required number is obtained by reversing the process of (*b*), § 152.

EXAMPLE 2. — Find the number whose logarithm is 3.6496.

The mantissa .6496 is not in the table, but the mantissas of the table between which it lies in value are .6493 and .6503.

Now .6493 = mantissa of log 446,
and .6503 = mantissa of log 447.
Subtracting, .6496 − .6493 = .0003,
and .6503 − .6493 = .0010.

Hence, an increase of .0010 in the mantissa .6493 causes an increase of 1 in the corresponding number. Therefore, an increase of .0003 will cause an increase of .0003 ÷ .0010, or .3, in the number.

Adding .3 to 446 gives 446.3.

Therefore, .6496 = mantissa of log 446.3.

Since the characteristic is 3, therefore

$$3.6496 = \log 4463.$$

To find the number corresponding to a given mantissa that is not in the table, take from the table the two mantissas between which the value of the given mantissa lies, and note the numbers corresponding to them. Subtract the smaller mantissa of the table from the larger and also from the given one. Divide the smaller remainder by the larger remainder, and add the quotient obtained to the number corresponding to the smaller mantissa of the table. This gives the sequence of figures of the required number.

EXERCISES

Find the numbers whose logarithms are:

1. 2.6739.
2. 1.7388.
3. 3.0828.
4. 0.7910.
5. $\bar{2}.8971$.
6. $\bar{3}.9777$.
7. $\bar{1}.6180$.
8. 0.2279.
9. 1.7582.
10. 3.8615.
11. 0.1847.
12. $\bar{1}.4609$.
13. $\bar{2}.6304$.
14. 5.6887.
15. 2.4783.
16. 3.6028.
17. $\bar{3}.7255$.
18. 2.8980.
19. 4.0096.
20. 1.4703.
21. $\bar{2}.9765$.
22. 0.8460.
23. $\bar{1}.4072$.
24. $\bar{5}.0220$.
25. 3.0392.
26. $\bar{4}.4756$.
27. 0.8735.
28. 1.5734.

154. General Principles of Logarithms. — Since a logarithm is an *exponent*, the following general principles of logarithms may be proved by the laws of exponents (see Chapter XIII):

I. *The logarithm of a product equals the sum of the logarithms of its factors. That is,*

$$\log_a mn = \log_a m + \log_a n.$$

For, let $a^x = m$, and $a^y = n$.

Then $mn = a^x \cdot a^y = a^{x+y}$.

Hence, $\log_a mn = x + y = \log_a m + \log_a n$.

Evidently, this principle may be extended to hold for a product of any number of factors. In general,

$$\log_a mnp \cdots = \log_a m + \log_a n + \log_a p + \cdots.$$

II. *The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor. That is,*

$$\log_a \frac{m}{n} = \log_a m - \log_a n.$$

For, let $a^x = m$, and $a^y = n$.

Then $\frac{m}{n} = a^x \div a^y = a^{x-y}$.

Hence, $\log_a \frac{m}{n} = x - y = \log_a m - \log_a n$.

III. *The logarithm of a power of a number equals the logarithm of the number, multiplied by the exponent of the power. That is,*

$$\log_a n^p = p \log_a n.$$

For, let $a^x = n$.

Then $n^p = (a^x)^p = a^{px}$.

Hence, $\log_a n^p = px = p \log_a n$.

IV. *The logarithm of a root of a number equals the logarithm of the number, divided by the index of the root. That is,*

$$\log_a \sqrt[r]{n} = \frac{\log_a n}{r}.$$

For, let $a^x = n$.

Then $\sqrt[r]{n} = \sqrt[r]{a^x} = a^{\frac{x}{r}}$.

Hence, $\log_a \sqrt[r]{n} = \frac{x}{r} = \frac{\log_a n}{r}$, or $\frac{1}{r} \log_a n$.

By use of the above principles it is possible to replace the operations of multiplication and division by those of addition and subtraction, raising a number to a power by multiplication, and extraction of a root by division.

Example 1. — Express $\log_a \frac{xy^2}{z\sqrt{w}}$ in terms of $\log_a x$, $\log_a y$, $\log_a z$, and $\log_a w$.

$$\log_a \frac{xy^2}{z\sqrt{w}} = \log_a xy^2 - \log_a z\sqrt{w} \qquad \text{(By II)}$$
$$= \log_a x + \log_a y^2 - \log_a z - \log_a \sqrt{w} \qquad \text{(By I)}$$
$$= \log_a x + 2\log_a y - \log_a z - \tfrac{1}{2}\log_a w. \qquad \text{(By III, IV)}$$

EXERCISES

Express the following in terms of $\log_a x$, $\log_a y$, $\log_a z$, and $\log_a w$:

1. $\log_a xyzw.$
2. $\log_a x^2y^2.$
3. $\log_a 2\,xy^2z^3.$
4. $\log_a \frac{x^2y}{z^3}.$
5. $\log_a \frac{5\,x}{yzw}.$
6. $\log_a \frac{8\,x^2y^2}{z^3w}.$
7. $\log_a \left(\frac{xy}{zw}\right)^2.$
8. $\log_a x^{\frac{2}{3}}y^{-\frac{3}{4}}.$
9. $\log_a x^{\frac{1}{3}}y^{\frac{1}{3}}z^{-\frac{1}{4}}.$
10. $\log_a \frac{\sqrt{x}}{\sqrt{y}}.$
11. $\log_a \frac{\sqrt{x}\sqrt{y}}{z\sqrt{w}}.$
12. $\log_a \frac{\sqrt[3]{xy^2}}{\sqrt{z}\sqrt[3]{w}}.$
13. $\log_a \frac{\sqrt[4]{zy^3}}{\sqrt[3]{x}}.$
14. $\log_a \sqrt{\frac{x}{y}} + \log_a \sqrt{\frac{z}{w}}.$
15. $\log_a \sqrt[3]{\frac{x}{y}}\sqrt[4]{\frac{y}{x}}.$

Express:

16. $\log \sqrt[3]{1728}$ in terms of $\log 1728$.
17. $\log \sqrt{2}\sqrt[3]{9}$ in terms of $\log 2$ and $\log 9$.
18. $\log \frac{\sqrt{98}}{\sqrt[3]{56}}$ in terms of $\log 98$ and $\log 56$.

155. Computation by Logarithms. — By use of the principles in § 154 and the processes of § 153 computations otherwise long and tedious may be performed with ease.

Example. — Find the value of $\sqrt[3]{\dfrac{621.3 \times 3.247}{71.8}}$.

Taking the logarithm,

$$\log \sqrt[3]{\frac{621.3 \times 3.247}{71.8}} = \tfrac{1}{3}(\log 621.3 + \log 3.247 - \log 71.8).$$

$$\begin{array}{rl}
\log 621.3 = & 2.7933 \\
\log 3.247 = & \underline{0.5115} \\
 & 3.3048 \\
\log 71.8 = & \underline{1.8561} \\
 & 3\overline{)1.4487} \\
 & 0.4829
\end{array}$$

Now, $0.4829 = \log 3.04.$

Hence, $\sqrt[3]{\dfrac{621.3 \times 3.247}{71.8}} = 3.04$, approximately.

In using a logarithm with a negative characteristic, it is convenient usually to add some number to the logarithm, in order to make the characteristic positive, then to indicate the subtraction of the number added.

Example. — Find the fifth root of 0.00327.

Taking the logarithm,

$$\begin{aligned}
\log \sqrt[5]{0.00327} &= \tfrac{1}{5} \log 0.00327 \\
&= \tfrac{1}{5}(\bar{3}.5145).
\end{aligned}$$

Adding 5 to $\bar{3}.5145$, then indicating the subtraction,

$$\begin{aligned}
\tfrac{1}{5}(\bar{3}.5145) &= \tfrac{1}{5}(2.5145 - 5) \\
&= 0.5029 - 1 \\
&= \bar{1}.5029.
\end{aligned}$$

Now, $\bar{1}.5029 = \log 0.31855.$

Hence, $\sqrt[5]{0.00327} = 0.31855$, approximately.

If the logarithm with a negative characteristic is to be divided by some number, as in the above example, the number added to and subtracted from the logarithm should be exactly divisible by the divisor.

Thus, if the logarithm $\bar{3}.6217$ is to be divided by 7, the number added and subtracted may be 7, 14, or 70, or any other multiple of 7.

Similarly,

$$\begin{aligned}
\tfrac{1}{4} \text{ of } \bar{3}.7248 &= \tfrac{1}{4}(37.7248 - 40) \\
&= 9.4312 - 10 \\
&= \bar{1}.4312.
\end{aligned}$$

EXERCISES

Make the following computations by logarithms:

1. 71.6×0.327.
2. 3.86×24.3.
3. 1.068×0.0039.
4. 3.412×72.81.
5. 3.1416×8.16.
6. $0.144 \div 0.98$.
7. $681.7 \div 4.235$.
8. 1728×3.1416.
9. 1.414×0.0632.
10. 4617×0.00396.
11. 0.0786×0.0657.
12. $1.875 \div 0.1897$.
13. $498 \div 62.4$.
14. $9.12 \div 0.374$.
15. $0.019 \div 3.1416$.
16. $2.0037 \div 666$.
17. $7.29 \div 2.468$.
18. 0.996^6.
19. 1.009^{10}.
20. 1.786^5.
21. 0.06814^7.
22. 1.19^8.
23. $\sqrt[3]{4.68}$.
24. $\sqrt[3]{93.46}$.
25. $\sqrt[5]{9.2618}$.
26. $\sqrt[6]{0.09984}$.
27. $\sqrt[7]{20035}$.
28. $0.068^{\frac{5}{2}}$.
29. $(\frac{61}{27})^6$.
30. $(7\frac{5}{12})^4$.
31. $0.64 \times 7.23 \times 92.5$.
32. $82.6 \times 0.047 \times 1.289$.
33. $\sqrt{69 - \sqrt{1492}}$.
34. $\dfrac{2.476 \times 73.81}{0.524 \times 6184}$.
35. $\dfrac{5334 \times 0.0237}{47.4 \times 3.1416}$.
36. $\sqrt{\dfrac{628 \times 3.1416}{9.16^3}}$.
37. $\sqrt[3]{\dfrac{6.8 \times 0.785}{3.86^2}}$.
38. $\sqrt[5]{\dfrac{0.027^4 \times 32.6 \times \sqrt[4]{542}}{\sqrt[3]{412} \times \sqrt[5]{714} \times \sqrt[6]{628}}}$.

39. The circumference of a circle is $2\pi R$. Find the circumference of a circle in which $R = 36.5$ inches.

40. The area of a circle is πR^2. Find the area of a circle in which $R = 7.34$ inches.

41. The volume of a sphere is $\frac{4}{3}\pi R^3$. Find the volume of a sphere in which $R = 16.6$ inches.

42. Find the radius of a circle whose area is 92 square feet.

43. Find the radius of a sphere whose volume is 62 cubic feet.

44. In a right triangle the square of the hypotenuse equals the sum of the squares of the other two sides. Find the hypotenuse of a right triangle whose other two sides are 14 inches and 18 inches, respectively.

45. The area of a triangle $=\sqrt{s(s-a)(s-b)(s-c)}$, where a, b, and c are the sides, and $s=\frac{1}{2}(a+b+c)$.

Find the area of a triangle whose sides are 13 inches, 16 inches, and 21 inches respectively.

46. Find the area of a triangular building lot whose dimensions are 80 feet by 120 feet by 96 feet.

47. The formula $S_{2n}=\sqrt{2-\sqrt{4-S_n^2}}$ is used in computing the value of π.

If $S_n=1$, compute S_{2n} to four decimal places.

48. In the formula of Ex. 47, if $S_n=0.5176$, compute S_{2n} to four decimal places.

49. The distance of Venus from the sun is $93{,}000{,}000\sqrt[3]{\left(\frac{226}{365}\right)^2}$ miles. By simplifying this expression, find to the nearest hundred thousands of miles the distance of Venus from the sun.

50. In a steam engine, the average velocity v of the piston head, in inches per second, is computed by the formula $v=1.7\sqrt[3]{s\sqrt{\frac{1}{15}p}}$, where s is the distance over which the piston moves (in inches), and p the number of pounds of pressure of steam in the cylinder.

If $s=28.75$ inches and $p=120$ pounds, find v.

51. The area A of the cross section of a chimney, in square feet, required to carry off the smoke is computed by the formula

$$A=\frac{0.06\,P}{\sqrt{h}},$$

where P is the number of pounds of coal burned per hour, and h is the height of the chimney in feet.

What should be the area of the cross section of a chimney 72 feet high to carry off the smoke if 750 pounds of coal are burned per hour?

52. If p dollars are invested at $r\%$ compound interest, compounded annually, the value, or *amount*, at the end of t years will be $p(1+r)^t$.

Find the amount at the end of 6 years of $6000 compounded annually at 4%.

53. Find the amount at the end of 8 years of $5500 compounded annually at $3\frac{1}{2}\%$.

54. If the interest is compounded semiannually, the formula for the amount in Ex. 52 becomes $p(1+\frac{1}{2}r)^{2t}$.

I invest $200 in a savings bank that pays 3% interest compounded semiannually. To what will the sum amount in 10 years?

156. Exponential Equations. — An equation in which the unknown number occurs as an exponent is called an **exponential equation.**

The root of an exponential equation may be obtained by the aid of logarithms.

Example 1. — Solve $5^x = 267$.

Taking the logarithms of both members,

$$x \log 5 = \log 267.$$

Hence, $$x = \frac{\log 267}{\log 5} = \frac{2.4265}{0.6990} = 3.47+.$$

Example 2. — Solve $5^{3x-1} = 4^{x-1}$.

Taking the logarithms of both members,

$$(3x-1)\log 5 = (x-1)\log 4.$$

Removing parentheses, transposing, etc.,

$$\begin{aligned} x &= \frac{\log 5 - \log 4}{3\log 5 - \log 4} \\ &= \frac{0.6990 - 0.6021}{2.0970 - 0.6021} \\ &= 0.0648. \end{aligned}$$

EXERCISES

Solve:

1. $3^x = 81$.
2. $5^x - 25 = 0$.
3. $2^{2x} = 64$.
4. $3^x = 15$.
5. $2^{3x} = 25$.
6. $4^{x-1} = 3^{x+1}$.
7. $2^{x-1} = 0.3^{2x-5}$.
8. $5^{2x} - 12^{x+1} = 0$.
9. $4^{x+1} = 8 \cdot 2^{x+2}$.
10. $\sqrt[3]{2^{x+2}} = \sqrt{2^{x-3}}$.
11. $\sqrt{3^{11-x}} = 3^{8-x}$.
12. $\sqrt[3]{3^{x+7}} = 2^x$.

13. In how many years will \$1 double itself at 4 %, interest compounded annually?

SUGGESTION. — See formula in Ex. 52, § 155. If t is the number of years required, show that $1.04^t = 2$.

14. In how many years will \$1 double itself at 3 %, interest compounded annually?

15. In how many years will \$1 double itself at 4 %, interest compounded semiannually? (See formula in Ex. 54, § 155.)

16. In how many years will \$2000 amount to \$5309.41, interest compounded annually at 5 %?

SUPPLEMENTARY EXERCISES

1. Show that the logarithm of 1 to any base is 0.

2. Show that the logarithm of the base itself is 1.

3. The logarithm of $\frac{1}{x}$ is called the **cologarithm** of x. Show that $\text{colog}\, x = -\log x$.

Find the values of:

4. $$\sqrt[3]{\frac{27^4 \times 0.028^2 \times 16.75^3}{\sqrt{629} \times \sqrt[4]{8710} \times \sqrt[5]{126.3}}}.$$

5. $$98.7\sqrt[5]{\frac{0.068 \times \sqrt[3]{28.59}}{\sqrt[3]{6} \times \sqrt[6]{206.4} \times \sqrt{0.0091}}}.$$

Solve:

6. $\sqrt[x]{6.82} = 2.36$.
7. $3^{2x} - 4(3^x) - 12 = 0$.
8. $2^{4x} - 3(2^{2x}) = 4$.
9. $2(4^{2x}) - 4^x - 6 = 0$.

10. The sum of money p that placed at interest compounded annually will amount to a given sum a in a number of years at a given rate is called the *present worth* of a. If p dollars at $r\%$ will amount to a dollar in t years, interest compounded annually, then $p = \frac{a}{(1+r)^t}$.

Find the present worth of \$12,000 at 3%, interest compounded annually for 10 years.

11. Find the present worth of \$2400 at 4%, interest compounded annually for 6 years.

12. Find the present worth of \$3200 at $3\frac{1}{2}$%, interest compounded annually for 12 years.

MISCELLANEOUS EXERCISES

Factor:

1. $(9n - a^2)^2 - a^2$.
2. $(2a - b)^2 - (a + 2b)^2$.
3. $4x^2 - 4a^2 + 4at - t^2$.
4. $N^4 + 9N^2 + 81$.
5. $V^5 - 11V^3 + 30V$.
6. $4k^2 + 6k + 2$.
7. $30R^2 + 56Rr + 24r^2$.
8. $p^4 - 11p^2q^2 + q^4$.
9. $12N^2 - 5N - 2$.
10. $2v^4 - 3v^2 - 14$.
11. $x^8 - 1$.
12. $a^5b - ab^5$.
13. $V^{16} - T^{16}$.
14. $m^{12} + n^{12}$.
15. $1 + 8a^9$.
16. $a^2 - ab - b - 1$.
17. $t^3 - t^2 - t + 1$.
18. $4 + 9y^4 - 37y^2$.
19. $(n^2 - n)^3 - 8$.
20. $H^3 + H^2 - H - 1$.
21. $v^3 - 7v^2 + 14v - 8$.
22. $a^2b^2 - a^2 - b^2 + 1$.
23. $(a^2 + b^2 - c^2)^2 - 4a^2b^2$.
24. $v^4 - v^2 + 12vw - 36w^2$.
25. $R^3 + R + N^3 + N$.
26. $h^2 - t^2 - h - t$.
27. $n^3 - 6n^2 + 11n - 6$.
28. $A^3 - 27A - 10$.
29. $m^5 - 32n^{10}$.
30. $x^5 - x^4 - x^3 + x$.
31. $e^{3n} + e^{2n} + e^n + 1$.
32. $3G^2 + (3a + b)G + ab$.

33. $ab^2 + ac^2 + a^2b + a^2c + bc^2 + b^2c + 2\,abc.$

34. $25\,n^2 - 20\,n + 4 - 4\,a^2 - 9\,b^2 - 12\,ab.$

35. $6\,x^{2n+6} - 25\,x^{n+3}y^{n-1} + 4\,y^{2n-2}.$

36. $xy^3 - x^3y + xz^3 - x^3z + yz^3 - y^3z.$

37. $n^2 + (a + b + c)\,n + ab + ac.$

38. $x^2 + a^2 - (b^2 + c^2) - 2(bc - ax).$

39. $mt^3 + ant^2 - b^2nt - ab^2m + nt^3 + amt^2 - b^2mt - ab^2n.$

40. Subtract $3\,n^3 - 4\,n + 2$ from $5\,n^2 - 2\,n - 1$, then add the difference to $5\,n^2 + 4$.

41. Simplify $a - [-3\,a - (a - 5) - (4 - 2\,a)]$.

42. Divide $a^2 + ab - 3\,ac - 2\,b^2 + 9\,bc - 10\,c^2$ by $a - b + 2\,c$.

43. Simplify $(r^n + 1 - r^{\frac{1}{n}})(r^{2n} - r^n - r^{\frac{1}{n}} - r^{\frac{2}{n}})$.

44. Find the H. C. F. of $3\,t^3 + 7\,t^2 + 3\,t + 2$ and $t^3 + 5\,t^2 + 6\,t$.

45. Find the L. C. M. of $7\,mx^2(x^3 - 3\,m^2x + 2\,m^3)$, $25\,n^3(x^2 - m^2)^2$ and $35\,m^3n\,(x^2 + mx - 2\,m^2)$.

46. Find the H. C. F. and the L. C. M. of $x^4 - y^4$, $x^3 + y^3$, and $x^3 + 2\,x^2y + 2\,xy^2 + y^3$.

47. Is $a^{15} + b^{15}$ divisible by $a + b$? By $a - b$? Give proof.

48. Find the L. C. M. of $2p^3 - 5p^2 - 22p - 15$ and $6p^4 - 21p^3 - 41p^2 - 14p - 30$.

49. Find the value of n for which $a^2 - a + 1$ is a factor of $6\,a^4 - 2\,a^3 + 2\,na^2 + 2\,a + n$.

Find the square root of:

50. $9 - 6\,n - 29\,n^2 + 22\,n^3 + 21\,n^4 - 20\,n^5 + 4\,n^6.$

51. $1 + 4\,y^3 - 2\,y^2 - 4\,y^5 - y^4 + 6\,y^6 - 4\,y^7 + y^8.$

52. $x^{-4} - 2\,x^{-3}y^{\frac{1}{2}} - 2\,x^{-1}y^{\frac{3}{2}} + 3\,x^{-2}y + y^2.$

53. What fraction must be added to $\dfrac{4\,n}{(n+1)(n-1)^2} - \dfrac{n+1}{(n-1)^2}$ to make it equal to 1?

Simplify:

54. $\dfrac{k+1}{12k^3+8k^2-3k-2} \div \dfrac{2k+1}{12k^3-10k^2-6k+4}$.

55. $\dfrac{1}{r^3-r} - \left(\dfrac{1}{r^3-1} - \dfrac{1}{r^3+1}\right)$.

56. $\dfrac{x^2(y-z)^3+y^2(z-x)^3+z^2(x-y)^3}{xy+yz+zx}$.

57. $\left(\dfrac{m-n}{m+n} - \dfrac{m^2-n^2}{m^2+n^2}\right) \div \left(\dfrac{m+n}{m-n} + \dfrac{m^2+n^2}{m^2-n^2}\right)$.

58. $\dfrac{x^2-1}{y+y^2} \times \left(1+\dfrac{y}{1-y}\right) \times \dfrac{y^2-1}{x+1}$.

59. $$\frac{1-\left(\dfrac{1}{1-n} - \dfrac{m^2+n^2-m+n}{1-n^2}\right)}{\left(\dfrac{m}{1+m}-1\right)\left(\dfrac{1-m^2}{1+n}\right)}.$$

60. If $\dfrac{a}{b+c}=x$, $\dfrac{b}{a+c}=y$, $\dfrac{c}{a+b}=z$, find the value of $\dfrac{x}{x+1}+\dfrac{y}{y+1}+\dfrac{z}{z+1}$.

61. If $a=\dfrac{m+n}{m-n}$ and $b=\dfrac{m-n}{m+n}$, find the value of $\dfrac{a^2+b^2}{a^2-b^2}$.

62. Simplify $$\frac{\dfrac{a+b}{ab}\left(\dfrac{1}{a}-\dfrac{1}{b}\right)-\dfrac{b+c}{bc}\left(\dfrac{1}{c}-\dfrac{1}{b}\right)}{\dfrac{a+c}{ac}\left(\dfrac{1}{a}-\dfrac{1}{c}\right)}.$$

63. Show that the equation $\dfrac{2nt}{t+2n^2+6n}=\dfrac{1}{\dfrac{t}{n^2+t^2-9}+\dfrac{n+3}{t}}$ reduces to $t^2(t^2-2nt+n^2-9)=0$.

64. If $5y^2-3y+(3-n)$ is a perfect square, what is the value of n?

65. For what value of n is $R^2-(6n-1)R+9n^2$ a perfect square?

66. Solve $x^2 - 1.4\,x + 0.2 = 0$, and compute the larger root to three decimal places.

67. Find to three decimal places the value of the larger root of $2\,x^2 - 0.5\,x - 1.2 = 0$.

68. Form the equation whose two roots are $3 + \sqrt{5}$ and $3 - \sqrt{5}$.

69. Find, without solving, the nature of the roots of $65\,x^2 - 19\,x + 2 = 0$.

Solve:

70. $x^2 + 8\,x + 6\sqrt{x^2 + 8\,x - 8} = 3.$

71. $\sqrt{2\,n + 6} - \sqrt{n - 4} = \sqrt{n + 4}.$

72. $\sqrt{3\,r + 4} + \sqrt{3\,r - 5} - 9 = 0.$

73. Solve the system $\begin{cases} 2\,v^2 + vt = 18, \\ t^2 - 2\,vt = 5. \end{cases}$

74. Solve the system $\begin{cases} x^2 + xy + y^2 = 49, \\ x^2 - xy + y^2 = 19. \end{cases}$

75. Solve the system $\begin{cases} a + b + 2\,c = 1, \\ 2\,a + 3\,b + 2\,c = 4, \\ 4\,a + 9\,b + 2\,c = 16. \end{cases}$

76. In the formula $S = \frac{1}{2}\,n\,[2\,a + (n - 1)\,d]$, solve for n in terms of the other letters.

Rationalize the divisors of:

77. $\dfrac{1}{1 + \sqrt{2}}.$ **78.** $\dfrac{\sqrt{5} - \sqrt{3}}{\sqrt{5} + \sqrt{3}}.$ **79.** $\dfrac{1}{a + \sqrt{ab} + b}.$

80. Find the value to two places of decimals of $\dfrac{9 - 2\sqrt{5}}{4 + \sqrt{5}}.$

81. Divide $\sqrt[3]{6}$ by $\sqrt{2}$, and express the result in its simplest form.

Simplify:

82. $(ab^{-2}c^{3})^{\frac{1}{2}} \times (a^{6}b^{3}c^{-2})^{\frac{1}{3}}$.

83. $(256\,m^{\frac{2}{3}}n^{-\frac{4}{3}})^{-\frac{3}{4}}$.

84. $\dfrac{a^{\frac{1}{2}}b^{-\frac{3}{2}}}{a^{\frac{2}{3}}b^{-\frac{1}{4}}}$.

85. $\left(\dfrac{x^{-2}y^{-4}z^{-1}}{x^{3}y^{2}z^{-2}}\right)^{-3}$.

86. $\sqrt{\dfrac{a}{b^{3}}} \times \sqrt[3]{\dfrac{b^{2}}{c^{2}}} \times \sqrt{\dfrac{ab}{c}}$.

87. Solve $\dfrac{\sqrt{2+x}}{\sqrt{2}+\sqrt{2+x}} = \dfrac{\sqrt{2-x}}{\sqrt{2}-\sqrt{2-x}}$.

88. If $\dfrac{a}{b} = \dfrac{c}{d}$, show that $\dfrac{a^{2}+b^{2}}{a(a-b)} = \dfrac{c^{2}+d^{2}}{c(c-d)}$.

89. If $\dfrac{a}{b} = \dfrac{c}{d}$, show that $ab+cd$ is a mean proportional between $a^{2}+c^{2}$ and $b^{2}+d^{2}$.

90. Show that either root of $x^{2}-q=0$ is a mean proportional between the roots of $x^{2}+px+q=0$.

91. If m is a mean proportional between a and b, show that $\dfrac{a-b}{2a-m-b} = \dfrac{a+m}{2a+m}$.

92. The second term of an A. P. is 10 and the seventh term is 25. Find the fiftieth term.

93. Insert three arithmetical means between 6 and 10.

94. Insert n arithmetical means between a and b.

95. Find the sum of 40 terms of the series 12, 9, 6, ….

96. In the series 15, $12\frac{1}{2}$, 10, …, which term is $-27\frac{1}{2}$?

97. How many terms of the series $3\frac{1}{2}$, 5, $6\frac{1}{2}$, … must be added to give 498?

98. Find the eighth term of 2, 3, $4\frac{1}{2}$, ….

99. Insert three geometrical means between 6 and 486.

100. Insert three geometrical means between a and b.

101. Find the sum of 6 terms of $3\sqrt{2}$, 6, $6\sqrt{2}$, ….

102. At each stroke of an air pump 10 % of the air in the bell jar is removed, and the air remaining expands and completely fills the jar again. The capacity of the jar is 360 cu. in. What part of the original quantity of air remains in the jar at the end of the eighth stroke?

103. Find the common fraction to which 6.4545 ⋯ is equal.

104. A flywheel whose circumference is 8 ft. makes 120 revolutions per minute. If, when the power is shut off, the wheel makes 98 % as many revolutions each second thereafter as it did the preceding second, how far will a belt on the wheel run by the time it is about to stop?

105. Find the common ratio of an infinite G. P. of which the first term is 1 and the sum of the terms $1\frac{1}{2}$.

106. Find the twelfth term in the expansion of $\left(\frac{\sqrt{a}}{2}-\frac{2\sqrt{b}}{3}\right)^{22}$.

107. Find by the Binomial Theorem the cube root of 121 to three places of decimals.

108. Expand $(1-n^{-\frac{1}{2}})^{-\frac{1}{2}}$ to five terms.

109. Simplify $\frac{(a^2bcd^{-1})^l(b^2acd^{-2})^m(c^2abd^{-3})^n}{(a^{-2}bcd)^l(b^{-2}acd^2)^m(c^{-2}abd^3)^n}\times\frac{(d^2)^{l+2m+3n}}{(a^lb^mc^n)^2}$.

110. Give the numerical values of $\sqrt[3]{-8}$, $27^{-\frac{1}{3}}$, $(\sqrt{-1})^6$.

111. Simplify $\sqrt[m]{a^{-x}}\div\sqrt[n]{a^{-y}}$.

112. If $a^{-\frac{7}{8}}=b$, $b^{-\frac{3}{14}}=c$, and $c^{-\frac{8}{3}}=a^{\frac{1}{2}}$, find a.

113. Simplify $\frac{8m-n}{2m^{\frac{1}{3}}-n^{\frac{1}{3}}}-2m^{\frac{1}{3}}n^{\frac{1}{3}}$.

114. In $a^2-2ab+b^2-4\sqrt{2}(a+b)+8$, substitute for a and b the values $a=\frac{m+n+1}{\sqrt{2}}$, $b=\frac{m-n+1}{\sqrt{2}}$, and simplify.

115. Show that if $n=\sqrt{a^2-b^2}$,

$$\log n=\tfrac{1}{2}\log(a+b)+\tfrac{1}{2}\log(a-b).$$

116. If $\log 3 = 0.4771$ and $\log 5 = 0.6990$, solve $3^{x+7} = 5^x$.

117. If $\log 2 = 0.3010$ and $\log 3 = 0.4771$, find the value of $\log \sqrt[5]{0.36}$.

118. The velocity of sound in a gas is computed by the formula $V = \sqrt{\frac{E}{D}} \times 1.41$. Solve for E. For D.

119. Eliminate R from the formulas $C = \frac{E}{R}$ and $C' = \frac{E}{R + r}$.

120. If $\frac{g}{g'} = \frac{\frac{1}{R^2}}{\frac{1}{(R+h)^2}}$, solve for h.

121. Solve $t = \frac{u + \sqrt{u^2 - 2\,gs}}{g}$ for u.

122. The velocity v at which water is discharged from a pipe is found from the formula

$$4\,v^2 + 5\,v - 2 = \frac{1200\,Hd}{l}.$$

If $H = 60$, $d = 6$, and $l = 1600$, find v.

123. If $y = 2\,\pi r (r - \sqrt{r^2 - R^2})$, solve for r. Find the value of y when $r = \frac{1}{2}$ and $R = 0.1$.

124. If $V = \frac{1}{6}\,h\{\frac{1}{2}\,b(e' + e'') + 4\,M\}$ and $M = \frac{1}{4}\,b(\frac{1}{2}[e + e'] + \frac{1}{2}[e + e''])$, show that $V = \frac{1}{6}\,hb(e + e' + e'')$.

125. If $\frac{S}{\frac{1}{2}(S + s)} = \frac{\sqrt{B}}{\sqrt{M}}$ and $\frac{s}{\frac{1}{2}(S + s)} = \frac{\sqrt{b}}{\sqrt{M}}$, show that

$$4\,M = B + b + 2\sqrt{Bb}.$$

126. By rationalizing the denominators and simplifying, show that $F = \frac{1}{3}\,B \times \frac{H\sqrt{B}}{\sqrt{B} - \sqrt{b}} - \frac{1}{3}\,b \times \frac{H\sqrt{b}}{\sqrt{B} - \sqrt{b}}$ reduces to

$$F = \frac{1}{3}\,H(B + b + \sqrt{Bb}).$$

N.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

TABLE OF SQUARE ROOTS AND CUBE ROOTS

Nos.	Square Roots	Cube Roots	Nos.	Square Roots	Cube Roots
1	1.000	1.000	**51**	7.141	3.708
2	1.414	1.259	**52**	7.211	3.732
3	1.732	1.442	**53**	7.280	3.756
4	2.000	1.587	**54**	7.348	3.779
5	2.236	1.709	**55**	7.416	3.802
6	2.449	1.817	**56**	7.483	3.825
7	2.645	1.912	**57**	7.549	3.848
8	2.828	2.000	**58**	7.615	3.870
9	3.000	2.080	**59**	7.681	3.892
10	3.162	2.154	**60**	7.745	3.914
11	3.316	2 223	**61**	7.810	3.936
12	3.464	2.289	**62**	7.874	3.957
13	3.605	2.351	**63**	7.937	3.979
14	3.741	2.410	**64**	8.000	4.000
15	3.872	2.466	**65**	8.062	4.020
16	4.000	2.519	**66**	8.124	4.041
17	4.123	2.571	**67**	8.185	4.061
18	4.242	2.620	**68**	8.246	4.081
19	4.358	2.668	**69**	8.306	4.101
20	4.472	2.714	**70**	8.366	4.121
21	4.582	2.758	**71**	8.426	4.140
22	4.690	2.802	**72**	8.485	4.160
23	4.795	2.843	**73**	8.544	4.179
24	4.898	2.884	**74**	8.602	4.198
25	5.000	2.924	**75**	8.660	4.217
26	5.099	2.962	**76**	8.717	4.235
27	5.196	3.000	**77**	8.774	4.254
28	5.291	3.036	**78**	8.831	4.272
29	5.385	3.072	**79**	8.888	4.290
30	5.477	3.107	**80**	8.944	4.308
31	5.567	3.141	**81**	9.000	4.326
32	5.656	3.174	**82**	9.055	4.344
33	5.744	3.207	**83**	9.110	4.362
34	5.830	3.239	**84**	9.165	4.379
35	5.916	3.271	**85**	9.219	4.396
36	6.000	3.301	**86**	9.273	4.414
37	6.082	3.332	**87**	9.327	4.431
38	6.164	3.361	**88**	9.380	4.447
39	6.244	3.391	**89**	9.433	4.464
40	6.324	3.419	**90**	9.486	4.481
41	6.403	3.448	**91**	9.539	4.497
42	6.480	3.476	**92**	9.591	4.514
43	6.557	3.503	**93**	9.643	4.530
44	6.633	3.530	**94**	9.695	4.546
45	6.708	3.556	**95**	9.746	4.562
46	6.782	3.583	**96**	9.797	4.578
47	6.855	3.608	**97**	9.848	4.594
48	6.928	3.634	**98**	9.899	4.610
49	7.000	3.659	**99**	9.949	4.626
50	7.071	3.684	**100**	10.000	4.641

INDEX

(Numbers refer to Sections)

Addition, 3, 4, 5, 6, 72, 80.
  elimination by, 54.
Antecedent, 112.
Arithmetical means, 130.
  progressions, 128.
Axioms, 43.
Axis, 51.

Binomial, 2.
  Theorem, 142.

Characteristic, 149.
Common denominator, 34.
  factor, highest, 28.
  multiple, lowest, 29.
Comparison, elimination by, 54.
Completing the square, 89.
Complex expression, 78.
  fraction, 39.
Conditional equation, 40.
Consequent, 112.
Constant, 115.
Continued fraction, 39.
Coördinates, 51.
Cube root, 62.
Cubic equation, 42.

Defective system, 107.
Degree of equation, 42.
Denominator, 30.
  common, 34.
Determinant, 58.
Determinate system, 56.
Direct variation, 116.
Discriminant, 93.
Dissimilar terms, 5.
Distribution, law of, 11.
Division, 16, 17, 18, 75, 83.
  synthetic, 18.

Elimination, 54.
  by addition or subtraction, 54.
  by comparison, 54.
  by substitution, 54.
Ellipse, 109.
Equations, 40.
  conditional, 40.
  cubic, 42.
  degree of, 42.
  discriminant, 93.
  equivalent, 44, 56.
  exponential, 156.
  fractional, 41.
  graph of, 51.
  homogeneous, 104.
  identical, 40.
  impossible, 100.
  inconsistent, 56.
  integral, 41.
  linear, 42.
  literal, 48.
  quadratic, 42.
  symmetrical, 106.
  system of, 52.
Equivalent equations, 44, 56.
Exponential equations, 156.
Exponents, fractional, 140.
  laws of, 13, 16, 137, 141.
  negative, 139.
  positive integral, 137.
  zero, 138.
Expressions, complex, 78.
  fractional, 36.
  integral, 36.
  literal, 2.
  mixed, 36.
  number, 2.
  radical, 69.
Extraneous roots, 100.
Extremes, 113.

Factoring, 19.
  general directions for, 27.
Factors, 19.
  highest common, 28.
  prime, 19.
  rationalizing, 75.
Formulas, 48.

Fourth proportional, 113.
Fractional equations, 41.
  exponents, 140.
  expressions, 36.
Fractions, 30.
  complex, 39.
  continued, 39.
  denominator of, 30.
  indeterminate, 125.
  numerator of, 30.
  reduction of, 33, 34.
  signs of, 31.
  terms of, 30.

Geometrical means, 134.
  progressions, 132.
Graph of an equation, 51.
Grouping, law of in addition, 3.
  law of in multiplication, 11.
  signs of, 10.

Highest common factor, 28.
Homogeneous equations, 104.
Hyperbola, 109.

Identical equations, 40.
Identity, 40.
Imaginary numbers, 63.
  typical form of, 79.
  unit, 79.
Impossible equations, 100.
  systems of, 56.
Inconsistent equations, 56.
Indeterminate fractions, 125.
  systems, 56.
Index of a root, 62.
Infinity, 121.
Integral equations, 41.
  expressions, 36.
Inverse variation, 117.
Isolating a surd, 99.

Law of distribution, 11.
  of grouping in addition, 3.
  of grouping in multiplication, 11.
  of order in addition, 3.
  of order in multiplication, 11.
Like terms, 5.
Limit of a variable, 120.
Linear equations, 42.
Literal equations, 48.
Literal expressions, 2.
  numbers, 2.
Logarithms, 147.
  characteristic of, 149.
  common, 148.
  mantissa of, 149.
Lowest common multiple, 29

Mantissa, 149.
Mean proportional, 113.
Means, 113.
  arithmetical, 130.
  geometrical, 134.
Mixed expressions, 36.
Monomial, 2.
Multiple, lowest common, 29.
Multiplication, 11, 12, 13, 14, 15, 74- 82

Negative exponent, 139.
Number expression, 2.
  imaginary, 63.
  literal, 2.
Numerator, 30.

Order, law of in addition, 3.
  law of in multiplication, 11.
  of a determinant, 58, 59.
  of a surd, 69.

Parabola, 109.
Pascal's Triangle, 142.
Polynomial, 2.
Principal root, 64.
Products, special, 15.
Progressions, 127.
  arithmetical, 128.
  geometrical, 132.
Proportion, 113.
  terms of, 113.
Proportional, fourth, 113.
  mean, 113.
  third, 113.

Quadratic equations, 42.
  complete, 86.
  pure, 86.
Quadratic surd, 69.

Radical expressions, 69.
  sign, 62.
Ratio, 112.

Rationalizing factor, 75.
Real numbers, 63.
Reduction of fractions, 33, 34.
Remainder Theorem, 25.
Root, extraneous, 100.
  index of, 62.
  principal, 64.
  square, cube, etc., 62.

Series, 126.
Signs of grouping, 10.
Similar terms, 5.
Simplest form of surd, 70.
Special products, 15.
Square, completing the, 89.
  root, 62.
Substitution, elimination by, 54.
Subtraction, 7, 8, 9, 72, 80.
  elimination by, 54.
Surd, 69.
  isolating a, 99.
  order of, 69.
  quadratic, 69.
  simplest form of, 70.
Symmetrical equations, 106.
Synthetic division, 18.
System of equations, 52.
  defective, 107.
System of equations, determinate, 56.
  impossible, 56.
  indeterminate, 56.
  solution of, 52.

Terms, 2.
  dissimilar, 5.
  of a fraction, 30.
  of a proportion, 113.
  of a series, 126.
  similar, 5.
Theorem, Binomial, 142.
  Remainder, 25.
Third proportional, 113.
Transposition, 45.
Triangle, Pascal's, 142.
Trinomial, 2.
Typical form of imaginary number, 79.

Unit, imaginary, 79.

Variable, 115.
  limit of, 120.
Variation, direct, 116.
  inverse, 117.
  joint, 118.

Zero exponent, 138.

STONE-MILLIS ELEMENTARY ALGEBRA

ANSWERS

FIRST COURSE

Pages 1, 2.

1. 180 sq. ft.
2. 108 sq. ft.
3. 26.52 sq. in.
4. 7 sq. ft.; 40 sq. ft.; 225 sq. ft.; 10,880 sq. ft.

Pages 3, 4.

1. ab; $2xy$.
2. 12.
3. 60.
4. 42.
5. 600.
6. 1600.
7. 3000.
8. 200.
9. 900.
10. 420.
12. 180.
13. 100.
14. 160.
15. 24.8.
16. 3.6.
17. 27.2.
19. 60.
20. 84.
21. 165.
22. 51.
23. 76.
24. 50.
25. 31.416.
26. 251.328; 376.992; 47.124.
27. 1200; 576; 1008.
29. 9408.
30. 9500.
31. 19,440.
32. 19,152.
33. 18.
34. $V = bh$.
35. $V = \frac{1}{3}bh$; 216 cu. in.
36. 50.2656; 314.16.
37. 113.0976 sq. in.
38. 10, 178.8— lb.

Pages 5, 6.

1. Bases a, y, C, A, 9, x; exponents, 3, 10, 6, 25, 5, n.
2. m^6; b^8; 2^{10}.
3. 8; 8; 27; 16; 81; 32.
4. 132.7326.
5. The area of a circle equals one fourth of 3.1416 times the square of the diameter; 15.9043+.
6. 1809.5616 sq. in.
7. 201,062,400 sq. mi., approximately.
8. 150.7968.
9. 113.0976.
10. 5276681625 cu. mi. approx.
11. 64 sq. in.
12. 18 sq. ft.

Page 7.

1. 26.
2. 48.
3. 64.
4. 192.
5. 544.
6. 36.
7. 46080.
8. $6\frac{1}{2}$.
9. 80.
10. 502.656.
11. 226.1952.
12. 5.
13. 10.
14. 50.
15. 25.
16. 20.
17. 30.
18. 3.
19. 8.
20. 15.
21. 9.
22. 40.
23. 21.

Pages 8, 9.

1. 1; monomial.
2. 2; binomial.
3. 3; trinomial.
4. 1; monomial.
5. 1; monomial.
6. 2; binomial.
7. 1; monomial.
8. 2; binomial.
9. 1; monomial.
10. 1; monomial.
11. 2; binomial.
12. 2; binomial.
13. 3; trinomial.
14. 2; binomial.
15. 6; polynomial.
16. 4; polynomial.
17. 3; trinomial.
18. 3; trinomial.
19. 1; monomial.
20. 4; polynomial.

Pages 9, 10.

1. 1.
2. 81.
3. 58.
4. 133.
5. 37.
6. 12.
7. 180.
8. 1.
9. 10348.4304.
10. 78.
11. 12601.3584.
12. 186.
13. 122.
14. 168.
15. $363\frac{1}{3}$.
16. 676.
17. $485\frac{1}{3}$.
18. 294.
19. 857.6568.
20. 1363.4544.
21. 10,348.4304
22. 6484.2624.

Pages 11, 12.

1. $9a$.
2. $8x$.
3. $21W$.
4. $13P$.
5. $17t$.
6. $20S$.
7. $20k$.
8. $20A$.
9. $17x$.
10. $18mn$.
11. $14abc$.
12. $32E$.
13. $12a$.
14. $17b$.
15. $91z$.
16. $8d$.
17. $6r_1$.
18. $4m'$.
19. $8\pi R^2$.
20. 0.
21. $8D$.
22. $32w$.
23. 22.
24. 6.
25. 2.
26. $3n$.
27. $6t$.
28. $3\pi D$.
29. Equal in each case.
30. Equal in each case.
32. $3N+6$.
33. $18z+6$.
34. $11R+4$.
35. $25+5v$.
36. $2m+2$.
37. $2F+8$.
38. $4K+1$.
39. $6s+6$.

Pages 12, 14.

1. 224.
2. $130\frac{1}{2}$.
3. $50\frac{2}{7}$.
4. $75\frac{3}{7}$.
5. 254.4696.
6. 50.2656.
7. $3218\frac{2}{7}$.
8. 1963.5.
9. 452.3904.
10. 2144.6353.
11. 100.
12. $581\frac{1}{3}$.
13. $1791\frac{3}{7}$.
15. $50.
16. $346.50.
17. $105.25.
18. $121.50.
19. $100.
20. $56.25.
21. $561.80; $595.51 –.
22. $2315.25.
23. 122.50.
24. 4; 10; 20.
25. 120; 720, 5040.
26. 40,320.
27. 27.712.
28. 1.2854 mi.
29. 60°; 108°; 120°; 150°.

Pages 14–18.

1. 1.22+.
2. 14.5+ ft. per sec.
3. 8 lb. per sq. ft.; 9600 lb.
4. 16 ft.; 64 ft.; 144 ft.; 256 ft.; 1600 ft.; 57,600 ft.; 1,440,000 ft.
5. 176 ft.
6. 944 ft.
7. $21\frac{1}{9}$°; $17\frac{7}{9}$°; $8\frac{8}{9}$°; $26\frac{2}{9}$°.
8. 508.7−.
9. 160.8+.
10. 30.7+.
11. $\frac{1}{600,000}$ in.
12. 1030 ft.
13. 1.12−.
14. 0.46+.
15. 76; $195\frac{5}{16}$.
16. 480 lb.
17. 1692+ gal.
18. 138.6+.
19. 2.5.

Pages 20, 21.

1. $n+10$; $n-10$; $10n$; $\frac{1}{10}n$.
2. $x-6$; $x+6$.
3. $3t-136$.
4. n; $\frac{1}{2}n+20$.
5. $\$2\frac{1}{6}y$.
6. v; $v+362$.
7. Width w; length $3w+2$.
8. 1.2 C.
9. $\$D+\$\frac{1}{5}D$.
10. $22\frac{1}{2}$ hr.; $\frac{900}{V}$ hr.

Pages 21, 22.

1. Let n=number of dollars received by partner investing the less. Then $n+2n=1200$.
2. Let c=number of dollars cost. Then $c+\frac{1}{4}c=1500$.
3. Let x ft.=width. Then $2x+6x=2240$.
4. Let n=number. Then $2n+8=40$.
5. Let n hr.=length of night. Then $n+n+6\frac{2}{15}=24$.
6. Let d degrees=angle A. Then $d+2d+3d=180$.

Page 24.—Oral.

1. $a=4$.
2. $P=14$.
3. $v=7$.
4. $R=7$.
5. $x=4$.
6. $A=12$.
7. $k=23$.
8. $t=3$.
9. $D=5$.
10. $B=24$.
11. $x=25$.
12. $M=62$.
13. $p=10$.
14. $R=8$.
15. $w=30$.
16. $x=1$.
17. $m=2$.
18. $y=3$.
19. $H=6$.
20. $a=9$.
21. $Q=8$.
22. $S=4$.
23. $D=3$.
24. $z=1\frac{1}{2}$.
25. $c=2$.
26. $y=100$.
27. $n=18$.
28. $A=48$.
29. $r=70$.
30. $Q=40$.
31. $V=360$.

Page 24.—Written.

1. $x=1.75$.
2. $D=4.746$.
3. $A=2.5$.
4. $P=12.08$.
5. $m=\frac{1}{9}$.
6. $d=3.02+$.
7. $C=7$.
8. $H=3\frac{11}{16}$.
9. $t=15\frac{16}{21}$.
10. $y=\frac{29}{48}$.
11. $r'=4$.
12. $v=2\frac{5}{19}$.
13. $W=14\frac{1}{12}$.
14. $D=17\frac{3}{8}$.

Pages 26, 27.

1. $a=3$.
2. $b=7$.
3. $P=2$.
4. $v=4$.
5. $D=3$.
6. $x=3$.
7. $A=2$.
8. $k=2$.
9. $M=3$.
10. $y=\frac{3}{8}$.
11. $R=2$.
12. $h=10$.
13. $z=1$.
14. $c=8$.
15. $V=1$.
16. $x=5$.
17. 6; 6.
18. 12.
19. 24 apples; $24a$; $15x$; $56n$; $30A$.
20. 4; 6; $6x$; $9t$.
21. $t=20$.
22. $n=180$.
23. $x=12$.
24. $a=10$.
25. $B=6$.
26. $s=18$.
27. $x=90$.
28. $W=6$.
29. $k=8$.
30. $R=14$.

Pages 28–31.

1. 12 yr.
2. 1600.
3. $500 and $2000.
4. $2.25 and $3.75.
5. 22½ doz.
6. A, $51; B, $24.
7. 20 and 25.
8. 7.
9. 6 hr.
10. 1.5 ohms.
11. 2 ohms.
12. Children's, 324; adult's, 512.
13. 1083 mi.
14. 9 boys; 15 girls.
15. 120 ft.
16. 6 mi. per hour.
17. 12 hr.
18. 24 in., 16 in.
19. 27 in., 35 in.
20. 25 ft., 33⅓ ft., 41⅔ ft.
21. 12½ ft., 16⅔ ft., 20⅚ ft.
22. 15, 20, 25.
23. 50¢ per pound.
24. $833.33⅓.
25. Angle $a=72°$, angle $b=72°$, angle $c=36°$.
26. 45°, 45°, 90°.
27. 12°.
28. 25°, 65°.
29. 60°, 120°, 180°.
30. 6.
31. 35, 36.
32. 15, 17.
33. 12, 24.
34. 48.
35. 16⅔.

Pages 31–33.

1. $nx+m$.
2. $D+\frac{pD}{100}$; $D+\frac{3pD}{100}$; $D+\frac{tpD}{100}$.
3. $c+\frac{12}{100}c$; $c+\frac{pc}{100}$.
4. $s=12$.
5. $F=20$.
6. $y=36$.
7. 10.
8. \$1.09$\frac{3}{8}$+.
9. 50¢.
10. 16$\frac{2}{33}$¢ per pound.
11. 19$\frac{3}{11}$¢.
12. \$8.12½.
13. 96.30−.
14. 97.27+.
15. 97.46−.
16. 92.93−.
17. 96.58−.
18. 152.
19. \$50.
20. 384.5− lb.

Page 36.

2. −1.
3. −5.
4. −2.
5. −9.
6. −8.
7. −\$25.
8. −\$150.
9. −\$75.
10. −\$75.

Pages 37–39.

1. \$5 profit; negative; −\$10.
2. −\$25; +\$40; −\$15; −\$6; +\$20.
3. \$5 debt.
4. +\$50; −\$18; −\$25; +\$60; −\$100; +\$20; −\$15.
5. +216 and −250; −34.
6. +1900, −675, −750, −650; −175.
7. +35; −20.
8. 200 lb.; −.
9. −250 lb.; −75 lb.
10. +20,000 T.; −20,000 T.
11. +165 lb. and −156 lb.; +9 lb.
12. 10° below zero; −10°; +.
13. Below; above.
14. −3.
15. −10; −6; −1; +5; +8; +2.
16. 31 B.C.; 14 A.D.
17. 287 B.C.; 212 B.C.
18. 4; −4; negative.
19. 50 yd. backward; +100 yd., −150 yd., −50 yd.
20. United States; Europe; Africa.
21. East; +77°.
22. −34°.

Pages 40, 41.

1. 12 spaces to the left of A; 10 spaces to the right of A; 2½ spaces to the left of A; 3½ spaces to the right of A; 40 spaces to the left of A; 100 spaces to the right of A.

Pages 44–47.

1. $28 profit.
2. $33 loss.
3. $9 profit.
4. $12 loss.
5. 20° rise.
6. 25° fall.
7. 9° fall.
8. $75 debt.
9. $360 debt.
10. +4; −15; +14; +5; −85; −8.
11. +7; −29; +30; +13; −29; −82.
12. +7; −11; −1; −9; −17; +36.
13. −$18; −$9; +5 lb.; +32 lb.; −4 mi.
14. −25° longitude; −8° latitude; 11 ft. backward.
15. −5.
16. +13.
17. −5.
18. +18.
19. +20.
20. −37.
21. +0.84; −22.45; −5.45; −2.675.
22. $-2\frac{1}{4}$; $+1\frac{7}{8}$; $-5\frac{11}{12}$; $+2\frac{7}{12}$.
23. −9.
24. −7.
25. +25.
26. −300.
27. +420.
28. +20; −5; +15.
29. −263 lb.
30. +10, +3, −4; +10+3−4; 9 miles per hour.
31. +12, −35, +8; −15.
32. −15°, +18°, −10°, +6°; −1°.
33. No change.
34. $+1\frac{2}{3}$°.
35. $-6\frac{2}{7}$°.
36. $+1\frac{2}{3}$°.
37. +25°.
38. −13°.

Pages 48–50.

1. +2.
2. −10.
3. +10.
4. −5.
5. +7.
6. +5.
7. $16 gain.
8. $275 deposit.
9. 30° fall.
10. 7° rise.
11. +13; +15; +12; +28; +23; +20; +6.
12. −8; −13; −40; −30; −15; −15; −15.
13. +9; +4; −2; −5; +5; −8; −15.
14. −3; −4; +2; +3; −5; +3; +1.
15. −10; +21; +2; +4; −20; −7; +15.
16. The subtraction of −2 from −3; −1.
17. The subtraction of +12 from +8; −4.
18. +10; +7; −2; +3; −6; −15; +15.
19. −10.10; +4.4; +1.4 ft.; −3.9 pk.; +3.3 lb.
20. $-\frac{1}{2}$; $+7\frac{7}{8}$; +5; $+4\frac{1}{3}$; $-21\frac{3}{4}$.
21. 6° rise.
22. 17° fall.
23. Montgomery, Ala., 112°; Little Rock, Ark., 118°; Denver, 134°; Washington, D.C., 119°; Boisé, Ida., 139°; Des Moines, 139°; Louisville, 127°; Boston, 115°; Duluth, 140°; Havre, Mont., 163°; New York, 106°; Chicago, 126°.
24. 76 yr.
25. 74 yr.
26. 90 yr.
27. 69 yr.
28. 1600 yr.
29. 101°.
30. 62°.

Pages 51, 52.

1. $+21$.
2. $+12$.
3. $+20$.
4. $+24$.
5. $+20$.
6. $+21$.
7. -10.
8. -18.
9. -35.
10. -24.
11. $+30$.
12. -28.
13. -48.
14. $+60$.
15. $+48$.
16. $+72$.
17. -96.
18. -84.
19. $+75$.
20. $+120$.
21. -140.
22. $+\frac{1}{4}$.
23. $-\frac{1}{2}$.
24. $-\frac{1}{4}$.
25. $+\frac{12}{25}$.
26. -1.
27. $+6.25$.
28. -2.56.
29. $-.56$.
30. $+1.43$.

Pages 53, 54.

1. $+6$.
2. -8.
3. $+60$.
4. -80.
5. -36.
6. $+120$.
7. $+4$.
8. -112.
9. -1.
10. -16.
11. $+120$.
12. -24.
13. $+16$.
14. -8.
15. $+9$.
16. $+1$.
17. -1.
18. -64.
19. -32.
20. -27.
21. $+256$.
22. -1.
23. $+256$.
24. -243.
25. $+8$.
26. $+36$.
27. $+200$.
28. -8.
29. $+72$.
30. -216.
31. -27.
32. $+36$.
33. $+6$.
34. $+36$.
35. -6.
36. $+36$.
37. $+27$.
38. -19.
39. $+8$.

Pages 54, 55.

1. -4.
2. $+5$.
3. -6.
4. -3.
5. -4.
6. $+9$.
7. -6.
8. $+9$.
9. $+9$.
10. $+10$.
11. -1.
12. $+1$.
13. -10.
14. -4.
15. -6.
16. $+12$.
17. -2.
18. -9.
19. $+7$.
20. $+12$.
21. -1.
22. $+\frac{1}{2}$.
23. $-1\frac{1}{2}$.
24. $-\frac{1}{2}$.
25. $+\frac{8}{9}$.
26. $+1\frac{1}{2}$.
27. -8.
28. $+7$.
29. $-.9$.
30. $+1.2$.
31. -1.
32. -72.
33. $+27$.
34. -16.
35. -54.
36. $-5\frac{1}{3}$.
37. $+4$.
38. -3.

Page 56.

1. $y=-14$.
2. $A=-3$.
3. $s=-2$.
4. $m=-3$.
5. $K=-5$.
6. $t=-3$.
7. $P=-5$.
8. $x=-9$.
9. $a=-3$.
10. $M=-2$.
11. $B=-\frac{5}{8}$.
12. $g=-1\frac{1}{4}$.
13. $W=-3\frac{1}{4}$.
14. $Q=-10$.
15. $b=-8$.
16. $x=-18$.
17. $R=-6$.
18. $t=-8$.
19. $p=-1$.
20. $z=-9$

Page 57.

1. $-6°$; the temperature fell 6° the first day.
2. $-\$150$; \$150 loss.
3. $-\$116.66\frac{2}{3}$; \$116.66$\frac{2}{3}$ loss.
4. Impossible.
5. Impossible.

Pages 57–59.

1. 60 yr.
2. $+17$, or 17 A. D.
3. $-7°$; 35° north latitude.
4. $+30$ yd., -120 yd., $+150$ yd., $+40$ yd., -300 yd; 200 yd. back.
5. $-\frac{1}{2}°$.
6. $-\frac{5}{8}$.
7. $-\frac{7}{10}$.
8. -1; $+64$; -432; -720.
9. $+1$.
10. $+336$.
11. $n=-\frac{1}{8}$.
12. $y=-\frac{85}{66}$.
13. $+4$; -8.
14. $+3$.
15. $+2$.
16. $+9$.
17. -7.
18. -5.
19. $+5$.
20. -1.
21. $+4$.
22. -2.
23. $+1$.
24. -4.
25. $+6$.
27. -4; -6; -14; $+10$; -2; $+2$; 0; -10.

Pages 60–62.

1. $7a$.
2. $-6n$.
3. $13ab$.
4. $-12P$.
5. $-8m^3n$.
6. $-8A$.
7. x^4.
8. PQ.
9. $7V^2$.
10. $-10b^2c^3$.
11. $2B^2$.
12. $-11xy^5$.
13. $-6a$; $-13x$; $-6K$; $10y^3$; $-6zw$.
14. $-4xyz$; a^4; $-9pq$; $67M$; $-V^2$.
15. $-9a^2bc$; $8s^3t^2$; $12ab^3$; $-3T$; $-7N^6$.
16. $1.2A$.
17. $-3.06x$.
18. $2.5n$.
19. $-1.9P^3$.
20. $-\frac{3}{8}y$.
21. 0.
22. $5mn$.
23. $\frac{1}{3}v^3$.
24. $-x$.
25. $4c^4$.
26. $4at^2$.
27. $-11v'$.
29. $7ab-5x^2$.
30. $2m^2-3mn$.
31. $4ab$.
32. $-y^3-3y^2z+4yz^2-5z^3$.
34. $-2a+3b-c$.
35. $h^2-2hk+k^2$.
36. $a^3+a^2b-ab^2-b^3$.
37. 480.
38. 540.
39. 750.
40. 6300.
41. 1400.
42. 2300.
43. 3360.
44. 9600.
45. 1680.
46. 850.
47. 1140.
48. 100π.

Pages 63, 64.

1. $7a+3b$.
2. $-2x-6y$.
3. $2u-v-3w$.
4. $3H-8K-10L$.
5. $-9m-4n-6p$.
6. $-7r-3r'-5r''$.
7. $7A-C$.
8. $4P-3Q-5R-2S$.
9. $-13a-11ab-12b$.
10. $-8x+4y$.
11. $a+2b+8c$.
12. $38x^2+2xy+5y^2$.
13. $8m^2-7m$.
15. 2188; 1579; 1679; 1788.

Pages 65, 66.

1. $4x^2-2x+2$.
2. $3a^2+4a+2$.
3. $4a^2$.
4. $14m+13n-11p$.
5. $-6u-13v+27w$.
6. $2r-7s+3$.
7. $2x^3+6x^2y-6xy^2-2y^3$.
8. $-3bm+2$.
9. $6s+2t+4$.
10. $7n^2-8n-5$.
11. $10a^2+11ab-9b^2$.
12. $14x^3+2x^2-9x+19$.
13. $2m^3+2m^2n-4mn^2-3n^3$.
14. $-4A^6+2A^4-2A^2-5$.
15. $5v^2+7vt-9t^2$.
16. $2ax-2by-2cz$.
17. $-3qr+2pr$.
18. $7k^2+6k+1$.
19. $-r+7s-18t$.
20. $3+6v'-21v''+3v'''$.
21. $24P+14W+2H$.
22. $7.9a-4b$.
23. $-.75x^2-9x+10.4$.
24. $5.2R^2-4.3m$.
25. $-7.7+4.5D+10.9D^2+8.5D^3$.
26. $1\frac{5}{6}x-\frac{5}{6}y$.
27. $2\frac{1}{4}m$.
28. $4H-1\frac{7}{10}K$.

Pages 66, 67.

1. $5x$; $4a$; $-10M^2$; $10y^3$; $-3ab$.
2. $5m^2n$; $-17xyz$; $30A$; $8P$; $-2ab^2$.
3. $5t^3$; $-9w^3v$; $3hk$; $28Q$; $-10abc$.
4. $9wv^4$; $13by^6$; $24x^2y^2z$; $14R^3$; $4v'$.
5. $13a$.
6. $-3x$.
7. $-2t$.
8. $-11P$.
9. $48at^2$.
10. $-9\pi r^2$.
11. $7x$.
12. $2y$.
13. $7m$.
14. $20A$.

Pages 68, 69.

1. $3x+2y$.
2. $3C+4D$.
3. $3T-6h$.
4. $2m+19a$.
5. $-b-21d$.
6. $-5s+7t$.
7. $5x+7y-21z$.
8. $-4p-7q+6r$.
9. $3v-4w$.
10. $-2f+g+h$.
11. $18p-6q+8r$.
12. $4ab$.
13. $2x^2+2$.
14. $-2F^4+6F^2+2$.
15. $-4ab-10bc+7ac$.
16. $7P+5Q-10R$.
17. $5v+v'-2v''$.
18. $5W-10S+18H$.
19. $-a^6+2a^2$.
20. x^3-5x^2+x+10.
21. $4a^2b+2ab^2+2b^3$.
22. $-3-3m+18m^2-2m^3$.
23. $2ab+2b^2$.
24. W^2+6WV^2.
25. $4+11t^4+6t^6$.
26. $-n^3+4n^2-2n+6$.
27. $-x-8x'+16x''$.
28. Wt^2-25.
29. $.8a+4.8b$.
30. $\frac{1}{4}m-\frac{3}{8}n$.
31. $-6a-11b-3c$.
32. $14x^2-12x+8$.
33. $-3m^2-mn-2n^2$.
34. $-6+A-2A^2$.
35. $-3t^2-2t-2$.
36. $-15+16v'+3v''$.
37. $-10+9a+6a^2+4a^3$.
38. $P^3-7P^2+13P-3$.
39. $D^2+9DR-4R^2$.
40. $-4x^2y+3xy^2+y^3$.
41. $4.5r+1.9t$.
42. $-1.8+22.4E-12.5E^2-9E^3$.
43. $5.6ab-7.5k+25$.
44. $-\frac{1}{3}x^2-\frac{1}{6}x-1$.
45. $3m^2-2m+5$.
46. $29W-26$.
47. a^2-2ab.
48. $10-S$.

Pages 70, 71.

1. $7r-3$.
2. $-6T-7$.
3. $12-w+w^2$.
4. $2-29C$.
5. $59-F$.
6. $3a-b+5$.
7. $3s+10$.
8. $6-6\pi$.
9. $5L+17$.
10. $-3M+2t$.
11. $a^2+2ab-b^2$.
12. $21y^2-9y$.
13. $4w-mn$.
14. B^2+4AC.
15. $r-3h$.
16. x^2+2x-7.
17. $6p+2q$.
18. $2.5W-4w+2.5$.
19. $P+Q-6R+1$.
20. $5x$.
21. $7v-4t$.
22. $3D-2$.
23. -1.
24. a.
25. $4t-2d$.
26. $2\frac{1}{2}at$.
27. $8m-7n$.
28. 0.
29. $y=2$.
30. $V=\frac{2}{3}$.
31. $k=2$.
32. $z=3\frac{3}{5}$.
33. $u=5\frac{3}{5}$.
34. 13; 7.
35. 12; −5.
36. 105°; 75°.
37. First, $1110; second, $1390.

Page 72.

1. $A+(B-C-D)$.
2. $xy+(yz-zw+wx)$.
3. $5+(-2n-9n^2+n^3)$.
4. $r+(-4s+4s^2+1)$.
5. $m^2+(6pq-q^2-9p^2$.
6. $x^2+2x+y^2+(-z^2+2zw-w^2)$.
7. $16-8P+P^2+(12M-9M^2-4)$.
8. $k+7+(10k-1-25k^2)$.
9. $18+(-B^2-b^2-Bb)$.
10. $1+(6rr'-r^2-9r'^2)$.
11. $A-(-B+C+D)$.
12. $xy-(-yz+zw-wx)$.
13. $5-(2n+9n^2-n^3)$.
14. $r-(4s-4s^2-1)$.
15. $m^2-(-6pq+q^2+9p^2)$.
16. $x^2+2xy+y^2-(z^2-2zw+w^2)$.
17. $16-8P+P^2-(-12M+9M^2+4)$.
18. $k+7-(-10k+1+25k^2)$.
19. $18-(B^2+b^2+Bb)$.
20. $1-(-6rr'+r^2+9r'^2)$.
21. $(an-3n)-(2t-bt)$.
22. $(10P+mP)-(2W+nW)$

Pages 72, 73.

1. $-6\sqrt{n}$.
2. $-6\sqrt{a+b}$.
3. $-6(x-y)$.
4. $-7(s-at^2)$.
5. $8(2\pi R+7)^2$.
6. $-5\sqrt[3]{8+\frac{1}{2}Wg^2}$.
7. $-8-4\sqrt{A}$.
8. $-4D+4\sqrt{w}-5\sqrt{t}$.
9. $2(W-V)+9W^2$.
10. $-2x^2-4xy-2y^2$.
11. $-r_1^2+12r_1r_2-7r_2^2$.
13. $(m+n-p)y$.
14. $(a-b-c+d)R^2$.
15. $(M+2N-3P+Q)t^3$
16. $(5+\pi-h-7)D^2$.
17. $(4a-3b+5)(x-y)$.
19. $(a-g)U+(-b-h)V+(-c+k)W$.
18. $(n+x)A+(m-y)B+(p+z)C$.
20. $3a-2b$.
21. $-5m+5n$.
22. $6A-6B$.
23. $[2x-y][-3y]$.
24. $[r-s+t][r+s-t]$.
25. $[6A+3][4A-3]$.
26. $[4r+3r'-7][4r-3r'+1]$.
27. $[7a-2b-c-3d][7a-2b+c+3d]$.

Pages 75, 76.

1. a^7.
2. x^9.
3. b^9.
4. M^{11}.
5. r^{11}.
6. V^5.
7 L^{11}.
8. t^{13}.
9. y^7.
10. n^{20}.
11. W^9.
12. T^{13}.
13. d^{13}.
14. S^{21}.
15. U^{14}.
16. z^{13}.
17. A^9.
18. C^{13}.
19. p^9.
20. x^7.
21. n^{16}.
22. B^{18}.
23. $12x^{12}$.
24. $8m^{12}$.
25. $24A^{15}$.
26. $42R^{12}$.
27. $-90y^{14}$.
28. $-24B^{14}$.
29. $48c^{14}$.
30. $96d^{12}$.
31. $x^{13}y^{13}$.
32. $-10a^4n^{10}$.
33. $-12A^4BC$.
34. $-84s^5r^8$.
35. $36x^5y^2z^6$.
36. $-48D^6W^3$.
37. $270M^5N^7P$.
38. $-224x^9y^3z^6$.
39. $80m^5n^8p^6$.
40. $42a^{12}b^3c^4$.
41. $-\frac{1}{4}x^7y^6$.
42. $-\frac{1}{2}H^3D^5$.
43. $-\frac{3}{4}r^6s^8$.
44. $\frac{3}{4}p^9q^{20}$.
45. $-\frac{5}{8}V^7$.
46. $-\frac{3}{10}c^{20}d^{18}$.
47. $2W^4t^9$.
48. $-3M^3V^6$.
49. $30Q^6t^4$.

Pages 77–79.

1. $5x+5y$.
2. $3a-3b$.
3. $7n-28$.
4. $6P+6$.
5. $8A-12$.
6. $12a+10b$.
7. $24M-27N$.
8. $-50E+70K$.
9. $4x^2-9xy$.
10. $12a-42b+6c$.
11. $4x^2-8x+4$.
12. $a^3b+2a^2b^2+ab^3$.
13. $15m-40m^2+20m^3$.
14. $-14a^4b-24a^2b^3$.
15. $60R^4t-16R^3t^2$.
16. $8x^5y^2+14x^2y^5$.
17. $12s^5-20s^4+36s^3$.
18. $9AB^2C-36A^2C^2$.
19. $-40v^3+16v^4-56v^5$.
20. $4L^6-8L^4+12L^2$.
21. $-6n^3p^3+8n^2p^3-24n^2p^4$.
22. $-5W^7+30W^6-10W^5+15W^4$.
23. $-6c^6-45c^2+12c^5$.
24. $8A^5B^2+16A^4B^3+8A^3B^4$.
25. $a^2b^2c+ab^2c^2+a^2bc^2$.
26. $-18D^4W^2+27D^3W^3-9D^2W^4$.
27. $2A^4B-2AB^4$.
28. $24x^{10}-48x^9+8x^8+64x^7-8x^6$.
29. $14a^2bc+6abc^2-8ac^2d$.
30. $20-5a-5a^2$.
31. $12x^2+32xy$.
32. $7t^3-7at^2$.
33. $5p^3+3p^2q$.
34. $A^3-5A^2B+6AB^2$.
35. $-6m^2+18m^3+10m^4$.
36. $3x^2y^5-9xy^6+21y^7$.
37. $120a^5b^3-8a^3b^5+8a^2b^6$.
38. $4m^6n^2-4m^5n^3+12m^4n^4-4m^3n^5$.
39. $2s^2-3s+1$.
40. $-\frac{1}{12}W^3+\frac{1}{9}W^2V-\frac{1}{18}WV^2$.
41. $\frac{1}{4}a^3b+\frac{1}{6}a^2b^2+\frac{1}{4}ab^3$.
42. $\frac{2}{9}x^5y^2-\frac{1}{6}x^4y^3+\frac{1}{9}x^3y^4-\frac{1}{12}x^2y^5$.
43. $-1.8m^5+3.75m^4n-5.1m^3n^2$.
44. $1.075p^5-.025p^4+.6p^3-2p^2$.
45. $a-6b$.
46. $3A-3B$.
47. $13s^2-11st-5t^2$.
48. $2a^2b$.
49. $27P-2P^2$.
50. $S^3+\frac{3}{8}S^2+\frac{1}{4}S$.
51. $-14B$.
52. $-2c$.
53. $-15P-10Q$.
54. $-2m^2n$.
55. $-48r-19t$.
56. $-2x^2y+2xy^2$.
57. $v^3-12v^2w+8w^3$.
61. 804.

Pages 81-84.

1. n^2+4n+4.
2. A^2-A-6.
3. $x^2-7x+12$.
4. $6a^2+5a-6$.
5. $5z^2+26z+24$.
6. $16t^2-16t-21$.
7. $24b^2+11b+1$.
8. $12R^2+7R+1$.
9. $6V^2+29V+35$.
10. $30y^2-53y+8$.
11. $-12D^2+14D+10$.
12. $4m^2-36$.
13. a^2-100.
14. $4W^2+12W+9$.
15. $6+17x+12x^2$.
16. $21+4S-S^2$.
17. $25-40b+16b^2$.
18. $10a^2+11ab-6b^2$.
19. $6p^2-8pt-8t^2$.
20. $25B^2+10BC+C^2$
21. $4k^2-12kc+9c^2$.
22. x^3-3x+2.
23. $A^3-8A^2+19A-20$.
24. $v^3+2v^2w+2vw^2+w^3$.
25. M^4-N^4.
26. $6x^2-5xy+x-6y^2+5y-1$.
27. $5x^3+17x^2-14x-8$.
28. $42-11R-20R^2$.
29. $2a^5-5a^4+13a^3-26a^2+26a-12$.
30. $12n^3-13n^2+19n-12$.
31. $12S^3+7S^2-6S+5$.
32. $4+12T+13T^2+6T^3+T^4$.
33. $8a^2+36ac-18b^2-6bc+40c^2$.
34. $3R^2-16RS+11RW+5S^2-13SW+6W^2$.
35. $\frac{1}{4}a^2+\frac{1}{3}a+\frac{1}{9}$.
36. $-6n^2-14np+15nr+12p^2-10pr$.
37. a^3+a^2-a-1.
38. $-6b^4+19b^3-17b^2+11b-12$.
39. $32+16D-18D^2-5D^3+2D^4$.
40. $-4w^5+8w^4+3w^3-17w^2+13w-3$.
41. $2x^2+5x-25$.
42. $10A^2+18A-4$.
43. $1-16t^2$.
44. $B^4-16A^2C^2$.
45. x^3-3x^2+3x-1.
46. $18-9v+4v^2-v^3$.
47. a^6-b^6.
48. $n^6+2n^4-19n^2-20$.
49. $2a^3-5a^2t-4at^2+3t^3$.
50. $\frac{1}{16}x^2-\frac{1}{25}y^2$.
51. x^3+2x^2-x-2.
52. $2a^3+3a^2-11a-6$.
53. $6B^3+5B^2-12B+4$.
54. $9V^3-15V^2-8V+16$.
55. m^4-n^4.
56. $16-d^4$.
57. $6p^3+p^2t-11pt^2-6t^3$.
58. S^6-1.
59. $x^{12}-z^{12}$.
60. $A^4+10A^3+35A^2+50A+24$.
61. $3t^2-4t+10$.
62. $8F^2+3F-2$.
63. $1-2b$.
64. $2x^2+2w^2$.
65. $-a^2+10a+12$.
66. $4AB$.
67. y^2-9y+6.
68. $-2vt$.
69. $4n^2-7n$.
70. $24ab$.
75. $w^2+13w+22$ sq. rd.
76. $w^2+10w-24$ sq. in., where w inches is the width.

Page 85.

1. a.
2. N^2.
3. R.
4. y^4.
5. t^6.
6. K^4.
7. v^2.
8. x.
9. S.
10. P.
11. x^9.
12. A^4.
13. m^2.
14. 1.
15. b^2.
16. z^4.
17. ab.
18. x^2,
19. m^4n.
20. b^2c^4.
21. $3x^4$.
22. $4v$.
23. $2R$.
24. $2g^2$.
25. $-3a^5$.
26. $-3t^4$.
27. $4P^3n$.
28. $-8M^5N$.
29. $-4a^3$.
30. $-4x^4y$.
31. -3.
32. $3D^2v$.
33. $-25xd^4$.
34. A.
35. $1\frac{1}{2}z^3$.
36. $-\frac{2}{3}m^4$.
37. $\frac{1}{6}t$.
38. $-\frac{1}{4}x^2y^2z^6$.
39. $-14F^2$.
40. $8D^2$.

Pages 86, 87.

1. $2m-3n$.
2. $4D+3F$.
3. $4x+6$.
4. $3a-4b$.
5. $6+7y$.
6. $2P-6Q$.
7. $9b-2c$.
8. $1+2x$.
9. $A-3$.
10. $6m+8n$.
11. $q-2t+3v$.
12. $-W^2+3Wg-2g^2$.
13. x^3-1.
14. a^2-ab+b^2.
15. $-3t^2+2$.
16. $x+y-z$,
17. $A-2B+C$.
18. $w^2+2wv-3v^2$.
19. $3p^4t-2p^2t^2+6$.
20. $3x^4+5x^2+2$.
21. $2a-c-5d$.
22. $zx-2x^2+3y^2$.
23. $2t^3-3P+P^2t$.
24. a^2+at-t^2.
25. $-MN+9$.
26. $-a^2+7b^2-2c$.
27. $n+\frac{1}{2}$.
28. $3+2x$.
29. $1-\frac{1}{2}g$,
30. $1+2t-3t^2$.

Pages 89, 90.

1. $a+2$.
2. $N-5$.
3. $x-3$.
4. $P+4$.
5. $t-2$.
6. $y-5$
7. $H+7+\dfrac{26}{H-8}$.
8. $b-4$.
9. $m+13+\dfrac{90}{m-10}$.
10. $x+1$.
11. $3x-2$.
12. $3K+\dfrac{17K-6}{4K-6}$.
13. $2V+5$.
14. $2A-3$.
15. $5R+2$.
16. $2p-1$.
17. $1+3a$.
18. $4-5D$.
19. $a+b$.
20. $x-2y$.
21. $W+5g$.
22. $m+n$.
23. $P-4Q$.
24. $x+7$.
25. $a+1$.
26. $y+11+\dfrac{-8}{y+4}$.
27. $6E^2+1$.
28. $r^2-3r+\dfrac{13r+3}{2r+3}$.
29. B^2+B+1.
30. $y^2-yz+z^2+\dfrac{-2z^3}{y+z}$.
31. W^2-V^2.
32. H^3-t^3.
33. $12R-1$.
34. $b+1$.
35. x^4+x^2+1.
36. $n^4+n^2a^2+a^4$,
37. $2y^2-5y+7$,

38. $4d^2+6dt+9t^2+\frac{54t^3}{2d-3t}$. **40.** $A^2+3A+15+\frac{80}{A-5}$.

39. $2q^2-3q+5+\frac{-10q}{2q^2+3q-5}$. **41.** $g^2+\frac{6k^2}{g^2-3k}$.

42. n^2+a^2+an. **43.** $4h^2-12h+9$. **44.** $x+y$. **45.** $2t+2$; 22

46. $(8t^2+8t+2)\div(4t+2)$; $2t+1$; 21.

47. $1+2x+2x^2+2x^3+\cdots$; unlimited number.

48. $1+n-n^2+n^3-\cdots$. **49.** $1+2y+2y^2+2y^3+\cdots$.

Pages 90, 91.

1. $6a^6$. **4.** $24t^{3a}$. **7.** $-24W^{2+d}v^{1+d}$. **10.** x^{nm}. **13.** y^{3m}.
2. x^{3n}. **5.** $10R^{5a}$. **8.** $-10z^{3a-1}$. **11.** w^{11}. **14.** B^5.
3. y^{3m}. **6.** $a^{2n+1}b^{2n+1}$. **9.** a^{5n}. **12.** M^{a-b}. **15.** $-2x^{3n}$
16. $9A^{3n-5}$. **17.** $-8x^a$. **18.** $-5v^5t$.
19. $a^{2n}+a^nb^n+a^nc^n$. **21.** $x^{2a}-2x^ay^b+y^{2b}$.
20. $x^ny^3-x^3y^{n+3}+x^3y^3z^{n+3}$. **22.** $P^{n+1}+PQ^n+P^nQ^n+Q^{2n}$.
23. $a^{2x}-b^{2y}$. **25.** $V^{3t}-1$. **27.** $5a^n-2a^{n+1}$. **29.** x^n+y^n.
24. $x^{4n}-y^{4n}$. **26.** a^v-a^{2v}. **28.** $1+M^2+M^4$. **30.** $x^4+x^2y^2+y^4$.
31. $W^{2n}+V^{2n}$. **32.** $a^{2m}-a^mb^m+b^{2m}$. **33.** $S^{3a}+S^{2a}T^a+S^aT^{2a}+T^{3a}$.
34. $x^2+2xy+y^2-xz-yz+z^2$. **36.** $m^2-mn-mp-np+n^2+p^2$.
35. $a^2-ab+ac+bc+b^2+c^2$. **37.** $u+v+w$. **38.** $2a+b-3c$.

Pages 93, 94.

1. $a=-5$. **8.** $W=2\frac{2}{3}$. **14.** $V=2$. **20.** $Y=-6$.
2. $x=0$. **9.** $P=15\frac{1}{7}$. **15.** $E=9$. **21.** $A=2\frac{10}{11}$.
3. $y=\frac{3}{4}$. **10.** $D=8$. **16.** $T=1\frac{1}{2}$. **22.** $u=-\frac{1}{4}$.
4. $R=\frac{1}{13}$. **11.** $m=5\frac{4}{5}$. **17.** $n=\frac{1}{2}$. **23.** $Q=12$.
5. $t=-4$. **12.** $K=-8$. **18.** $K=7$. **24.** $w=-2$.
6. $v=\frac{1}{4}$. **13.** $c=5$. **19.** $S=-1$. **25.** $a=2$.
7. $d=21$. **26.** $z=1$.

Pages 95–98.

1. 17; 8. **2.** $22\frac{4}{7}$; $11\frac{3}{7}$. **3.** 36; 28. **4.** 54; 32. **5.** 17; 19. **6.** 9; 10; 11.
7. Coffee, 35¢ per pound; tea 80¢ per pound. **9.** 8 Posts; 26 two-cent papers.
10. 28 of lemonade; 15 of cider.
8. 32 one-cent; 19 two-cent. **11.** 200 gal. **12.** 6. **13.** $37\frac{1}{2}$; $90\frac{3}{8}$

14. $37\frac{1}{2}$ lb. of 28-cent, and $12\frac{1}{2}$ lb. of 36-cent coffee.
15. $3\frac{1}{3}$ lb. of 50-cent, and $6\frac{2}{3}$ lb. of 65-cent tea.
16. 2 lb. of 25-cent, and 6 lb. of 45-cent spice.
17. 44 doz. at 28¢; 36 doz. at 30¢.
18. $66\frac{2}{3}$ parts.
19. 6 qt.
20. $2\frac{2}{3}$ qt.
21. 8 qt.
22. $8\frac{3}{4}$ gal. cream; $1\frac{1}{4}$ gal. milk.
23. $12\frac{8}{51}$ gal. cream; $7\frac{43}{51}$ gal. milk.
24. 25 oz.
25. 4 oz.
26. $1300 at 5%; $500 at 6%.
27. $2800 at 4%; $2200 at 5%.
28. $1400 at 8%; $2800 at 6%.
29. 35 nickels; 16 dimes.
30. 11 quarters; 18 dimes.
31. 18 yr.
32. Son's age 10 yr.; father's age 40 yr.
33. 38 in. and 12 in.
34. $42\frac{6}{7}$ in. and $17\frac{1}{7}$ in.
35. 8 in.
36. 10 rd. by 16 rd.
37. 18 ft.
38. 36 ft. by 78 ft.

Pages 98, 100.

1. $3,500,000.
2. $8500.
3. $12,000.
4. $6000.
5. $1200.
6. 7 yr.
7. 3%.
8. $6.75.
9. $1.56–.
10. $4.80.
11. $2.55.
12. $6.79+.
13. $14.
14. 35¢.
15. $20.

Pages 100–103.

1. 180 mi.
2. $39\frac{7}{37}$ mi. per hour.
3. $4\frac{8}{9}$ sec.
4. $5\frac{1}{3}$ hr.
5. 3 hr.
6. 8 hr.
7. $8\frac{1}{2}$ hr.; $21\frac{1}{4}$ mi. from starting point of slower.
8. $3\frac{8}{19}$ hr.
9. $3\frac{1}{2}$ hr.
10. $\frac{11}{40}$ hr.; 6.6 mi.
11. 1080 mi.
12. $6\frac{3}{7}$ hr.
13. A, 9 mi. per hour;
B, 12 mi. per hour.
14. $32\frac{1}{4}$ mi. per hour;
$42\frac{1}{4}$ mi. per hour.
15. 2816 ft.
16. $1\frac{3}{5}$ sec.; 2200 ft.
17. $\frac{4}{5}$ sec.; 1320 ft.
18. $21\frac{9}{11}$ min. after 4.
19. $38\frac{2}{11}$ min. after 7.
20. $16\frac{4}{11}$ min. after 6.
21. $54\frac{6}{11}$ min. after 4.
22. $10\frac{10}{11}$ min. after 8.
23. $65\frac{5}{11}$ min.
24. 4 mo.
25. $334\frac{89}{154}$ da.

Pages 104, 105.

1. 72 gm.; 1440 gm.
2. 289 gm.; 72.25 gm.
3. 600 cu. cm.
4. $83\frac{1}{3}$ cu. cm.
5. 550 cu. cm.
6. 470588+ cu. cm.
7. 2724+ cu. cm.
8. 853− cu. cm.
9. 2114− cu. cm.
10. 28.2− cu. cm.
11. 13.57+.
12. 7.3.

Pages 106, 107.

1. 2 lb.
2. 12 lb.
3. $66\frac{2}{3}$ lb.
4. 5 ft.
5. $5\frac{23}{41}$ ft. from one end.
6. $25\frac{1}{3}$ in.
7. $6\frac{6}{7}$ in. from stone.
8. $22\frac{2}{9}$ in. from 240 lb. weight.
9. $25\frac{5}{7}$ in. from one carrying the more.
10. $27\frac{7}{9}$ in. and $22\frac{2}{9}$ in.
11. $25\frac{3}{5}$ in. from end nearer smaller horse.

Page 108.

1. $2\times10+5$; $3\times10^2+4\times10+7$; $4\times10^3+10^2+9\times10+6$.
2. $1000d+100c+10b+a$.
3. 36.
4. 84.
5. 86.
6. 47.
7. 42.
8. 74.
9. 321.

Pages 110, 111.

1. $a=5, b=2$.
2. $M=2, N=3$.
3. $E=1, d=-4$.
4. $x=-2, y=-3$.
5. $n=5, k=1$.
6. $P=4, R=2$.
7. $z=5, c=7$.
8. $T=-2, H=-3$.
9. $W=5, V=-5$.
10. $m=4, n=3$.
11. $t=3, a=7$.
12. $G=-6, v=-3$.

Pages 111–113.

1. \$3000 at 4%; \$2600 at 8%.
2. \$800 at 6%; \$1350 at 7%.
3. 20 dimes; 30 nickels.
4. $6\frac{18}{19}$ ft. from end nearer boy weighing 80 lb.
5. $27\frac{3}{11}$ in. from end nearer smaller horse.
6. 30 lb. of 28-cent; 10 lb. of 36-cent.
7. 450 cu. cm. copper; 150 cu. cm. zinc.
8. 3.2− cu. cm. gold; 6.8+ cu. cm. nickel.
9. 37.
10. 4 ft.; 6 ft.
11. Present route, 13,090 mi.; Panama route, 5294 mi.
12. Present route, 13,040 mi.; Panama route, 10,088 mi.
13. *Mauretania*, 790 ft.; *Britannica*, 207 ft.
14. New *Deutschland*, 16,000 tons; old *Deutschland*, 717 tons.
15. 9.52+ oz. beans; 14.65+ oz. pork.

Pages 114–116.

1. $t=\frac{I}{pr}$.
2. $t=\frac{s}{v}$.
3. $H=\frac{2A}{B}$.
4. $B'=\frac{2A-HB}{H}$.
5. $F=\frac{9}{5}C+32$.
6. $l=\frac{33000\text{ H. P}}{pan}$.
7. $n=\frac{s+360}{180}$.
8. $x=\frac{cy+b}{2a}$.
9. $A=1$.
10. $V=3S$.
11. $t=\frac{2a-1}{4-a}$.
12. $R=\frac{1+k}{2k}$.
13. $x=7a$; $y=4a$.
14. $x=17n$; $y=13n$.
15. $x=-11b$; $y=7$.
16. $x=11n$; $y=-4m$.
17. Square, 6 in.; rectangle, 9 in. by 4 in.
18. 7.
19. 16 yr.
20. 86½°.
21. 77½°.
22. 84 lb. nitrogen; 26 lb. phosphoric acid; 61 lb. potash.
23. 42 lb. tin; 14 lb. zinc.
24. 1650 ft. per second.
25. 30 lb.; 24 lb.
26. Speed of current, 2 mi. per hour; speed of crew in still water, 4 mi. per hour.
27. Local, 30 ft. per second; express, 45 ft. per second.

Page 118.

1. a^8.
2. R^{12}.
3. w^{20}.
4. k^{10}.
5. p^{14}.
6. n^{24}.
7. B^{16}.
8. V^6.
9. x^{18}.
10. A^{30}.
11. L^{12}.
12. t^{15}.
13. S^{24}.
14. x^{18}.
15. r^{21}.
16. Q^{36}
17. v^{30}.
18. N^{45}.
19. p^{60}.
20. b^{48}.
21. x^8y^4.
22. a^2b^{10}.
23. y^4z^{12}.
24. $r_1{}^4r_2{}^4r_3{}^4$.
25. $W^{12}g^4$.
26. H^8p^{14}.
27. a^6t^9.
28. $V^{12}y^{12}$.
29. A^9n^{15}.
30. $w^{18}n^{12}$.
31. $a^{12}s^{21}v^9$.
32. $9x^4y^{10}$.
33. $25M^2N^8$.
34. $4a^4b^8c^{12}$.
35. $64w^{20}v^4$.
36. $121A^2N^{16}$.
37. $225x^2y^8z^{14}$.
38. $36v^6t^8$.
39. $81S^{10}H^4$.
40. $256m^4n^{16}p^6$.
41. $27D^9W^{18}$.
42. $125x^{12}T^3$.
43. $8a^9n^6b^{12}$.
44. $64p^6t^{12}y^3$.
45. $216V^9n^6g^6$.
46. $4A^8$.
47. $9m^6n^{14}$.
48. $64E^{10}H^{12}$.
49. $4a^{18}b^{18}c^{18}$.
50. $100R^2S^8V^8$.
51. $M^{12}W^{12}$.
52. $9a^{10}b^8c^6$.
53. $-x^6y^6$.
54. $-8a^6t^{12}$.
55. $-27B^{18}T^6$.
56. $-64H^6g^{21}v^{27}$
57. $-8P^{15}V^6R^3$.
58. $-27L^6T^{12}$.
59. $-8w^6v^6$.
60. $72a^{12}$.
61. $72n^{22}$.

Pages 120, 121.

1. ±1; ±2; ±4; ±9; ±11; ±7; ±12; ±6.

2. $\pm a^2$; $\pm W^4$; $\pm x^3$; $\pm P^{10}$; $\pm n^8$; $\pm A^6$; $\pm y^5$.

3. $\pm M^7$; $\pm D^{11}$; $\pm S^9$; $\pm k^{16}$; $\pm 3x^3$; $\pm 4a^2$.

4. $\pm 7B^5$; $\pm 5v^4$; $\pm 10R^6$; $\pm 6a^2b^3$; $\pm 9x^4y$.

5. $\pm 4W^3V^6$; $\pm 3m^4n$; $\pm 5d^3t^2$; $\pm 11h^2p^2$; $\pm 15r_1^2r_2^4$.

6. $\pm 12a^3t^5$; $\pm 7w^7d^4$; $\pm 13x^2y^3z$; $\pm 25R^2m^8$.

7. $\pm 14a^5bc^2$; $\pm 16H^4T^4a$; $\pm 21A^7x^{10}y^3$; $\pm 20m^{20}n^{15}$.

8. 1; 2; 3; 4; 5; 6; 8; 10.

9. -1; -2; -3; -4; -5; -10.

10. A^2; n^4; t; W^5; x^7; P^{12}; r^6.

11. v^7; y^3; M^8; a^{11}; $-D^3$; $-z^4$; $-L^2$.

12. $-w^6$; $-T^{12}$; $-2x^2$; $-3b^6$; $-5P^3$.

13. $4R^2$; $3v^2t^3$; $-2m^4n$; $-5r_1^2r_2^2$; $-A^4t^3d$.

14. $-pq^2r^3$; $6M^3a^5$; $-3x^4y^3z^2$; $7V^5$; $-2FD^4$.

15. $7d$. **16.** $10x$. **17.** $4E$. **18.** 4 in.

Pages 121, 122.

1. $x=\pm5$.
2. $t=\pm9$.
3. $A=\pm6$.
4. $n=\pm10$.
5. $y=\pm4$.
6. $R=\pm8$.
7. $B=\pm11$.
8. $F=\pm15$.
9. $a=\pm2$.
10. $M=\pm3$.
11. $p=\pm4$.
12. $D=\pm2$.
13. $v=\pm7$.
14. $S=\pm12$.
15. $W=\pm5$.
16. 2 sec.
17. 1 sec.; 3 sec.; 6 sec.; 10 sec.
18. 2 in.
19. 3 in.

Pages 123, 124.

1. n^2-9.
2. A^2-36.
3. v^2-4.
4. x^2-25.
5. k^2-16.
6. n^2-49.
7. w^2-v^2.
8. $4a^2-9$.
9. $25P^2-1$.
10. $16S^2-49$.
11. $100-9R^2$.
12. $1-x^4$.
13. $36-4v^2$.
14. $4x^2-25z^2$.
15. $81M^2-9T^2$.
16. $25a^2-16b^2$.
17. $36W^2-49V^2$.
18. $144H^2-25a^2$.
19. a^4-81.
20. x^6-16.
21. $4A^4-1$.
22. $49s^4-16t^4$.
23. $1-36x^8$.
24. a^6-b^6.
25. x^8-y^8.
26. $P^{12}-Q^{12}$.
27. $16-A^2$.
28. $1-36x^2$.
29. $4M^2-49N^2$.
30. $25t^4-144$.
31. $81-100D^6$.
32. a^4-1.
33. $16-W^4$.
34. $1-x^8$.
35. $a-b$.
36. $x+y$.
37. $D+1$.
38. $4+R$.
39. $3h-5t$.
40. $9A^2+1$.
41. y^2-6.
42. $12n^3+1$.
43. $8S^2-5$.
44. $6x+7y^2$.
45. $n=2$.
46. $x=-6$.
47. $A=50$.
48. $R=-2$.

Page 124.

1. 399.
2. 896.
3. 396.
4. 1599.
5. 1591.
6. 884.
7. 2496.
8. 2491.
9. 4899.
10. 4896.
11. 3591.
12. 6391.
13. 6375.
14. 8096.
15. 9999.
16. 9991.
17. 12099.
18. 14396.
19. $3.96.
20. $3.91.
21. $8.96.
22. 2496 lb.
23. 3591 lb.
24. 6391 sq. rd.

Pages 125, 126.

1. $a^2+9a+20$.
2. x^2+4x+3.
3. $M^2+8M+12$.
4. $y^2+11y+28$.
5. n^2+7n+6.
6. $v^2+11v+24$.
7. $t^2-5t-36$.
8. $A^2+10A+21$.
9. $D^2+10D+25$.
10. $m^2+15m+56$.
11. $K^2+11K+10$.
12. $x^2+14x+48$.
13. $a^2-8a+15$.
14. $W^2-10W+16$.
15. R^2-7R+6.
16. $t^2-15t+36$.
17. $B^2-19B+48$.
18. $N^2-16N+63$.
19. $d^2-17d+30$.
20. $S^2-18S+72$.
21. $v^2-14v+40$.
22. $d^2-26d+160$.
23. $a^2+2a-15$.
24. $x^2+4x-96$.
25. $n^2+4n-32$.
26. $V^2+6V-27$.
27. $t^2+3t-40$.
28. $b^2-5b-84$.
29. $x^2-12x-45$.
30. y^2+5y-6.
31. $H^2-5H-36$.
32. $R^2-5R-150$.
33. $A^2+10A-56$.
34. $m^2+4m-192$.
35. $u^2-2u-99$.
36. $C^2+11C-126$.
37. x^4+7x^2+10.
38. a^6+8a^3-9.
39. P^4-14P^2+48.
40. y^8-4y^4-77.
41. c^4-6c^2-40.
42. $4a^2+16a+15$.
43. $9x^2+27x+14$.
44. $36t^2-30t+4$.
45. $64x^2+8x-6$.
46. $16W^2+12W-10$.
47. $25y^2+25y-14$.
48. $4c^2-30c+36$.
49. $64v^2+56v-144$.
50. $4x^4+16x^2+15$.
51. $49A^6+28A^3-12$.
52. $144W^4-48W^2-320$.
53. $4a^2+14ab+12b^2$.
54. $25x^2-40xy+12y^2$.
55. $36k^2-12kt-15t^2$.
56. $16m^2+24mn-27n^2$.
57. $81D^2+45Dv-24v^2$.
58. $49v^2-49vw-18w^2$.
59. $9a^4+48a^2b^2+48b^4$.
60. $25t^4-10t^2s^2-24s^4$.
61. $64B^4-24AB^2C-4A^2C^2$.
62. $16c^4-68c^2d^4-110d^4$.
63. $1-2a-15a^2$.
64. $x+3$.
65. $n-3$.
66. $A-5$.
67. $v-3$.
68. $T+3$.
69. $S-6$.
70. $W+9$.
71. $y+5$.
72. $r-7$.
73. $m-9$.
74. $x=3$.
75. $v=13$.
76. $n=25$.

Page 127.

1. 1056.
2. 2652.
3. 1806.
4. 4224.
5. 1638.
6. 1684.
7. 2544.
8. 5624.
9. 7221.
10. 9016.
11. 8544.
12. 7917.
13. 12656.
14. 19602.
15. 63503.
16. $10.88.
17. 2016 mi.
18. $12.24.
19. 9016 sq. ft.
20. 1120 lb.

Pages 128, 129.

1. n^2+6n+9.
2. $x^2+10x+25$.
3. P^2+4P+4.
4. $a^2-8a+16$.
5. $W^2-12W+36$.
6. t^2-6t+9.
7. $m^2+2mn+n^2$.
8. $x^2-2xy+y^2$.
9. $4a^2+12ab+9b^2$.
10. $25r^2-40rs+16s^2$.
11. $9C^2+48CD+64D^2$.
12. $49w^2-42wv+9v^2$.
13. $25B^2+90BT+81T^2$.
14. $100b^2-20bc+c^2$.
15. $81F^2+144FC+64C^2$.
16. $16K^2-56KL+49L^2$.
17. $144x^2+24xy+y^2$.
18. $A^2-30AB+225B^2$.
19. $121W^2+66Wa+9a^2$.
20. $16R^2-72Rn+81n^2$.
21. $25A^2+160AK+256K^2$.
22. x^4+2x^2+1.
23. a^4-6a^2+9.
24. n^4+10n^2+25.
25. t^4-20t^2+100.
26. $x^4-2x^2y^2+y^4$.
27. $A^4+4A^2B^2+4B^4$.
28. $9t^4-24t^2s^2+16s^4$.
29. $25w^6+20w^3v^3+4v^6$.
30. y^8+2y^4+1.
31. $16k^4-8k^2t+t^2$.
32. $z^{12}+10z^6+25$.
33. $144-168a^2+49a^4$.
34. $81+270R^3+225R^6$.
35. $4p^6-4p^3q^6+q^{12}$.
36. $1-16d^8+64d^{16}$.
37. $B^4-8AB^2C+16A^2C^2$.
38. $x^4y^4+10x^2y^2+25$.
39. $A^4-8A^3+16A^2$.

42. $t=4$. 43. $V=-1\frac{1}{2}$. 44. $k=1\frac{1}{2}$. 45. $u=-3$.

Page 129.

1. 1681.
2. 3481.
3. 5184.
4. 1444.
5. 4624.
6. 7921.
7. 6084.
8. 6889.
9. 8464.
10. 9801.
11. 9604.
12. 10201.
13. 11881.
14. 39601.
15. 21904.
16. 248004.
17. 62001.
18. 90601.
19. 252004.
20. 998001.

Page 130.

1. $a^2+b^2+c^2-2ab+2ac-2bc$.
2. $4x^2+9y^2+z^2+12xy+4xz+6yz$.
3. $m^2+4n^2+p^2-4mn-2mp+4np$.
4. $A^2+9B^2+C^2+6AB-2AC-6BC$.
5. $4u^2+36v^2+w^2-24uv+4uw-12vw$.
6. $A^4+4A^3+6A^2+4A+1$.
7. $x^4-4x^3+10x^2-12x+9$.
8. $r^4+s^4+t^4+2r^2s^2-2r^2t^2-2s^2t^2$.
9. $a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$.
10. $x^2+y^2+z^2+w^2-2xy+2xz-2xw-2yz+2yw-2zw$.
11. $a^6-2a^5-a^4+4a^3-a^2-2a+1$.
12. $1+4P-2P^2-4P^3+25P^4-24P^5+16P^6$.

Pages 130, 131.

1. x^2+xy+y^2.
2. m^2+mn+n^2.
3. w^2+wv+v^2.
4. x^2+x+1.
5. $1+P+P^2$.
6. D^2+2D+4.
7. s^2+3s+9.
8. $B^2+4B+16$.
9. $25+5y+y^2$.
10. $a^2+2ab+4b^2$.
11. $9t^2+3t+1$.
12. $d^2+4ds+16s^2$.
13. $4w^2+6wv+9v^2$.
14. $25M^2+5MN+N^2$.
15. $16A^2+20AB+25B^2$.
16. $100r^2+10r+1$.
17. $9b^2+18bc+36c^2$.
18. $16H^2+28HD+49D^2$.
19. x^2y^2+xy+1.
20. $4+2ab+a^2b^2$.
21. a^4+a^2+1.
22. $m^4+m^2n^2+n^4$.
23. $1+t^3+t^6$.
24. $4a^2+6ab^2+8b^4$.
25. $16W^4+4W^2g+g^2$.
26. $M^6+M^3N^3+N^6$.
27. $1+2y^4+4y^8$.
28. a^3-b^3.
29. m^3-n^3.
30. W^3-V^3.
31. x^3-1.
32. $8a^3-1$.
33. $27x^3-8y^3$.
34. $64s^3-125t^3$.
35. x^6-1.
36. $1-8D^6$.

Page 132.

1. x^2-xy+y^2.
2. W^2-Wg+g^2.
3. m^2-mn+n^2.
4. a^2-a+1.
5. y^2-2y+4.
6. $9-3t+t^2$.
7. $16-4N+N^2$.
8. $x^2-5x+25$.
9. $1-2b+4b^2$.
10. $w^2-3wv+9v^2$.
11. $P^2-5PQ+25Q^2$.
12. $4m^2-2mn+n^2$.
13. $9D^2-3Dt+t^2$.
14. $16x^2-20xy+25y^2$.
15. $100K^2-10Ka+a^2$.
16. $36m^2-18mn+9n^2$.
17. $b^2-7bd+49d^2$.
18. $4R^2-10Rc+25c^2$.
19. x^4-x^2+1.
20. $1-a^3+a^6$.
21. $W^4-W^2B+B^2$.
22. $4y^4-6y^2z^2+9z^4$.
23. $25P^4-5P^2+1$.
24. $36-6at^2+a^2t^4$.
25. $49-14v^3t^2+4v^6t^4$.
26. $H^6-H^3K^3+K^6$.
27. $9-3u^4+u^8$.

28. m^3+n^3.
29. x^3+1.
30. $1+a^3$.
31. W^3+V^3.
32. $27P^3+1$.
33. $64S^3+1$.
34. a^3+125b^3.
35. $1+x^6$.
36. y^6+z^9.
37. $8+27A^3$.

Pages 133–135.

1. a^{2n}; S^{4n}; x^{6n}; $A^{2n}B^{2n}$; y^{2n+2}; t^{2n-2}; a^{4n+6}.
2. $9X^{2a}$; $25X^{2a+2}Y^{2a-2}$; $144a^{4x}b^{6x}$; $49V^{4r-2}W^{4r+2}$.
3. $\pm X^n$; $\pm Y^{3a}$; $\pm P^{2n}Q^{5n}$; $\pm t^{n+1}$; $\pm v^{2x+3}$.
4. $\pm 4D^{3a+4}$; $\pm 5X^{4n-1}Y^{4n+1}$; $\pm 9a^{2n}t^{n+2}$.
5. X^{3n}; a^{6t}; $-P^{9v}$; $W^{3n}V^{3n}$; t^{3n+3}; t^{3n-3}; $-y^{9n+6}$.
6. $8A^{3n}$; $27x^{3\ +3}y^{3v-3}$;$-64M^{12a}N^{6a}$; $-125c^{6v-3}d^{6v+3}$.
7. w^n; Q^{2t}; $r_1{}^{4p}$; $-A^{2d}$; $-2x^ny^{2n}$.
8. $3x^{2n}t^{3m}$; $-4N^tg^{4t}$; $-5a^{3n}t^{6n}$.

9. $x^{2n}-1$.
10. $V^{2t}-9$.
11. $a^{2n+2}-4$.
12. $x^{2a}-y^{2b}$.
13. $A^{2n}+10A^n+21$.
14. $D^{4t}+2D^{2t}-24$.
15. $N^{2x}+7N^x-8$.
16. $9y^{4n}+12y^{2n}-12$.
17. $a^{2n}+2a^n+1$.
18. $x^{2n}-2x^ny^n+y^{2n}$.
19. $V^{2t}+2V^tT^v+T^{2v}$.
20. $4y^{6a}+12y^{3a}w^{2a}+9w^{4a}$.
21. $25H^{8t}-10H^{4t}d^{4t}+d^{8t}$.
22. $9p^{4d}+6p^{2d}q^{2c}+q^{4c}$.
24. $x^2+2xy+y^2-1$.
25. $a^4+2a^3+a^2-1$.
26. $m^2-2mn+n^2-p^2$.
27. $b^2-2bc+c^2-36$
28. $S^2+4St+4t^2-9$.
29. $a^2-10ab+25b^2-c^2$.
30. $N^2+6Nv+9v^2-25w^2$.
31. $a^2+2ac+c^2-4b^2$.
32. $P^2+2PR+R^2-Q^2$.
33. x^4+x^2+1.
34. $a^4+a^2b^2+b^4$.
36. $16n^2-9x^2-6xy-y^2$.
37. $4u^2-9v^2+6vw-w^2$.
38. $9P^2-16T^2+8TV-V^2$.
39. $a^2+2ab+b^2-c^2-2cd-d^2$.
40. $x^2-4xy+4y^2-t^2+2tz-z^2$.
41. $R^2+4RS+4S^2-4T^2+4TU-U^2$.
42. $x^6-2x^5+x^4-x^2+2x-1$.
43. $1+2a+a^2-a^4-2a^5-a^6$.
45. $x^2-2xy+y^2+5x-5y+6$.
46. $p^2+2pq+q^2-6p-6q-160$.
47. $4W^2-12WV+9V^2+10W-15V-24$.
48. $x^4+2x^3+5x^2+4x-12$.
49. $t^4-2t^3-3t^2+4t-140$.
50. $n^4+2n^3-2n^2-3n+2$.

Pages 137, 138.

1. $2(a+b)$.

2. $3(x-2y)$.

3. $4(B^2-4AC)$.

4. $5n(R+2Q)$.

5. $3d(x^2-y^2)$.

6. $A^2(A-4)$.

7. $\pi(R^2+R'^2)$.

8. $x^4(x^2+5)$.

9. $m^2(n^2-m^2)$.

10. $2\pi R(R+H)$.

11. $s^2t^2(s^4+t^4)$.

12. $5w^2(w-2v)$.

13. $9x^3(2x^2-1)$.

14. $c^2n(c^2+2n)$.

15. $7P^3(P+3)$.

16. $x(x^2-x+1)$.

17. $2V_1(2V_1+3V_2+2V_3)$.

18. $H^4(H^2-6Hb+2b^2)$.

19. $vt(v+2t+t^2)$.

20. $2nm^3(4n^2+3nm+2m^2)$.

21. $3a^2b(a^2-2ab+b^2)$.

22. $2\pi(R^2+r^2+Rr)$.

23. $a^2b^2c^2(a^2b^3+b^2c^4+a^3c^2)$.

24. $6w^3d(4w^2d^2-2+7w^3d)$.

25. $2v^6(3v^2+4v^4-8)$.

26. $-x(x^2+x+1)$.

27. $-1(a^2+ab+b^2)$.

28. $2x^2(2x^4-3x^2y^2+4xy^3-6y^4)$.

29. $-2M^2(5M^6-6M^4+8M^2-4)$.

30. $7a^3y^2(8y^3-2ay^2+4a^2y-15a^3)$.

33. 1350; 7360; 71875; 78.54; 1024; 313600.

35. 621; 224; 624; 1216; 2024; 3024; 4216; 5621; 7216; 9021; 7209; 7224; 5609; 5625; 9009; 9024.

36. To square any number ending in 5, take the product of the number of tens by a number one greater for the hundreds, and annex 25.

37. 625; 2025; 1225; 4225; 7225; 3025; 9025; 5625; 11025; 13225.

Pages 139, 140.

1. $(x+y)(3+n)$.

2. $(P-Q)(a+b)$.

3. $(a+b)(v-t)$.

4. $(m-n)(2x-3y)$.

5. $(h-k)(5w+u)$.

6. $(x-y)(a-b)$.

7. $(A-5)(s-t)$.

8. $(r-R)(p+q)$.

9. $(N-M)(B^2-4AC)$.

10. $(w-v)(x-1)$.

11. $(a+b)(c+d)$.

12. $(m+y)(n+x)$.

13. $(v+w)(t+2)$.

14. $(g-h)(W+k)$.

15. $(m+n)(p-q)$.

16. $(a-b)(s-t)$.

17. $(u-R)(v+S)$.

18. $(n+t)(m-s)$.

19. $(x+y)(x+z)$.

20. $(p-q)(p+r)$.

21. $(M-N)(M+P)$.

22. $(v-u)(2+v)$.

23. $(s+y)(3r-2x)$.

24. $(2-m)(t+2)$.

25. $(h^2-m^2)(k^2-1)$.

26. $(x-y)(W^2-V^2)$.

27. $(F+G)(C+2)$.

28. $(q-3)(p-2)$.

29. $(R-1)(R-r)$.

30. $(2-c)(a^2+x^2)$.

Pages 141, 142.

1. Yes.
2. Yes.
3. Yes.
4. No.
5. No.
6. No.
7. No.
8. Yes.
9. No.
10. No.
11. No.
12. No.
13. Yes.
14. Yes.
15. No.
16. Yes.
17. $x^2+2xy+y^2$.
18. $a^2-2ab+b^2$.
19. $m^2+2mn+n^2$.
20. $4A^2+4A+1$.
21. $16-24t+9t^2$.
22. $25v^2-30vw+9w^2$.
23. $W^2+49+14W$.
24. $36+x^2-12x$.
25. $a^2-20a+100$.
26. $2x^2+1+x^4$.
27. $4P^3+1+4P^6$.
28. $r_1^2-12r_1r_2+36r_2^2$.
29. $4N^2-32N+64$.
30. b^4+42b^2+441.
31. $(x+3)^2$.
32. $(R-2)^2$.
33. $(t+4)^2$.
34. $(5-a)^2$.
35. $(1+6n)^2$.
36. $(7W-2)^2$.
37. $(v+1)^2$.
38. $(9-A)^2$.
39. $(x^2+2)^2$.
40. $(4-3D^2)^2$.
41. $a(x-3y)^2$.
42. $2(n+5m)^2$.
43. $t(S-15V)^2$.
44. $(4H+7d)^2$.
45. $(5b-6c)^2$.
46. $v(2w+9v)^2$.
47. $(M-25N)^2$.
48. $(3r_1+5r_2)^2$.
49. $x(x^2-y^2)^2$.
50. $a^2(a+2b^2)^2$.
51. $(P-3Q^3)^2$.
52. $(9m^2+2n)^2$.
53. $(5W^2-4g)^2$.
54. $x(xy+1)^2$.
55. $(cd+mn)^2$.
56. $(Rr-4t)^2$.
57. $4v(3v+1)^2$.
58. $z^2(5+9z)^2$.
59. $(8t_1-7t_2)^2$.
60. $36A(A+2)^2$
61. $-1(1+3a)^2$.

Pages 143, 144.

1. $(t+2)(t+3)$.
2. $(a+4)(a+1)$.
3. $(n+2)(n+6)$.
4. $(P+2)(P+7)$.
5. $(x+5)(x+4)$.
6. $(S+8)(S+1)$.
7. $(v+2)(v+8)$.
8. $(a+9)(a+1)$.
9. $(R+6)(R+4)$.
10. $(k+3)(k+7)$.
11. $(V+2)(V+9)$.
12. $(d+3)(d+8)$.
13. $(M+5)(M+6)$.
14. $(b+2)(b+10)$.
15. $(y+3)(y+9)$.
16. $(A+4)(A+8)$.
17. $(r+6)(r+9)$.
18. $(m+7)(m+10)$.
19. $(c+3)(c+10)$.
20. $(p+6)(p+8)$.
21. $(B-3)(B-2)$.
22. $(N-4)(N-1)$.
23. $(w-6)(w-2)$.
24. $(Q-7)(Q-2)$.
25. $(z-5)(z-4)$.
26. $(t-9)(t-1)$.
27. $(D-2)(D-8)$.
28. $(V-3)(V-7)$.
29. $(a-4)(a-6)$.
30. $(T-8)(T-1)$.
31. $(y-2)(y-15)$.
32. $(H-5)(H-9)$.
33. $(W-3)(W-1)$.
34. $(m-4)(m-8)$.
35. $(c-5)(c-7)$.
36. $(B-11)(B-5)$.

37. $(t-7)(t-1)$.
38. $(F+5)(F-3)$.
39. $(x+3)(x-1)$.
40. $(R+5)(R-2)$.
41. $(K+4)(K-1)$.
42. $(s+7)(s-5)$.
43. $(y+6)(y-3)$.
44. $(a+7)(a-4)$.
45. $(C+6)(C-2)$.
46. $(m+7)(m-2)$.
47. $(b+8)(b-5)$.
48. $(x+9)(x-4)$.
49. $(B-6)(B+3)$.
50. $(g-8)(g+5)$.
51. $(m-7)(m+2)$.
52. $(T-6)(T+2)$.
53. $(x+2y)(x+3y)$.
54. $(a+2b)(a+5b)$.
55. $(w-3v)(w-2v)$.
56. $(r-5s)(r-2s)$.
57. $(p-7q)(p+4q)$.
58. $(C-7D)(C+2D)$.
59. $(y+7z)(y-2z)$.
60. $(ab-6)(ab+3)$.
61. $(mn-7)(mn+5)$.
62. $(H-24)(H+5)$.
63. $(V-6)(V+5)$.
64. $a(b-10)(b+7)$.
65. $x(y+10)(y-9)$.
66. $w^2(v+11)(v-9)$.
67. $d^3(m+8)(m-7)$.
68. $2s(t-5)(t-23)$.
69. $3(x-20)(x-10)$.
70. $6c(b+10)(b+11)$.
71. $t^2(a+27)(a+3)$.
72. $4w(n-12)(n+5)$.
73. $6v(y+17)(y-2)$.
74. $(6+a)(1-a)$.

Pages 146, 147.

1. $(a+1)(10a+7)$.
2. $(3t+5)(t+2)$.
3. $(3x+1)(x+3)$.
4. $(m+2)(5m+2)$.
5. $(2V+5)(V+2)$.
6. $(3A+7)(A+1)$.
7. $(2r+3)(4r+5)$.
8. $(3B-5)(B-2)$.
9. $(y-1)(10y-7)$.
10. $(3S-1)(S-3)$.
11. $(5g-2)(g-2)$.
12. $(5-4x)(3-2x)$.
13. $(2P-5)(P-2)$.
14. $(5-c)(1-2c)$.
15. $(2-3k)(2-k)$.
16. $(4R-9)(R+3)$.
17. $(3m-2)(2m+7)$.
18. $(3D+5)(D-1)$.
19. $(3a+4)(a-1)$.
20. $(3v-2)(v+1)$.
21. $(4H-5)(H+1)$.
22. $(4b+9)(b-3)$.
23. $(3n+2)(2n-7)$.
24. $(3L-5)(L+1)$.
25. $(3F-4)(F+1)$.
26. $(3R+2)(R-1)$.
27. $x(2a+7)(a-4)$.
28. $d(2c-5)(3c+7)$.
29. $(3z+4)(2z-5)$.
30. $2(5x+6)(2x-7)$.
31. $3(4g+1)(3g-2)$.
32. $2a(2y-9)(y+2)$.
33. $5g(3W-4)(W+1)$.
34. $v^2(5w-7)(6w+1)$.
35. $t(2Q+5)(Q-5)$.
36. $(2C+15)(2C+1)$.
37. $(4x+3y)(x+5y)$.
38. $(2M+N)(M+7N)$.
39. $(d-k)(2d-k)$.
40. $(2E-5F)(3E-4F)$.
41. $(7b-2c)(2b-c)$.
42. $(7A-n)(A-4n)$.
43. $(3R+2t)(7R-4t)$.
44. $(2r_1-9r_2)(r_1+2r_2)$.
45. $S(3T-5T')(T+T')$.
46. $n^2(4G+d)(3G-2d)$.
47. $2x(2b-5d)(b+2d)$.
48. $4(3Y-4Z)(Y+Z)$.
49. $(3a^2+2)(a^2-1)$.
50. $3t(2w+9v^2)(w-2v^2)$.

Pages 147, 148.

2. $(N+2)(N-2)$.
3. $(a+3)(a-3)$.
4. $(V+1)(V-1)$.
5. $(t+4)(t-4)$.
6. $(1-P)(1+P)$.
7. $(2-w)(2+w)$.
8. $(k-5)(k+5)$.
9. $(4-S)(4+S)$.
10. $(H-8)(H+8)$.
11. $(7-m)(7+m)$.
12. $(10-v)(10+v)$.
13. $(x-9)(x+9)$.
14. $(A+12)(A-12)$.
15. $(1-4x)(1+4x)$.
16. $(1-7t)(1+7t)$.
17. $(3+2B)(3-2B)$.
18. $(h+k)(h-k)$.
19. $(9m+8n)(9m-8n)$.
20. $(x-7y)(x+7y)$.
21. $(4R-5D)(4R+5D)$.
22. $(2c-5d)(2c+5d)$.
23. $(6E-F)(6E+F)$.
24. $(9W-7t)(9W+7t)$.
25. $(C+10D)(C-10D)$.
26. $(4x-11y)(4x+11y)$.
27. $2(5a+4b)(5a-4b)$.
28. $y^2(x+1)(x-1)$.
29. $w(w+1)(w-1)$.
30. $6a(t+2)(t-2)$.
31. $4m^2(2m+3)(2m-3)$.
32. $t(H+5t)(H-5t)$.
33. $7g^2(2V+g)(2V-g)$.
34. $(xy+1)(xy-1)$.
35. $4(1+3at)(1-3at)$.
36. $8W^2(5P+W)(5P-W)$.
37. $(m^2+n^2)(m+n)(m-n)$.
38. $(4+T^2)(2+T)(2-T)$.
39. $(a^2+1)(a+1)(a-1)$.
40. $\pi(r_1+r_2)(r_1-r_2)$.
41. $2\pi R(R+2H)(R-2H)$.
42. Area of shaded border $=A^2-a^2=(A+a)(A-a)$.
43. Area of ring $=\pi R^2-\pi r^2=\pi(R+r)(R-r)$.
44. 28.2744 sq. in.
45. 4105+lb.
46. 300; 280; 720; 300; 760; 460.

Pages 149, 150.

1. $(a+b+1)(a+b-1)$.
2. $(x-y+z)(x-y-z)$.
3. $(S+T+5)(S+T-5)$.
4. $(h-k+4t)(h-k-4t)$.
5. $(2m+n+6)(2m+n-6)$.
6. $(1+A+B)(1-A-B)$.
7. $(2+2w+v)(2-2w-v)$.
8. $(5a+b-c)(5a-b+c)$.
9. $(6r_1+r_2-4r_3)(6r_1-r_2+4r_3)$.
10. $(7V+3T-S)(7V-3T+S)$.
11. $(x+y+a+b)(x+y-a-b)$.
12. $(M-N+W-V)(M-N-W+V)$.
13. $(C+D+4E-F)(C+D-4E+F)$.

14. $(1-x+v-t)(1-x-v+t)$.

15. $(m+n+b)(m+n-b)$.

16. $(R-2A+B)(R-2A-B)$.

17. $(1+x+y)(1+x-y)$.

18. $(r_1+3r_2+r_3)(r_1+3r_2-r_3)$.

19. $(x^2+2x+1)(x^2+2x-1)$.

20. $(W-3D+2M)(W-3D-2M)$.

21. $(1-4x+z)(1-4x-z)$.

22. $(1+a+b)(1-a-b)$.

23. $(2+M-3V)(2-M+3V)$.

24. $(1+h-7k)(1-h+7k)$.

25. $(A+3B+5C)(A-3B-5C)$.

26. $(4U+2V-9W)(4U-2V+9W)$.

27. $(S+T-2a)(S-T+2a)$.

28. $(4r+2p-3q)(4r-2p+3q)$.

29. $(A-4B-2C)(A-4B+2C)$.

30. $(1+t-3m)(1-t+3m)$.

31. $(a+b+c+d)(a+b-c-d)$.

32. $(x-2+m-3n)(x-2-m+3n)$.

33. $(1+p+x-y)(1+p-x+y)$.

34. $(W-5g+2a-3t)(W-5g-2a+3t)$.

35. $(M-6d+2N+k)(M-6d-2N-k)$.

36. $(x^2-8n+E-5m^2)(x^2-8n-E+5m^2)$.

Page 151.

2. $(m-n)(m^2+mn+n^2)$.

3 $(x-2)(x^2+2x+4)$.

4. $(A-1)(A^2+A+1)$.

5. $(1-P)(1+P+P^2)$.

6. $(2-r)(4+2r+r^2)$.

7. $(3-t)(9+3t+t^2)$.

8. $(2w-v)(4w^2+2wv+v^2)$.

9. $(T-3U)(T^2+3TU+9U^2)$.

10. $(4y-1)(16y^2+4y+1)$.

11. $(M-5N)(M^2+5MN+25N^2)$.

12. $(2r_1-3r_2)(4r_1^2+6r_1r_2+9r_2^2)$.

13. $(4g-3a)(16g^2+12ga+9a^2)$.

14. $(6V-5V_1)(36V^2+30VV_1+25V_1^2)$.

15. $(ab-1)(a^2b^2+ab+1)$.

16. $x(3x-1)(9x^2+3x+1)$.

17. $a^2(5-a)(25+5a+a^2)$.

18. $2P(t-2P)(t^2+2tP+4P^2)$.

19. $3y(3y-2)(9y^2+6y+4)$.

20. $(m-n^2)(m^2+mn^2+n^4)$.

21. $(2A^2-3)(4A^4+6A^2+9)$.

22. $(v^2-4w)(v^4+4v^2w+16w^2)$.

23. $(5-Z^4)(25+5Z^4+Z^8)$.

24. $(3R^3-V)(9R^6+3R^3V+V^2)$.

25. $4\pi r(r-2h)(r^2+2rh+4h^2)$.

26. Volume of shell $=\frac{1}{6}\pi D^3-\frac{1}{6}\pi d^3=\frac{1}{6}\pi(D-d)(D^2+Dd+d^2)$.

27. 19.37+ cu. in.; 79.58+ cu. in.; 381.18+ cu. in.; 862.37− cu. in.

Page 152.

1. $(m+n)(m^2-mn+n^2)$.
2. $(A+1)(A^2-A+1)$.
3. $(1+p)(1-p+p^2)$.
4. $(w+2)(w^2-2w+4)$.
5. $(H+4)(H^2-4H+16)$.
6. $(3+v)(9-3v+v^2)$.
7. $(5+t)(25-5t+t^2)$.
8. $(D+6)(D^2-6D+36)$.
9. $(1+2a)(1-2a+4a^2)$.
10. $(1+3x)(1-3x+9x^2)$.
11. $(4N+1)(16N^2-4N+1)$.
12. $(2a+b)(4a^2-2ab+b^2)$.
13. $(y+3z)(y^2-3yz+9z^2)$.
14. $(W+4D)(W^2-4WD+16D^2)$.
15. $(3S+2T)(9S^2-6ST+4T^2)$.
16. $(5r_1+3r_2)(25r_1^2-15r_1r_2+9r_2^2)$.
17. $(4a+3t)(16a^2-12at+9t^2)$.
18. $2(2x+3y)(4x^2-6xy+9y^2)$.
19. $(ab+1)(a^2b^2-ab+1)$.
20. $(3+mn)(9-3mn+m^2n^2)$.
21. $P^2(3+P)(9-3P+P^2)$.
22. $x(2x+5)(4x^2-10x+25)$.
23. $3D(3D+2)(9D^2-6D+4)$.
24. $(v+t^2)(v^2-vt^2+t^4)$.
25. $(2N^2+1)(4N^4-2N^2+1)$.
26. $5(2a^2+3b)(4a^4-6a^2b+9b^2)$.
27. $\pi(D+d)(D^2-Dd+d^2)$.
28. $2(3R_1^3+R_2)(9R_1^6-3R_1^3R_2+R_2^2)$.
29. $3\pi r(r+3h)(r^2-3rh+9h^2)$.
30. $(w^2+v^2)(w^4-w^2v^2+v^4)$.

Pages 153–155.

1. $A^2(A^3+3A+1)$.
2. $2xy(x+y)(x+2y)$.
3. $(2w+3v)^2$.
4. $n(m+3)^2$.
5. $(P+6)(P+2)$.
6. $(t-6)(t-2)$.
7. $(R+6)(R-2)$.
8. $(n-6)(n+2)$.
9. $a(a+3)(a+5)$.
10. $M(M-3)(M-5)$.
11. $d(d+5)(d-3)$.
12. $v^2(1+v)(1-v)$.
13. $5xy(x-y)^2$.
14. $a(1-a)(1+a+a^2)$.
15. $D(D-5)(D+3)$.
16. $(M+6)(M+7)$.
17. $(6-y)(7-y)$.
18. $(W-7)(W+6)$.
19. $(a+7)(a-6)$.
20. $t(t-2)^2$.
21. $(2-B)(4+2B+B^2)$
22. $x(x+8)(x-6)$.
23. $x(x-8)(x+6)$.
24. $(W-5g)(W-4g)$.
25. $(W+5g)(W+4g)$.
26. $(H^2+4t^2)(H+2t)(H-2t)$.
27. $(a^2+8)(a+1)$.
28. $(1+2m-v)(1-2m+v)$.
29. $y(x+7)(x-4)$.
30. $(N+8)(N+9)$.
31. $t(a+8)(a+9)$.
32. $(A+10)(A-5)$.
33. $(A-10)(A+5)$.
34. $(2x+1)(x+1)$.

35. $(t-5)(t^2+5t+25)$. **37.** $(12L-M^2)(12L+M^2)$.

36. $(az-bw)(az+bw)$. **38.** $(a-b)(x-1)$.

39. $(v-5)(v-10)$. **40.** $(xy-14)(xy+4)$. **41.** $(2a-1)(a-1)$.

42. $(Q^2t^3+3V)(Q^2t^3-3V)$. **44.** $(m+n^2)(m^2-mn^2+n^4)$.

43. $(x-1)(y-1)$. **45.** $(2A+B)(A+2B)$.

46. $(v+2)(v^2-2v+4)$. **48.** $(D^2-5)(D^2-2)$. **50.** $(V^2+13)(V^2-2)$.

47. $(wv+3t)(wv+2t)$. **49.** $(x^2y^2+5)(x^2y^2+2)$. **51.** $(3a+1)(2a+3)$.

52. $(4w-5)(3w-2)$. **67.** $(t+5)(t^2-5t+25)$.

53. $(3+T)(9-3T+T^2)$. **68.** $(5m+n)(m-2n)$.

54. $4M^2(3+4N)(3-4N)$. **69.** $3(3y-1)(9y^2+3y+1)$.

55. $(3x+y)(x+2y)$. **70.** $(4H-1)(3H+2)$.

56. $(9B^2+4C^2)(3B+2C)(3B-2C)$. **71.** $(6+v)(36-6v+v^2)$.

57. $(a-2)(b-3)$. **72.** $ab(3x-1)(x+1)$.

58. $(2m-5n)(m-2n)$. **73.** $(2D+1)(4D^2-2D+1)$.

59. $V(V+7)(V-6)$. **74.** $(6s-5t)(s-t)$.

60. $(x+7y)(x-5y)$. **75.** $(9x^2+10v^2)(9x^2-10v^2)$.

61. $(a+3)(a-2)(a^2-a+6)$. **76.** $(N-2)(N+a)$.

62. $(p+1)(p-1)(q+1)(q-1)$. **77.** $(2x+a+b)(2x-a-b)$.

63. $(7w+1)(w+5)$. **78.** $(3E+1)(9E^2-3E+1)$.

64. $(2A-1)(A+3)$. **79.** $(2A+13B)(2A+3B)$.

65. $(2A-3)(A-1)$. **80.** $(5v+t)(25v^2-5vt+t^2)$.

66. $(a+2b+c)(a+2b-c)$. **81.** $(2a-2b+y+z)(2a-2b-y-z)$.

82. $(8B-1)(B+2)$. **83.** $(r_1+3r_2+1+2r_3)(r_1+3r_2-1-2r_3)$.

84. $5(2M+3V)(4M^2-6MV+9V^2)$.

85. $(H-t)(H+t+v)$. **88.** $(5-z)(3+z)$.

86. $(3x-2)(x+1)$. **89.** $(x-z)(y-z)$.

87. $(A-1)(A+1)(A+n)$. **90.** $(3x-4)(3x-5)$.

91. $(D-6)(D^2+6D+36)$.

92. $(W+V)(W-V)(W^2-WV+V^2)(W^2+WV+V^2)$.

93. $m(n+10)(n-3)$. **97.** $(5Q+t)(Q-16t)$.

94. $(1+a^2-2t^2)(1-a^2+2t^2)$. **98.** $(x-a)(x+1)(x^2-x+1)$.

95. $(3a+3b-d)(3a-3b+d)$. **99.** $(v+w)(v^2+w^2)$.

96. $(v-6t^2)(v^2+6vt^2+36t^4)$. **100.** $(R-r)(R+2)$.

Page 156.

1. $2a^2b^2$.
2. $3a^3b^3c^2$.
3. $5W^4V^3$.
4. $8mnp$.
5. $8M^2N^5$.
6. y^6z^2.
7. $7wv^3t^2$.
8. $7CDP^3$.
9. $6mp^2$.
10. $4(v+t)^2$.
11. $7(E-F)^4$.
12. $2(x-1)$.
13. $2(y+1)^2$.
14. $13(m-n)(m+n)^3$.
15. $A-1$.
16. $1-t$.
17. $V-3$.
18. $a-b$.
19. $2x(x+2)$.
20. $R-3$.
21. $w+1$.
22. $2a-3$.
23. $r_1r_2(r_2-r_1)$.
24. $A+B$.
25. $2m(m+2n)$.

Pages 157, 158.

1. $6a^2b^3$.
2. $30x^3y^4$.
3. $24w^6v^2u^3$.
4. $162M^2N^6$.
5. $42p^3q^3r^2$.
6. $84A^5B^3C^3$.
7. $100a^2m^2n^3$.
8. $60R^5r^8$.
9. $2(x+y)^3$.
10. $3(A-B)^3(A+B)^2$.
11. $2n^2(n-1)(n+1)$.
12. $t^3(1+t)^2(1-t)$.
13. $(S+1)(S-1)^2$.
14. $(D+1)(D-1)(D-2)$.
15. $y(1-y)(1+y)(1+2y)$.
16. $(P-Q)(P+Q)(2P+Q)$.
17. $(t+2)(t+3)(t+4)$.
18. $(v+6)(v-5)(v-1)$.
19. $(A+2)(2A-1)(3A+1)$.
20. $(2x-5)(x+2)(2x+3)(x-1)$.
21. $2N(N+1)(N+2)(N+3)$.
22. $2k^2(k+1)(k-1)(k^2+k+1)$.
23. $(P+1)(P-1)(P^2+1)(P+2)$.
24. $(1-d)(1+d)(1+d+d^2)$.
25. $2z^2(1-z)(1+z)(z-2)(z^2+1)$.
26. $12B^2(B+7)(B-2)^2$.
27. $(a-b)(a+b)(a^2+b^2)(a^2+ab+b^2)$.
28. $(a+b)^2(a+c)$.
29. $(x-y)(x+y)(x+v)$.
30. $(A^2+B^2)^2(A^2-B^2)^2$.

Pages 159–161.

1. 2, 4.
2. 2, −6.
3. 3, −3.
4. 4, −4.
5. $2\frac{1}{2}$, $-2\frac{1}{2}$.
6. $2\frac{1}{3}$, $-2\frac{1}{3}$.
7. 8, −8.
8. 2, −2.
9. 1, −1.
10. 2, −2.
11. 7, −7.
12. −1, 9.
13. 11, −2.
14. 18, −3.
15. −2, −10.
16. 2, −3.
17. −4, −10.
18. 5, 6.
19. 2, −7.
20. 3, 4.
21. 2, −5.
22. 3, 3.
23. 9, −6.
24. 2, $\frac{1}{2}$.
25. −3, $\frac{1}{2}$.
26. 1, $-\frac{1}{5}$.
27. $-\frac{2}{7}$, $1\frac{2}{3}$.
28. 0, 1, −1.
29. 0, 1, −3.
30. 0, 5, −2.
31. 0, −2, $-\frac{1}{2}$.
32. −1, −3, 3.
33. 1, 2, −2.
34. $\frac{1}{3}$, 3, −3.
35. 2, 4, −4.
36. 2, −2, 1, −1.
37. 0, 0, 3, −3.
38. 0, 0, 3, 4.
39. 3, 3, −3, −3.
40. 1, −1, 3, −3.
41. Division by $n-1$ is not allowable since it equals 0.

42. 8, or -13.
43. 13 and 14.
44. 9 and 11.
45. 7, 8, and 9.
46. Altitude 24 in., base 12 in.
47. Altitude 8 ft., base 15 ft.
48. Altitude 9 ft., base 12 ft.
49. Altitude 6 in., bases 8 in. and 10 in.
50. Altitude 12 in., bases 12 in. and 8 in.
51. 20 in. square.
52. 16 in. by 24 in.
53. 10 rd.
54. 6 ft. **55.** 10 in. **56.** 2 in. **57.** 3 in. **58.** 10 in. **59.** 10 in. **60.** 16.

Pages 162, 163.

1. $4B^3(A^2-2B^2)$.
2. $x^n(a+b)$.
3. $v^n(1-w)$.
4. $a^{m-1}t(a+t)$.
5. $P^x(P^x-5Q^x)$.
6. $r^{n-1}s^{n-1}(s^2+r^2)$.
7. $5v^{a-1}y^{a-1}(v-2y)$.
8. $4M^{2n}(2M+3N)$.
9. $(ab+a-b)(ab-a+b)$.
10. $(m-n)(m+n+x)$.
11. $(V-W)(V+W+t)$.
12. $(x-a)(1+x-a)$.
13. $(x-y)(x+y-1)$.
14. $(m^2+1)(m+1)(m-1)(n^2+1)(n+1)(n-1)$.
15. $2r_2^2(r_1+r_2)(r_1-r_2)$.
16. $(A+B)(t^2+t+1)$.
17. $(2+d)(a-b+c)$.
18. $(n+5)(y+1)(y+2)$.
19. $(p-q)(p+2)^2$.
20. $(x+y-z-w)(x-y-z+w)$.
21. $(W^2+V^2)(W^2-V^2+1)$.
22. $(A-B+C)(A^2+AB+B^2)$.
23. $(m+n+p)(m+n-p)(m+p-n)(-m+p+n)$.
24. $(a-b)(a+b+c)$.
25. $(w-u+v)(w-u-v)$.
26. $(x^n-y^n)^2$.
27. $a^n(1-3t^n)^2$.
28. $-1(3P-1)^2$.
29. $-1(t+2)^2(t-2)^2$.
30. $-1(x-y^2)^2$.
31. $-1(2M+3N)^2$.
32. $(a+b+2)(a+b+5)$.
33. $(x-y-10)(x-y+4)$.
34. $(ax+b)(x+1)$.
35. $(ax+b)(x-1)$.
36. $(2y+b)(y+2a)$.
37. $(2W-n)(W-m)$.
38. $(av+1)(v+b)$.
39. $(x^n+y^n)(x^n-y^n)$.
40. $(2A^{2n}+3B^{3n})(2A^{2n}-3B^{3n})$.
41. $(w^n+v^n)(w^{2n}-w^nv^n+v^{2n})$.
42. $(1-Q^n)(1+Q^n+Q^{2n})$.
43. $(x-y)(x^2+xy+y^2)(x+y)(x^2-xy+y^2)$.
44. $(1-n)(1+n+n^2)(1+n)(1-n+n^2)$.
45. $(D-1)(D^2+D+1)(D+1)(D^2-D+1)$.
46. $(a-2b)(a^2+2ab+4b^2)(a+2b)(a^2-2ab+4b^2)$.
47. $(R^2+r^2)(R^4-R^2r^2+r^4)(R+r)(R^2-Rr+r^2)(R-r)(R^2+Rr+r^2)$.
48. $a, 2a$.
49. $n, -n$.
50. $2v, 2v$.
51. $3t, -\frac{1}{2}t$.
52. $w+v, -w-v$.
53. $3+a-2b, 3-a+2b$.
54. $9r, -7r$.
55. $b, -b, -n$.
57. 1768; 2196; 2538; 4154; 4316; 79870; 76638; 133614; 236160; 230232; 9612.

Pages 165, 166.

1. $-\frac{3}{5}$. **2.** $-\frac{2}{5}$. **3.** $-\frac{4}{5}$. **4.** $-\frac{10}{13}$. **5.** $\frac{4}{7}$.

6. $\frac{3x}{5y}$. **8.** $-\frac{7n}{9t^2}$. **10.** $-\frac{d}{M^3}$. **12.** $\frac{at^2}{vg^2}$. **14.** $\frac{x}{a-b}$.

7. $-\frac{2a^2}{10b^2}$. **9.** $\frac{1}{V^2}$. **11.** $-\frac{100}{P^4}$. **13.** $\frac{-1}{a-b}$. **15.** $\frac{n-m}{a-b}$.

16. $\frac{6}{x-y}-\frac{2}{x-y}$. **18.** $\frac{2x+1}{x^2-1}+\frac{3x}{x^2-1}-\frac{1-x}{x^2-1}$.

17. $\frac{a}{a-b}-\frac{b}{a-b}+\frac{a+b}{a-b}$. **19.** $\frac{5at}{a^2-t^2}+\frac{a^2}{a^2-t^2}+\frac{a^2+t^2}{a^2-t^2}$.

20. $\frac{1+r}{1-r}+\frac{r}{1-r}-\frac{1}{1-r}$.

21. $+\frac{x}{y}$. **23.** $+\frac{2x}{3y}$. **25.** $+\frac{-1}{A}$. **27.** $+\frac{-\pi r^4}{\pi r^4}$.

22. $+\frac{-w}{v}$. **24.** $+\frac{-R}{N}$. **26.** $+\frac{5}{n}$. **28.** $+\frac{-vg^2}{100}$.

Pages 167, 168.

1. $\frac{5}{6}$. **3.** $\frac{5}{6}$. **5.** $\frac{3}{5}$. **7.** $\frac{7}{9}$. **9.** $\frac{3}{10}$. **11.** $\frac{b}{c}$. **12.** $\frac{1}{2x}$.

2. $\frac{3}{4}$. **4.** $\frac{3}{4}$. **6.** $\frac{2}{3}$. **8.** $\frac{18}{25}$. **10.** $\frac{1}{12}$.

13. $\frac{2m}{3n}$. **16.** $\frac{xz}{2}$. **19.** $\frac{2M}{9v^3}$. **22.** $\frac{a-b}{a+b}$. **25.** $\frac{A^2-AB}{A+B}$.

14. $\frac{3p}{4q^3}$. **17.** $\frac{w}{4v}$. **20.** $\frac{4}{R}$. **23.** $\frac{x}{n+m}$. **26.** $\frac{R-1}{2y}$.

15. $\frac{2c}{3ad}$. **18.** $\frac{t}{3r}$. **21.** $\frac{1}{2R}$. **24.** $\frac{t}{v^2+t^2}$. **27.** $\frac{t+5}{t+2}$.

28. $\frac{1+p}{1-p}$. **29.** $\frac{2n+1}{2n-1}$. **30.** $\frac{a+b}{a^2+ab+b^2}$. **31.** $\frac{1}{1+x+x^2}$.

32. $\frac{v-w}{v+w}$. **33.** $\frac{V+2}{V^2}$. **34.** $\frac{b}{c^2}$. **35.** $\frac{M^3}{M^2+N^2}$. **36.** $\frac{r-1}{r+2}$.

37. $\frac{r_1-r_2}{r_1r_2}$. **39.** $-\frac{D+1}{D+2}$. **41.** $\frac{R-1}{R-2}$. **43.** $\frac{5a+t}{6a-5t}$.

38. $\frac{m^2-n^2}{m^2+n^2}$. **40.** $\frac{c-d}{c+d}$. **42.** $\frac{2x-1}{x+4}$. **44.** $\frac{P^2-P+1}{P+7}$.

Page 169.

1. $2\frac{1}{3}$. 3. $4\frac{3}{8}$. 5. $3\frac{12}{17}$. 7. $2\frac{10}{27}$. 9. $8\frac{8}{75}$.

2. $4\frac{4}{5}$. 4. $5\frac{8}{17}$. 6. $4\frac{3}{56}$. 8. $2\frac{11}{25}$. 10. $3\frac{481}{682}$.

11. $x+1+\frac{2}{x-1}$. 14. $x+1+\frac{2x}{x^2-x-1}$. 17. $M-N+\frac{2N^2}{M+N}$.

12. $A+\frac{A-7}{2A-4}$. 15. $w+1-\frac{w-6}{w^2-w+2}$. 18. $r^2+2r+3-\frac{1}{r+1}$.

13. $n-3-\frac{5}{n-2}$. 16. $P+3+\frac{9}{P-3}$. 19. $D^2-2D+4+\frac{8}{D+2}$.

20. $a^3+2a^2+4a+8+\frac{48}{a-2}$. 23. $n^2-4n+7-\frac{8}{n+1}$.

21. $3A+3B+\frac{10B^2}{3A-2B}$. 24. $S^4+2S^3+4S^2+8S+16+\frac{16}{S-2}$.

22. $v^2-vt+t^2-\frac{2t^3}{v+t}$. 25. $E+7+\frac{21E+3}{E^2-3E}$.

Pages 170, 171.

1. $\frac{12}{20}$. 2. $\frac{48}{56}$. 3. $\frac{36}{64}$. 4. $\frac{6}{64}$. 5. $\frac{35}{126}$. 6. $\frac{36}{72}, \frac{54}{72}, \frac{60}{72}, \frac{63}{72}, \frac{66}{72}, \frac{68}{72}$

7. $\frac{a^4bc^2}{a^3b^3c^3}$. 9. $\frac{M^2-2MN+N^2}{M^2-N^2}$. 11. $\frac{v^3+2v^2t+2vt^2+t^3}{v^3-t^3}$,

8. $\frac{x+1}{x^2-1}$. 10. $\frac{R^2+2R}{R^2+3R+2}$. 12. $\frac{3x^2-4xy}{6x^2+7xy-20y^2}$.

13. $\frac{a^2b-ab^2}{a^3b-ab^3}, \frac{a^2b+ab^2}{a^3b-ab^3}, \frac{a^2b-b^3}{a^3b-ab^3}, \frac{a^4-ab^2}{a^3b-ab^3}$.

14. $\frac{x(x-y)(x+y)^2}{(x-y)^2(x+y)^2}, \frac{y(x-y)^2(x+y)}{(x-y)^2(x+y)^2}, \frac{(x+y)^3}{(x-y)^2(x+y)^2}, \frac{(x-y)^3}{(x-y)^2(x+y)^2}$.

Pages 171, 172.

1. $\frac{6}{12}, \frac{8}{12}, \frac{9}{12}$. 4. $\frac{63}{72}, \frac{40}{72}, \frac{66}{72}$. 7. $\frac{120}{240}, \frac{144}{240}, \frac{105}{240}, \frac{100}{240}$.

2. $\frac{8}{24}, \frac{20}{24}, \frac{21}{24}$. 5. $\frac{56}{63}, \frac{27}{63}, \frac{15}{63}$. 8. $\frac{45}{240}, \frac{54}{240}, \frac{192}{240}, \frac{160}{240}$.

3. $\frac{45}{60}, \frac{50}{60}, \frac{24}{60}$. 6. $\frac{8}{60}, \frac{9}{60}, \frac{45}{60}$.

9. $\frac{4x}{6x^2}, \frac{3}{6x^2}, \frac{5x}{6x^2}$. 11. $\frac{8a^4}{12a^2b^2c}, \frac{18b^4}{12a^2b^2c}, \frac{3c^3}{12a^2b^2c}$

10. $\frac{9t}{24t^4}, \frac{20t}{24t^4}, \frac{6}{24t^4}$ 12. $\frac{z^2}{xyz}, \frac{x^2}{xyz}, \frac{y^2}{xyz}$.

13. $\frac{A^2-A}{A^2-1}, \frac{A^2+A}{A^2-1}, \frac{1}{A^3-1}.$ **14.** $\frac{6n+3}{2n^3+7n^2+7n+2}, \frac{5n+5}{2n^3+7n^2+7n+2}.$

15. $\frac{V^2-4V+3}{V^3+2V^2-11V-12}, \frac{V^2+4V+3}{V^3+2V^2-11V-12}.$

16. $\frac{8R^2+22R-21}{24R^3+34R^2-59R+15}, \frac{6R^2+7R-20}{24R^3+34R^2-59R+15}.$

17. $\frac{a^3+a^2b+ab^2+b^3}{a^4-b^4}, \frac{2a^3-2a^2b+2ab^2-2b^3}{a^4-b^4}, \frac{a^2-b^2}{a^4-b^4}.$

18. $\frac{2(c+1)^2}{(c+1)^3}, \frac{c(c+1)}{(c+1)^3}, \frac{2c^2}{(c+1)^3}.$ **19.** $\frac{w^3-wv^3}{vw^2-v^3}, \frac{w^3+w^2v}{vw^2-v^3}, \frac{wv^2}{vw^2-v^3}.$

20. $\frac{6x^2+6xy}{6(x^2-y^2)}, \frac{3xy+3y^2}{6(x^2-y^2)}, \frac{x^2}{6(x^2-y^2)}.$

21. $\frac{p^2-10p+25}{p^3-2p^2-13p-10}, \frac{p^2+2p+1}{p^3-2p^2-13p-10}, \frac{p^2+4p+4}{p^3-2p^2-13p-10}.$

22. $\frac{30S}{180S^3+30S^2-60S}, \frac{12S-6}{180S^3+30S^2-60S}, \frac{15S+10}{180S^3+30S^2-60S}.$

Pages 174–176.

1. $\frac{19}{12}$. **2.** $\frac{11}{24}$. **3.** $\frac{2}{5}$. **4.** $\frac{13}{36}$. **5.** $\frac{95}{84}$. **6.** $\frac{239}{420}$. **7.** $\frac{49}{90}$. **8.** $\frac{125}{144}$.

9. $\frac{23a}{12}.$

10. $\frac{13P}{12}.$

11. $\frac{7v}{20}.$

12. $\frac{27R-14}{35}.$

13. $\frac{ad-bc}{bd}.$

14. $\frac{7A-2}{2A^2}.$

15. $\frac{x^2+y^2+z^2}{xyz}.$

16. $\frac{30n+7}{14n^2}.$

17. $\frac{r_1+r_2+r_3}{r_1r_2r_3}.$

18. $\frac{7R+4\pi R+6H}{4\pi R^2}.$

19. $\frac{x^2+y^2}{x^2-y^2}.$

20. $\frac{-2a}{1-a^2}.$

21. $\frac{2m^2+2n^2}{m^2-n^2}.$

22. $\frac{-4v}{v^2-1}.$

23. $\frac{1+s+4s^2}{s^3-s}.$

24. $\frac{3x^2+3}{2x^2-5x+2}.$

25. $\frac{4v}{w+v}.$

26. $\frac{-4}{a^3+3a^2-13a-15}.$

27. $\frac{2M^2+4MN+20N^2}{M^3+3M^2N-6MN^2-8N^3}.$ **28.** $\frac{2b^2-18bc+6c^2}{8b^4-6b^3c-71b^2c^2+54bc^3-9c^4}.$

29. $\frac{4R^2}{R^2-H^2}.$ **30.** $\frac{4y^2-yz-32z^2}{2y^4+5y^3z-6y^2z^2-19yz^3-10z^4}.$ **31.** $\frac{2a^2-8}{a^2-1}.$

32. $\frac{1+2P}{1-P^2}$.

33. $\frac{4v^2}{v^2-u^2}$.

34. $\frac{5-6x}{1-x^2}$.

35. $\frac{M+2N}{M^2-N^2}$.

36. $\frac{1}{2+y}$.

37. $\frac{3R^2}{R^2-1}$.

38. $\frac{1+3b-b^2-b^3}{b^3-1}$.

39. $\frac{4ax+4bx-4ab}{4b^2-x^2}$.

40. $\frac{6a^2b-12b^3}{a^4-5a^2b^2+4b^4}$.

41. $\frac{-8n^2+24}{n^4-10n^2+9}$.

42. $\frac{2rS^2-2rs^2+2R^2s-2r^2s}{R^2S^2-R^2s^2-r^2S^2+r^2s^2}$.

43. $\frac{14y^2+130}{y^4-26y^2+25}$.

44. $\frac{2t^4-2a^4+2a^2t^4-2a^2}{t^8-a^4t^4-t^4+a^4}$.

45. $\frac{4A^3+2A^2+4A-5}{A^6-A^2}$.

Page 177.

1. $\frac{x+1}{x}$.

2. $\frac{2B+B^2}{1+B}$.

3. $\frac{1}{1-r}$.

4. $\frac{3T+4}{T+2}$.

5. $\frac{mn}{m-n}$.

6. $\frac{w^2-14}{w-3}$.

7. $\frac{\pi H^2+\pi R^2}{H^2}$.

8. $\frac{r_1^2}{r_1+r_2}$.

9. $\frac{a^2+b^2}{a-b}$.

10. $\frac{B^2-4A^2C}{4A}$.

11. $\frac{y^3-y^2+2y}{y^2-1}$.

12. $\frac{5-t-5t^2}{1-t^2}$.

13. $\frac{2b^3}{a+b}$.

14. $\frac{1+4V^4}{1+V+V^2}$.

15. $\frac{2u^3-3uv^2-v^3}{u^3+v^3}$.

16. $\frac{4p^2+5pq+2q^2}{(p+q)^2}$.

17. $\frac{B^2+Bb+b^2}{B^2}$.

Pages 178, 179.

1. $\frac{5}{14}$.

2. $\frac{1}{6}$.

3. $\frac{2}{17}$.

4. $\frac{3}{14}$.

5. $\frac{2}{7}$.

6. $\frac{1}{21}$.

7. $\frac{1}{13}$.

8. $\frac{7}{12}$.

9. $\frac{4}{21}$.

10. $\frac{AC}{BD}$.

11. $\frac{w}{v}$.

12. $\frac{3y}{5x}$.

13. $\frac{n^8}{20}$.

14. $\frac{9pr}{2}$.

15. $\frac{\pi^2R^3}{6}$.

16. 1.

17. $\frac{5n}{t}$.

18. $\frac{4}{9x^2}$.

19. $\frac{2}{y}$.

20. $\frac{y^2}{z^3}$.

21. $\frac{d}{b}$.

22. $\frac{gt^2}{2}$.

23. $\frac{b}{a^2+ab}$.

24. $\frac{M-N}{5N^2}$.

25. $\frac{1}{x-1}$.

26. $\frac{a+b}{a^2+b^2}$.

27. $\frac{1}{m+2n}$.

28. $\frac{2s+3}{t-1}$.

29. A^2-1.

30. $\dfrac{x-2y}{x^3+xy+y^2}$.

31. v^3.

32. $\dfrac{x^2-8x+7}{x^2+9x+20}$.

33. $\dfrac{r-5}{r^3+7r^2+7r+6}$.

34. $\dfrac{R}{R^2+2RH+H^2}$.

35. $\dfrac{3m^2-7mn+4n^2}{3m^2-13mn-10n^2}$.

36. $\dfrac{V+3}{-V^2+7V-12}$.

37. $\dfrac{a+c}{b+d}$.

38. $-y-2$.

39. $\dfrac{M}{R}\times\dfrac{R}{N}=\dfrac{M}{N}$, and $\dfrac{b}{b'}\times\dfrac{a}{a'}=\dfrac{ab}{a'b'}$.

Pages 180, 181.

1. $\dfrac{b}{b-a}$.

2. $\dfrac{-D-R}{D^2}$.

3. $\dfrac{uv^2-u^3}{v^3}$.

4. $\dfrac{r_1{}^4-r_2{}^4}{r_1{}^2r_2}$.

5. $\dfrac{9h^2+3hb}{b^2}$.

6. $\dfrac{M^2N^2-1}{N^2}$.

7. $\dfrac{1-2x+x^2}{x^2}$.

8. $\dfrac{y^2+2yz+z^2}{yz}$.

9. $4k^2-12k+9$.

10. $\dfrac{8E}{E-1}$.

11. $\dfrac{A^5+A^4B-2A^3B^2}{A^3+A^2B+AB^2+B^3}$.

12. $\dfrac{1+t}{t}$.

13. $\dfrac{2P^2-1}{P^4+2P^3+P^2}$.

14. $\dfrac{1}{s+t}$.

15. $\dfrac{p^2+pq-q^2}{2p^2q}$.

Pages 181, 182.

1. $\frac{5}{2}$. **2.** $\frac{32}{3}$. **3.** $\frac{33}{4}$. **4.** $\frac{15}{2}$. **5.** $\frac{40}{3}$. **6.** AB. **7.** $3xyz$.

8. $\dfrac{3v}{4}$.

9. $\dfrac{R^2+RH}{2}$.

10. $\dfrac{n^3+7n^2-36}{n-1}$.

11. $\pi R^2+2\pi RH+\pi H^2$.

12. $\dfrac{2t^3+2t^2+2t}{t-6}$.

13. $2v^2-4v$.

14. $\dfrac{2x^3-3x^2y}{4x+y}$.

15. $\dfrac{r_1r_2}{r_1-3r_2}$.

16. $2a$.

17. $4y^2-3y$.

18. m^2-2m-5.

19. $16P-24$.

20. $2g+14$.

21. $33-11D$.

22. $22R-2$.

23. $17k+17$.

24. $2x^2-2y^2$.

25. $10-5M$.

Pages 183, 184.

1. $\frac{4}{5}$. **3.** $\frac{5}{18}$. **5.** $\frac{5}{9}$. **7.** $\frac{15}{34}$. **9.** $\dfrac{2m}{3}$. **10.** $\dfrac{\pi}{4R}$

2. $\frac{7}{6}$. **4.** $\frac{11}{2}$. **6.** $\frac{7}{20}$. **8.** $\frac{27}{4}$.

11. $\frac{2v}{3t}$.
12. $\frac{8}{3P}$.
13. $\frac{3xw}{yz}$.
14. $\frac{5H^3}{7L^3}$.
15. $\frac{3r_1^2r_2}{5R_1R_2}$.
16. $\frac{a}{a-1}$.
17. $\frac{6M-2}{2M-M^2}$.
18. $3W+3V$.
19. $\frac{x+1}{x+5}$.
20. $\frac{R^3+R^2-R-1}{R^3-17R^2+95R-175}$.
21. $\frac{2t}{a^2-at+t^2}$.
22. $\frac{1}{3b-3c}$.
23. $\frac{2mn-n^2}{mn^2-m^3}$.
24. $\frac{A+B}{A}$.
25. $\frac{ab-a^2}{b}$.
26. $\frac{n+3}{3n^3+8n^2+8n+5}$.
27. $\frac{x^3-4x^2y-3xy^2+18y^3}{x^3-2x^2y+4xy^2-8y^3}$.
28. p^2-q^2.
29. $\frac{t^2}{2t+v}$.
30. $\frac{vt+t^2}{v^2-vt+t^2}$.

Page 185.

1. $\frac{15}{2}$.
2. $\frac{10}{7}$.
3. $\frac{3}{4}$.
4. 10.
5. $\frac{B}{A}$.
6. x^3.
7. w.
8. $\frac{x}{m-n}$.
9. $\frac{ay}{ay+b}$.
10. $\frac{R+H}{R-H}$.
11. $\frac{u+v}{u}$.
12. $\frac{M+N}{M-N}$.
13. $\frac{2}{c-2}$.
14. $\frac{1}{r_2^2+r_1r_2}$.
15. $\frac{H^2}{\pi H-\pi R}$.
16. $\frac{2x+1}{x^2+x+1}$.
17. $\frac{A-4}{A-5}$.
18. y.
19. -1.
20. $\frac{1}{2R^2-1}$.
21. $\frac{r_1^2-r_2^2}{r_1r_2^2}$.

Pages 185–188.

2. $\frac{2}{(x-y)(x-z)}$.
3. 1.
4. $\frac{u^2-uw+v^2-vw+w^2-wu}{(u-v)(v-w)(w-u)}$.
5. $\frac{2r_2+2r_3}{(r_1-r_2)(r_1-r_3)}$.
6. 1.
7. 1.
8. $\frac{1}{(x-1)(x-3)}$.
9. $\frac{2a-26}{(a-1)(a-3)(a-6)}$.
10. $\frac{2}{3}$.
11. $\frac{b^4}{a^4}$
12. $\frac{n-3}{n}$
13. 1.
14. $\frac{t^3+9t^2-30t-88}{t^3-18t^2+71t-78}$.
15. $\frac{3P^2+8P+4}{P^2+2P-3}$.
16. $\frac{a}{a^3+a^2-a+2}$.
17. 1.
18. -1.
19. $\frac{3n+7}{3n+9}$.
20. $\frac{n}{y^2+n^2}$.
21. $3x+2y$.
22. $\frac{8A^2-14AB+3B^2}{3A^2+8AB+4B^2}$.

Pages 190–192.

1. 3.
2. 6.
3. 1.
4. 6.
5. $-1\frac{1}{2}$.
6. 1.
7. $2\frac{1}{15}$.
8. 12.
9. −1.
10. 2.
11. 6.
12. 5.
13. 15.
14. 15.
15. $2\frac{1}{11}$.
16. $2\frac{3}{4}$.
17. 1.
18. 1.
10. −1.
20. $-\frac{4}{5}$.
21. 25.
22. $-1\frac{1}{2}$.
23. $-\frac{4}{5}$.
24. 5.
25. 6.
26. 5.
27. $\frac{1}{4}$.
28. 4.
29. $-2\frac{4}{5}$.
30. 3.
31. $1\frac{1}{2}$.
32. $\frac{1}{11}$.
33. 1.
34. 4.
35. 1.
36. $2\frac{1}{7}$.
37. −8.
38. $9\frac{2}{3}$.
39. 3.
40. 43.
41. 6.
42. 4.

Pages 192–197.

1. 248; 17.
2. 128; 32.
3. 446; 122.
4. 48; 36.
5. 126; 22.
6. 14 ft.; 4 ft.
7. 24 in.
8. 60 ft. from one stake and 90 ft. from the other.
9. 7.
10. 5.
11. 6.
12. 2.
13. 8.
14. 5.
15. $2\frac{2}{5}$ da.
16. $9\frac{9}{19}$ da.
17. $6\frac{6}{7}$ hr.
18. $3\frac{1}{13}$ hr.
10. 20 hr.
20. $10\frac{2}{7}$ hr.
21. $4\frac{4}{9}$ da.
22. $43\frac{7}{11}$ da.
23. $3\frac{15}{16}$ da.
24. 6 da.
25. 15 min.
26. $1\frac{1}{21}$ mi. per hour.
27. $7\frac{6}{71}$ mi. per hour.
28. 35 mi. and 40 mi. per hour.
29. Motorcyclist 15 mi. and automobile 25 mi. per hour.
30. 4 mi. and 6 mi. per hour.
31. 6 mi. per hour.
32. 20 mi. per hour.
33. 30 mi. per hour.
34. 15 mi. per hour.
36. 2.

Pages 107–202.

1. $a=\frac{nx-b}{n}$.
2. $x=\frac{cy+b}{2a}$.
3. $a=1$.
4. $t=\frac{v-1}{4}$.
5. $V=\frac{2+2R}{2-R}$.
6. $y=mn$.
7. $t=A^2+AB+B^2$.
8. $r=\frac{1}{m-n}$.
9. $P=\frac{QR}{QR-Q-R}$.
10. $d=N-1$.
11. $t=\frac{5y}{3}$.
12. $A=5\frac{5}{6}C$.
13. $P=-W$.
14. $c=-2d$.
15. $r=\frac{cR}{C}$.
16. $s=\frac{aS}{C}$; $S=\frac{sC}{a}$; $a=\frac{sC}{S}$; $C=\frac{aS}{s}$.
17. $S=\frac{st}{T}$; $s=\frac{ST}{t}$; $t=\frac{ST}{s}$; $T=\frac{st}{S}$.

18. $L=2AT$; $A=\frac{L}{2T}$. **19.** $V=\frac{vR^2H}{r^2h}$; $v=\frac{Vr^2h}{R^2H}$; $H=\frac{Vr^2h}{vR^2}$; $h=\frac{vR^2H}{Vr^2}$.

20. $T=\frac{tR(H+R)}{r(h+r)}$; $t=\frac{Tr(h+r)}{R(H+R)}$; $H=\frac{Tr(h+r)-tR^2}{tR}$; $h=\frac{tRH+tR^2-Tr^2}{Tr}$.

21. (1) $V=RS$, $S=\frac{V}{R}$; (2) $R=\frac{S}{2\pi H}$, $H=\frac{S}{2\pi R}$, $S=2\pi RH$;

(3) $H=\frac{V}{\pi R^2}$, $V=\pi R^2H$; (4) $H=\frac{T-2\pi R^2}{2\pi R}$, $T=2\pi R^2+2\pi RH$;

(5) $T=2\pi R^2+2\pi RH-S$, $S=2\pi R^2+2\pi RH-T$, $H=\frac{S+T-2\pi R^2}{2\pi R}$.

22. $H=\frac{RT-RS}{S}$; $R=\frac{HS}{T-S}$; $T=\frac{HS+RS}{R}$; $S=\frac{RT}{H+R}$.

23. $M=\frac{Fr}{v^2}$; $r=\frac{Mv^2}{F}$. **25.** $w=\frac{Dw'}{D-1}$; $w'=\frac{w(D-1)}{D}$.

24. $w=\frac{fgr}{v^2}$; $g=\frac{wv^2}{fr}$; $r=\frac{wv^2}{fg}$. **26.** $E=CR$; $R=\frac{E}{C}$.

27. $E=C(R+r)$; $R=\frac{E-Cr}{C}$; $r=\frac{E-CR}{C}$.

28. $n=\frac{CR}{E-Cr}$; $E=\frac{CR+nCr}{n}$; $R=\frac{nE-nCr}{C}$; $r=\frac{nE-CR}{nC}$.

29. $n=\frac{Cr}{E-CR}$; $E=\frac{nCR+Cr}{E}$; $R=\frac{nE-Cr}{nC}$; $r=\frac{nE-nCR}{C}$.

30. $R=\frac{r_1r_2}{r_1+r_2}$. **31.** $R=\frac{r_1r_2r_3}{r_1r_2+r_1r_3+r_2r_3}$.

32. $r=\frac{S-a}{S-l}$; $a=S-Sr+rl$; $l=\frac{a-S+Sr}{r}$. **33.** $r=\frac{4h^2+w^2}{8h}$. **34.** 10 ft.

Pages 202–205.

1. $-\frac{2}{3}$.
2. $1\frac{1}{3}$.
3. $1\frac{2}{11}$.
4. $1\frac{15}{38}$.
5. 9.
6. 6.
7. $\frac{5}{7}$.
8. $\frac{17}{19}$.
9. 13.
10. 8.
11. 4.
12. $2\frac{1}{2}$.
13. $3\frac{1}{2}$.
14. $-4\frac{1}{2}$.
15. 5.
16. -7.
17. $-2\frac{1}{4}$.
18. $1\frac{1}{2}$.
19. $\frac{1}{2}$.
20. $-\frac{3}{11}$.

21. -10. **22.** $x=\frac{2ac^2-4b^2c-bc^2}{2b^3+b^2c}$. **23.** $t=\frac{ac}{2c-a}$.

24. $n=\frac{ab+bc-b^2}{a}$. **25.** $n=xy$. **26.** $R=\frac{r_1r_2r_3r_4}{r_1r_2r_3+r_1r_2r_4+r_1r_3r_4+r_2r_3r_4}$.

27. $t=\dfrac{mcE+m'c'E-mcT}{m'c'}$. **28.** $V=\dfrac{m-n}{m+n}$.

29. He divided 20 by 4 less than the result given.

31. $W=\dfrac{wv}{V-v}$; $V=\dfrac{Wv+wv}{W}$; $w=\dfrac{WV-Wv}{v}$. **32.** $4\frac{4}{5}$ ft. per second.

Pages 206–208.

1. $\frac{1}{2}$. **3.** $\frac{18}{1000}$. **5.** $\frac{5}{8}$. **7.** $\frac{1}{6}$.

2. $\frac{1}{20}$. **4.** $\frac{2}{5}$ copper; $\frac{3}{5}$ tin. **6.** $\frac{560}{84000}$; $6\frac{2}{3}$. **8.** $\frac{196}{25}$; 7.8+.

9. Cast iron 7.2; brass 8.4−; gold 19.2+; oak 1.0+; ash 0.8+; cork 0.2+.

10. 3.14+. **11.** 3.14+. **12.** $\dfrac{1}{x-y}$. **13.** $\dfrac{aa'}{bb'}$. **14.** $\frac{7}{12}$.

15. $\frac{12}{15}$, $\frac{21}{25}$, $\frac{5}{6}$. **18.** 29,120. **20.** 25 and 60.

16. A $915; B $1220. **19.** 12 and 28. **21.** $2\frac{1}{2}$ and $4\frac{1}{2}$.

17. A $400; B $800.

Page 209.

1. 3. **2.** $1\frac{1}{2}$. **3.** $5\frac{1}{4}$. **4.** 54. **5.** 12. **6.** 48. **7.** 144. **8.** $21\frac{1}{3}$. **9.** $14\frac{2}{3}$.

10. $\dfrac{bc}{a}$. **11.** $\dfrac{xz}{y}$. **12.** $\dfrac{tw}{v}$. **13.** $\dfrac{2ab}{b-a}$. **14.** r^2. **15.** $\dfrac{b^2}{ac}$.

16. Family of three members, $48; other family $64.

Page 210.

1. 15. **2.** 15. **3.** 40. **4.** 3. **5.** a^3. **6.** mn^2. **7.** $1-t^3$.

8. 12. **9.** 18. **10.** $9x^5$. **11.** Pt^2. **12.** $\dfrac{b^2}{a}$. **13.** $1-n-n^2+n^3$.

14. $\frac{1}{10}$. **16.** ± 6. **17.** ± 14. **18.** ± 3. **19.** $\pm n^3$.

20. $\pm 10A^2$. **21.** $\pm(R+R^2)$.

Pages 211–213.

1. $3\frac{1}{3}$ in.; 5 in. **3.** 72 ft. **5.** 773 yd. 1 ft.

2. 60 rd.; 75 rd. **4.** 67 ft. 1 in. **6.** 4800 ft.

7. 60 yd.
8. (1) 2 in.; (2) $16\frac{2}{3}$ in.; (3) $12\frac{3}{5}$ in.
9. 16 in.
10. *AD*, $17\frac{7}{9}$ ft.; *DC*, $22\frac{2}{9}$ ft.
11. *AD*, $7\frac{1}{5}$ in.; *DB*, $4\frac{4}{5}$ in.
12. *AD*, $33\frac{1}{3}$ yd.; *DB*, $26\frac{2}{3}$ yd.
13. 10 in.
14. $450\frac{2}{3}$ in.
15. 6 in.; $4\frac{1}{2}$ in.; *AD* 16 in. and *DB* 4 in., or *AD* 4 in. and *DB* 16 in.

Pages 213–215.

1. 120 lb.
2. 60 lb.
3. 960 lb.
4. 12 in. from point of application of resistance.
5. 150 lb.
6. $16\frac{1}{5}$ lb.
7. $266\frac{2}{3}$ lb.
8. 100 lb.
9. 48 lb.
10. $120\frac{5}{6}$ lb.
11. 50 lb.
12. 18 tons.

Page 220.

1. $d=35t$.
2. $c=30w$.
3. $c=5w$.
4. $i=.06pt$.
5. $v=32t$.
6. 60 lb.
7. 9 ohms.
8. $R=\frac{8}{15}L$.
9. 8; $9x=2y$.
10. $5P=7V$; 25.

Pages 229–231.

1. $m+n$.
2. 16, 24, 32.
3. $\frac{3+4n}{3+5n}$.
4. $\frac{x+7y}{x+8y}$.
5. $\pm 6a^2(a-b)$.
6. $\frac{1}{(v+w)^2}$.
7. $\frac{3}{4}=\frac{15}{20}$; $\frac{3}{15}=\frac{4}{20}$; $\frac{4}{20}=\frac{3}{15}$.
8. $\frac{x+v}{5}=\frac{6}{x-y}$.
13. $37\frac{1}{2}$.
14. 144 ft.; 256 ft.; 1600 ft.
15. $\frac{1}{2}$.
16. 16 cu. in.
17. 2.4+ in.

Pages 232, 233.

1. $a=5$, $b=-6$.
2. $m=12$, $n=-2$.
3. $x=4$, $y=3$.
4. $s=5$, $t=7$.
5. $w=6\frac{7}{13}$, $v=-1\frac{2}{13}$.
6. $A=-2$, $B=-3$.
7. $x=-1$, $y=4$.
8. $p=2$, $q=3$.
9. $R=3$, $r=7$.
10. $M=2$, $N=5$.
11. $h=12$, $k=11$.
12. $x=7$, $y=10$.
13. $r_1=1$, $r_2=\frac{1}{2}$.
14. $B=-3$, $C=11$.
15. $m=8$, $n=-2$.
16. $W=-2\frac{4}{5}$, $w=29\frac{2}{5}$
17. $f=2\frac{1}{5}$, $t=0$.
18. $x_1=13$, $x_2=17$.
19. $a=14$, $b=14$,
20. $v=16$, $w=12$.

Page 234.

1. $m=2,\ n=3.$

2. $x=12,\ a=1.$

3. $r=2,\ s=1.$

4. $r_1=13,\ r_2=17.$

5. $A=4,\ B=-5.$

6. $p=2\frac{12}{17},\ q=7\frac{2}{17}.$

7. $t=-2,\ v=8.$

8. $M=2,\ N=1.$

9. $e=35,\ f=20.$

10. $R=7,\ T=9.$

11. $V=15,\ n=8.$

12. $x=-1,\ x'=3.$

13. $h=2,\ k=5.$

14. $B=4,\ Q=-3.$

15. $y=\frac{2}{3},\ z=\frac{8}{5}.$

16. $L=-2,\ Q=1\frac{2}{3}.$

17. $W=3,\ D=4.$

18. $a=5,\ t=-7.$

10. $x=1,\ y=\frac{1}{3}.$

20. $V=2,\ P=3.$

21. $t=1,\ T=-4.$

22. $R=1\frac{37}{145},\ r=-1\frac{101}{145}.$

23. $w=\frac{4}{5},\ W=-1\frac{1}{5}.$

Pages 235, 236.

1. $a=3,\ b=2.$

2. $x=8,\ y=4.$

3. $m=3,\ n=1.$

4. $x_1=17,\ x_2=13.$

5. $R=6,\ S=2.$

6. $V=1,\ t=5.$

7. $a=-3,\ z=-7.$

8. $p=9,\ q=2.$

9. $D=-11,\ Q=7.$

10. $g=8,\ k=1.$

11. $A=1\frac{57}{59},\ B=\frac{5}{59}.$

12. $r_1=1,\ r_2=-1\frac{1}{2}.$

13. $T=7,\ t=3.$

14. $v=5,\ s=7.$

15. $M=4,\ b=3.$

16. $A=5,\ p=-5.$

17. $x=8,\ y=1.$

18. $a=\frac{2}{3},\ m=4.$

19. $d=-2\frac{14}{51},\ y=7\frac{40}{51}.$

20. $r=-11,\ t=7.$

21. $a=24\frac{2}{7},\ k=5\frac{5}{7}.$

22. $a=5,\ b=4.$

23. $v=-4,\ w=1\frac{1}{2}.$

Page 236.

1. $x=7,\ y=3.$

2. $v=7,\ w=4.$

3. $R=-2,\ t=-3.$

4. $M=-3,\ b=24.$

5. $a=4,\ b=\frac{2}{3}.$

6. $p=-7\frac{1}{5},\ q=-17\frac{3}{5}.$

7. $n=7\frac{1}{6},\ z=-\frac{29}{30}.$

8. $L=6,\ d=8.$

9. $k=\frac{246}{271},\ m=\frac{729}{1084}.$

10. $x=\frac{4}{11},\ y=-\frac{1}{11}.$

11. $V=-1\frac{11}{31},\ P=1\frac{15}{31}.$

Pages 237–242.

1. 21, 8.

2. −6, 15.

3. 24, 96.

4. 8, 56.

5. 6, 27.

6. $\frac{14}{21}$.

7. $4\frac{2}{7}$ ft., $5\frac{5}{7}$ ft.

8. $86\frac{2}{13}$ ft. from one end.

9. $3\frac{1}{2}$ ft. from one end.

10. $4\frac{1}{2}$ ft. from C.

11. $81\frac{1}{4}$ lb., 65 lb.

12. 24 lb., 18 lb.

13. 48 lb., 72 lb.

14. 36 oz., 54 oz.

15. A $1800, B $4500.

16. $7500 at 5%, $4500 at 6%.

17. $920 at 5%, $1080 at 4%.

18. $9000 in 3%, $7200 in 4% bonds.

19. N. Y. City 4's $8000, U. S. Steel 5's $14500.

20. 10 dimes, 40 nickels.

21. 12 \$5 bills, 4 \$10 bills.

22. $14\frac{2}{7}$ lb. of 25-cent, $35\frac{5}{7}$ lb. of 32-cent.

23. 750 cu. cm. copper, 250 cu. cm. zinc.

24. 83.

25. $126\frac{2}{3}°$, $53\frac{1}{3}°$.

26. 36°, 72°, 72°.

27. 10 ft., 18 ft.

28. Length 15 in., width 8 in.

29. Length 1004 ft., width 760 ft.

30. 12 ft., 9 ft.

31. $22\frac{8}{11}$ in., $6\frac{9}{11}$ in.

32. 2 mi. per hour.

33. $2\frac{1}{2}$ mi. per hour, 4 mi. per hour.

34. 15 mi. per hour, 35 mi. per hour.

35. 6 mi. per hour, 42 mi. per hour.

36. 90 ft. per sec., 60 ft. per sec.

37. Passenger 540 ft., freight 720 ft.

38. Passenger 480 ft., freight 576 ft.

39. 16 gal. cream, 4 gal. milk.

40. $9\frac{55}{57}$ gal. cream, $6\frac{2}{57}$ gal. milk.

41. 42 lb. tin, 14 lb. zinc.

42. 28.85+ oz. bread, 10.03+ oz. mutton.

43. No, because it would require a negative quantity of bread.

44. 200 children, 320 adults.

45. $m=5\frac{5}{9}$ in., $n=4\frac{4}{9}$ in.

Pages 242, 243.

1. $x=a,\ y=b.$

2. $x=a-b,\ y=a-b.$

3. $x=m,\ y=\frac{n}{m}.$

4. $x=\frac{R^2+S^2}{2R+S},\ y=\frac{2S-R}{2R+S}.$

5. $x=-1,\ y=A+B.$

6. $x=\frac{p^2}{p^2+q^2},\ y=\frac{1}{p^2+q^2}.$

7. $x=\frac{1}{r_1+r_2},\ y=\frac{1}{r_1+r_2}.$

8. $x=a^2,\ y=b.$

9. $x=2p,\ y=\frac{1}{2}q.$

10. $x=R_1+R_2,\ y=R_1.$

11. $x=\frac{1}{p+m+n},\ y=\frac{1}{p+m+n}.$

12. $x=\frac{tw}{v+t},\ y=\frac{vw}{v+t}.$

13. $x=\frac{Q^2+R}{P+Q},\ y=\frac{PQ-R}{P+Q}.$

14. $x=18a-24b,\ y=36b-24a.$

15. $x=a-b,\ y=a+b.$

16. $x=\frac{1}{2}m,\ y=n.$

17. $x=2\frac{7}{17}v,\ y=3\frac{12}{17}v.$

18. $\frac{3m+12n}{m+3n}.$

19. $\frac{qs}{1+q},\ \frac{s}{1+q}.$

20. $\frac{Pl}{P+W},\ \frac{Wl}{P+W}.$

21. $\frac{100d-12000y}{x-y}$ dollars at $x\%$, $\frac{12000x-100d}{x-y}$ dollars at $y\%$.

22. $\frac{aw-nw}{m-n}$ gal. milk, $\frac{mw-aw}{m-n}$ gal. cream.

Pages 245, 246.

1. $x=7$, $y=3$.
2. $x=4$, $y=10$.
3. $x=2$, $y=8$.
4. $x=4\frac{6}{13}$, $y=1\frac{5}{13}$.
5. $x=\frac{1}{2}$, $y=1$.
6. $x=3$, $y=2$.
7. $x=2$, $y=3$.
8. $x=-1$, $y=4$.
9. $x=5$, $y=1$.
10. $x=-2$, $y=-3$.
11. $x=7$, $y=9$.
12. $x=4$, $y=3$.

Pages 247–249.

1. $m=1, n=2$.
2. $p=-2$, $q=3$.
3. $A=\frac{1}{5}, B=\frac{1}{2}$.
4. $w=3$, $v=4$.
5. $r_1=3$, $r_2=2$.
6. $P=5$, $Q=2$.
7. $R=6$, $r=4$.
8. $x_1=-\frac{1}{2}$, $x_2=-\frac{1}{3}$.
9. $a=\frac{25}{134}$, $b=-\frac{10}{33}$.
10. $t=\frac{1}{5}$, $T=\frac{1}{2}$.
11. $A=2, B=7$.
12. $x=\frac{32}{405}$, $y=-\frac{16}{19}$.
13. $x=\dfrac{ab'-a'b}{b'c-bc'}$, $y=\dfrac{ab'-a'b}{ac'-a'c}$.
14. $x=\dfrac{5}{4m}$, $y=\dfrac{5}{n}$.
15. $x=m$, $y=n$.
10. 20 min., 16 min.
17. $10\frac{25}{32}$ hr., $17\frac{2}{11}$ hr.
18. 56 hr., 105 hr.
19. A 30 da., B 24 da.
20. $53\frac{1}{3}$ ohms, 160 ohms.

Pages 250–252.

1. $a=1, b=3, c=5$.
2. $P=6$ $Q=1, R=2$.
3. $u=1, v=\frac{1}{2}, w=\frac{1}{3}$.
4. $r_1=\frac{1}{5}, r_2=\frac{1}{6}, r_3=\frac{1}{7}$.
5. $A=-7, B=8, C=1$.
6. $x=1\frac{1}{2}, y=-1\frac{1}{2}, z=-\frac{1}{2}$.
7. $h=4, k=5, l=6$.
8. $p=2, q=3, r=4$.
9. $a=20, b=10, c=30$.
10. $x=2, y=\frac{2}{3}, z=\frac{2}{5}$.
11. $m=1\frac{1}{3}, n=-1\frac{1}{4}, p=-1\frac{1}{5}$.
12. 8 in., 8 in., 12 in.
13. 24 in., 20 in., 16 in.
14. $61\frac{2}{3}$°, $61\frac{2}{3}$°, $56\frac{2}{3}$°.
15. $P=48°, Q=64°, R=68°$.
16. $x=9, y=5, z=7$.
17. Vertices 9 in., 15 in., 23 in. from points of contact.
18. 4 in., 6 in., 9 in.
19. 0.168 in., 0.392 in., 0.452 in.
20. 432.
21. 29.6+ oz. bread, 2.9− oz. butter, 8.7+ oz. beef.
22. Impossible.
23. 8.9 oz., 4.5+ oz., 2.8+ oz.
24. 1sts 4, 2ds 2, 3ds 1

Pages 253, 254.

1. $M=6, N=8$.
2. $W=1\frac{1}{2}, w=-4$.
3. $r_1=8, r_2=10$.
4. $a=0, b=0$.
5. $x=1, y=1$.
6. $x=10, y=5$.
7. $x=\dfrac{17a}{33c}$, $y=\dfrac{17b}{16c}$.
8. $x=\dfrac{7mn-m+6n}{2m^2+2n^2+2m-2n}$, $y=\dfrac{7mn-m+6n}{m^2-6n^2}$.

9. $a=1, b=-1, c=2, d=3$.
10. $r=6, s=6, t=6, w=6$.
11. $x=1, y=2, z=3, w=4$.
12. $p=2, q=-1, r=-3, s=5$.
13. 384.
14. $62\frac{1}{2}$¢, $82\frac{1}{2}$¢.
15. 4 lb. of first, 3 lb. of second.
16. Altitude 9 in., base 12 in.
17. 6 persons, $2.

Pages 257, 258.

1. $2a+5$.
2. $1-8y$.
3. m^2-5n.
4. x^2-x+2.
5. $2t^2+5t-7$.
6. $1+b-b^2+b^3$.
7. $7A^4+3A^2-2$.
8. r^2-5r-2.
9. $a^2+2ab+b^2$.
10. $2w^2-w+4$.
11. $M^2-3MN+2N^2$.
12. x^2-xy+y^2.
13. $3p^2-2pq-5q^2$.
14. $7-3V+2V^2$.
15. x^3-x^2+x-1.
16. $m^3-3m^2n+3mn^2-n^3$.
17. $v^3-2v^2w+3vw^2-4w^3$.
18. $2P^3-5P^2+4P-3$.

Page 260.

1. 86.
2. 43.
3. 76.
4. 162.
5. 203.
6. 132.
7. 127.
8. 9.82.
9. 3.5.
10. 12.1.
11. 11.2.
12. 2.15.
13. 2.236.
14. 1.732.
15. 0.774.
16. 1.095.
17. 5.291.
18. 0.670.
19. 6.090.
20. 0.866.
21. 749.84.

Pages 262, 263.

1. $2\sqrt{2}$.
2. $2\sqrt{3}$.
3. $2\sqrt{6}$.
4. $3\sqrt{3}$.
5. $3\sqrt{2}$.
6. $5\sqrt{2}$.
7. $2\sqrt{7}$.
8. $4\sqrt{3}$.
9. $5\sqrt{3}$.
10. $3\sqrt{6}$.
11. $6\sqrt{2}$.
12. $5\sqrt{5}$.
13. $4\sqrt{2}$.
14. $4\sqrt{5}$.
15. $11\sqrt{2}$.
16. $13\sqrt{2}$.
17. $12\sqrt{3}$.
18. $10\sqrt{3}$.
19. $\frac{1}{2}\sqrt{2}$.
20. $\frac{1}{3}\sqrt{6}$.
21. $\frac{2}{7}\sqrt{7}$.
22. $\frac{3}{10}\sqrt{5}$.
23. $\frac{4}{21}\sqrt{7}$.
24. $\frac{1}{12}\sqrt{6}$.
25. $\frac{5}{6}\sqrt{3}$.
26. $\frac{2}{5}\sqrt{5}$.
27. $\frac{1}{8}\sqrt{14}$.
28. $\frac{1}{21}\sqrt{105}$.
29. $\frac{2}{15}\sqrt{6}$.
30. $\frac{1}{24}\sqrt{66}$.
31. $a\sqrt{b}$.
32. $2b^2\sqrt{a}$.
33. $5m\sqrt{mn}$.
34. $7x^2\sqrt{xy}$.
35. $2t^3v\sqrt{2v}$.
36. $5P^2\sqrt{2t}$.
37. $3Wg\sqrt{3W}$.
38. $k^4v^2\sqrt{5v}$.
39. $3B^3\sqrt{5AB}$.
40. $a(a-b)\sqrt{a}$.
41. $\frac{ab^2}{cd}\sqrt{c}$.
42. 2.828, 4.242, 7.070, 8.484, 5.656, 15.554, 18.382, 0.707.
43. 3.464, 5.196, 10.928, 8.660.

Pages 263, 264.

1. $3\sqrt{2}$.
2. $3\sqrt{3}$.
3. $3\sqrt{2}+3\sqrt{3}$.
4. $\sqrt{5}$.
5. $2\sqrt{2}$.
6. $2\sqrt{2}$.
7. $6\sqrt{2}$.
8. $2\sqrt{5}$.
9. $4\sqrt{7}$.
10. $5\sqrt{11}$.
11. $5\sqrt{2}$.
12. $2\sqrt{3}$.
13. $\sqrt{5}$.
14. $10\sqrt{6}$.
15. $6\sqrt{2}$.
16. $\frac{3}{2}\sqrt{2}$.
17. $\frac{31}{3}\sqrt{3}$.
18. $-\frac{2}{5}\sqrt{5}$.
19. $\frac{26}{5}\sqrt{5}$.
20. $-\frac{20}{3}\sqrt{3}$.
21. $5a^2\sqrt{a}$.
22. $-2xy\sqrt{x}$.
23. $8a^2b\sqrt{b}$.
24. $4n\sqrt{mn}$.
25. $3v^2\sqrt{2vt}$.
26. $24pq\sqrt{3pq}$.
27. $(2r+t)\sqrt{t}$.
28. $(u+v+w)\sqrt{uvw}$.
29. $(a+ab+ac)\sqrt{bc}$.

Pages 264, 265.

1. $\sqrt{15}$.
2. $\sqrt{14}$.
3. $\sqrt{33}$.
4. $\sqrt{65}$.
5. $2\sqrt{3}$.
6. $3\sqrt{5}$.
7. $3\sqrt{7}$.
8. $10\sqrt{3}$.
9. $2\sqrt{7}$.
10. $35\sqrt{3}$.
11. 6.
12. $44\sqrt{3}$.
13. $3-2\sqrt{5}$.
14. $6+4\sqrt{3}$.
15. $20-9\sqrt{6}$.
16. $16-6\sqrt{11}$.
17. $6+2\sqrt{2}+3\sqrt{3}+\sqrt{6}$.
18. $5\sqrt{3}-4\sqrt{5}+\sqrt{15}-4$.
19. $2\sqrt{7}-4$.
20. $-4-8\sqrt{3}$.
21. 1.
22. $11+2\sqrt{30}$.
23. $11-3\sqrt{14}$.
24. $\sqrt{6}+4\sqrt{22}+\sqrt{33}+44$.
25. $12-\sqrt{14}$.
26. $2\sqrt{3}+3\sqrt{2}-2\sqrt{6}-6$.
27. $r\sqrt{s}$.
28. $ab\sqrt{ab}$.
29. $2x^4\sqrt{y}$.
30. $p^4q\sqrt{q}$.
31. $10w^2v\sqrt{2v}$.
32. $mn\sqrt{21}$.
33. x^2-y.
34. $r^2+s-2r\sqrt{s}$.
35. $2a^2-6b+a\sqrt{b}$.
36. $m-n$.

Page 266.

1. $\frac{1}{2}\sqrt{2}$.
2. $\sqrt{3}$.
3. $2\sqrt{5}$.
4. $\frac{1}{5}\sqrt{30}$.
5. $\sqrt{3}$.
6. $\sqrt{2}$.
7. $\frac{4}{3}\sqrt{6}$.
8. $2\sqrt{2}$.
9. $\frac{1}{2}\sqrt{10}$.
10. $\sqrt{a}$.
11. $\frac{1}{m}\sqrt{mn}$.
12. $\frac{1}{x}\sqrt{x^2+x}$.
13. $\frac{2}{p}\sqrt{2p}$.
14. $\frac{1}{2}\sqrt{6v}$.
15. $10r\sqrt{2r}$.
16. $2-\sqrt{2}$.
17. $2\sqrt{3}+2$.
18. $6\sqrt{3}-6\sqrt{2}$.
19. $8\sqrt{5}+8\sqrt{3}$.
20. $12\sqrt{7}+12\sqrt{6}$.
21. $2\sqrt{21}+2\sqrt{19}$.
22. $2+\sqrt{3}$.
23. $9+4\sqrt{5}$.

24. $4\frac{1}{2}-\frac{1}{2}\sqrt{77}$.
25. $\frac{1}{2}+\frac{1}{2}\sqrt{3}$.
26. $\frac{39}{42}-\frac{11}{42}\sqrt{15}$.
27 $-3\frac{3}{5}-\frac{37}{35}\sqrt{14}$.
28. $\frac{a-\sqrt{b}}{a^2-b}$.
29. $\frac{x\sqrt{x}+x\sqrt{y}}{x-y}$.
30. $\sqrt{m}+\sqrt{n}$.

Pages 267, 268.

1. $y^2+2y+\frac{1}{y}$.
2. $R+4-\frac{2}{R}$.
3. $\frac{a}{b}+b^2+\frac{b}{a}$.
4. $n-\frac{t}{2}-\frac{3t^2}{2n}$.
5. 0.84.
6. 0.70.
7. 0.92.
8. 0.85.
9. 0.86.
10. 0.93.
11. 9.05.
12. $4\frac{1}{8}$; $4\frac{1}{9}$.
13. $7\frac{1}{14}$; $7\frac{1}{15}$.
14. $8\frac{1}{16}$; $8\frac{1}{17}$.
15. $10\frac{1}{10}$; $10\frac{2}{21}$.
16. $12\frac{1}{8}$; $12\frac{3}{25}$.
17. $\frac{1}{x}\sqrt{1+2x+3x^2}$
18. $\frac{1}{mn}\sqrt{m^3+n^3}$.
19. $\frac{1}{2A}\sqrt{B^2-4AC}$.
20. $\frac{3}{4}\sqrt{2}+\frac{1}{2}-\frac{1}{4}\sqrt{6}-\frac{1}{2}\sqrt{3}$.
21. $\frac{2}{11}\sqrt{15}+\frac{1}{11}\sqrt{5}-\frac{3}{11}\sqrt{3}+\frac{4}{11}$.
22. $\frac{1}{2}\sqrt{2}+\frac{1}{2}\sqrt{5}-\frac{1}{2}$.
23. $\frac{8}{5}\sqrt{3}+\frac{4}{5}\sqrt{2}-\frac{1}{5}\sqrt{42}-\frac{1}{5}\sqrt{7}$.
24. $\frac{1}{2}+\frac{1}{6}\sqrt{6}-\frac{1}{6}\sqrt{15}$.
25. $\frac{4}{11}-\frac{1}{11}\sqrt{5}-\frac{3}{11}\sqrt{3}-\frac{2}{11}\sqrt{15}$.
26. $\frac{(x+\sqrt{xy}-\sqrt{xz})(x+y-z-2\sqrt{xy})}{x^2+y^2+z^2-2xy-2xz-2yz}$.
27. $\frac{(a-b+\sqrt{ac}-\sqrt{bc})(a+b-c-2\sqrt{ab})}{a^2+b^2+c^2-2ab-2ac-2bc}$.
28. $\frac{(1-m-\sqrt{n}+\sqrt{mn})(1+m-n-2\sqrt{m})}{1+m^2+n^2-2m-2n-2mn}$.

Page 270.

1. 3, 4.
2. 2, −5.
3. −2, −7.
4. −3, $\frac{1}{2}$.
5. $-\frac{2}{3}$, $-1\frac{1}{2}$.
6. $\frac{1}{5}$, $-3\frac{1}{2}$.
7. 4, $-\frac{1}{6}$.
8. $\frac{2}{7}$, $-\frac{1}{3}$.
9. 4, 2.
10. 11, −7.
11. 1, −2.
12. 4, 7.
13. $-\frac{8}{5}$, $\frac{1}{3}$.
14. 3, $-2\frac{1}{5}$.
15. −2, $1\frac{7}{8}$.
16. 2, $-1\frac{5}{6}$.
17. −10, $\frac{5}{6}$.
18. 3, $-1\frac{2}{5}$.
19. 6, −1.
20. −1, −9.
21. 28, −20.
22. $\frac{1}{5}$, $8\frac{1}{2}$.
23. 5, $-1\frac{1}{2}$.
24. 3, $-2\frac{1}{3}$.
25. $\frac{1}{2}$, −3.
26. 2, −2.
27. 6, −3.

Pages 271, 272.

1. ± 2.
2. ± 5.
3. ± 13.
4. ± 7.
5. $\pm\frac{1}{2}\sqrt{30}$.
6. $\pm\sqrt{-9}$.
7. $\pm\sqrt{-4}$.
8. $\pm\frac{1}{2}\sqrt{7}$.
9. $\pm\frac{2}{3}\sqrt{3}$.
10. $\pm\sqrt{-2}$.
11. $\pm\sqrt{-\frac{2}{5}}$.
12. ± 25.
13. $\pm 1\frac{1}{3}$.
14. $\pm\sqrt{7}$.
15. ± 2.
16. $\pm\sqrt{-4}$.
17. $\pm\frac{1}{6}\sqrt{13}$.
18. $\pm 1\frac{1}{2}$.
19. $\pm\frac{1}{3}\sqrt{2}$.
20. $\pm\sqrt{-\frac{19}{11}}$.
21. $\pm\sqrt{5}$.
22. ± 3.
23. ± 7.
24. ± 1.
25. ± 5.
26. $\pm\frac{1}{3}\sqrt{30}$.
27. 0.
28. $\pm\sqrt{-14}$.
29. ± 2.
30. ± 2.64.
31. ± 1.58.
32. ± 1.22.
33. ± 0.51.
34. ± 1.92.
35. ± 2.91.
36. ± 1.67.
37. ± 3.05.
38. ± 4.47.
39. ± 4.12.
40. ± 3.60.

Pages 273, 274.

1. 2, 4.
2. 2, -6.
3. 9, -1.
4. 11, -2.
5. -4, -10.
6. 2, -3.
7. 5, 6.
8. -7, 2.
9. $-6\pm\sqrt{21}$.
10. $-\frac{1}{4}\pm\frac{1}{4}\sqrt{33}$.
11. 3, $-\frac{1}{2}$.
12. $\frac{2}{3}$, $1\frac{1}{2}$.
13. $-\frac{1}{4}$, 3.
14. $\frac{2}{5}$, $2\frac{1}{2}$.
15. 7, $-1\frac{1}{3}$.
16. $-\frac{3}{4}\pm\frac{1}{4}\sqrt{41}$.
17. $\frac{1}{10}\pm\frac{1}{10}\sqrt{141}$.
18. $\frac{7}{12}\pm\frac{1}{12}\sqrt{13}$.
10. $\frac{1}{3}\pm\sqrt{3}$.
20. 14, -8.
21. $3\frac{1}{2}\pm\frac{1}{2}\sqrt{-11}$.
22. 4, -3.
23. ± 1.
24. 7, $-6\frac{1}{3}$.
25. $3\frac{3}{4}\pm\frac{1}{12}\sqrt{1185}$.
26. ± 5.
27. $-3\frac{3}{4}\pm\frac{1}{4}\sqrt{793}$.
28. $-\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$.
29. $-\frac{1}{6}\pm\frac{1}{6}\sqrt{85}$.
30. $\frac{1}{3}\pm\frac{1}{3}\sqrt{37}$.
31. $2\pm\frac{1}{3}\sqrt{3}$.
32. 3, $-\frac{4}{5}$.
33. 4, 11.
34. 3, $-6\frac{1}{2}$.
35. 5, -3.
36. 0, 4.
37. $2\pm\frac{1}{3}\sqrt{3}$.
38. $5\frac{23}{32}\pm\frac{1}{32}\sqrt{209}$.
39. 4.54, -1.54.
40. 18.28, -3.28.
41. 0.20, -0.53.
42. -1.77, -6.23.
43. 0.16, -6.16.
44. 0.45, -0.36.
45. 13.85, -2.52.
46. 2.05, -2.30.
47. -1, 1.4.
48. 2.05, -2.55.

Pages 276, 277.

1. $5, -2$.
2. $-7, -3$.
3. $-2, 1\frac{1}{2}$.
4. $-2, \frac{1}{3}$.
5. $-1, \frac{5}{6}$.
6. $7, \frac{1}{7}$.
7. $\frac{-1\pm\sqrt{-79}}{8}$.
8. $\frac{2}{3}, 1\frac{1}{2}$.
9. $\frac{2}{5}, 3\frac{1}{2}$.
10. $8, \frac{1}{3}$.
11. $\frac{-3\pm\sqrt{-23}}{4}$.
12. $\frac{1\pm\sqrt{-34}}{5}$.
13. $\frac{1\pm\sqrt{-179}}{30}$.
14. $4\pm 2\sqrt{3}$.
15. $2, 1\frac{1}{2}$.
16. $\frac{3\pm\sqrt{-11}}{4}$.
17. $\frac{1\pm\sqrt{337}}{42}$.
18. $\frac{1\pm\sqrt{217}}{18}$.
19. $-1, 1\frac{1}{11}$.
20. $\frac{-3\pm\sqrt{13}}{2}$.
21. $\frac{-1\pm\sqrt{17}}{2}$.
22. $1\pm\sqrt{-9}$.
23. $\frac{-1\pm\sqrt{129}}{8}$.
24. $\frac{3\pm\sqrt{-7}}{4}$.
25. $-1\pm 2\sqrt{2}$.
26. $3, -\frac{1}{3}$.
27. $\frac{-1\pm\sqrt{-3}}{2}$.
28. $3, -6\frac{1}{2}$.
29. $6, -1$.
30. $1, 1$.
31. $\frac{1\pm\sqrt{-3}}{2}$.
32. $-2, \frac{2}{3}$.
33. $2, -\frac{1}{3}$.
34. $2\pm\sqrt{5}$.
35. $7, 3\frac{4}{5}$.
36. $0, -5$.
37. $3, \frac{3}{5}$.
38. $3, \frac{3}{4}$.
39. $-4\frac{3}{5}, \frac{1}{2}$.
40. $3, -5$.
41. $4, -\frac{4}{7}$.
42. $4, -1$.
43. $4, -1\frac{3}{4}$.
44. $5, -3\frac{1}{2}$.
45. $2, -24$.
46. $6.27, -1.27$.
47. $1.20, -1.45$.
48. $1.38, -1.13$.
49. $2.16, -4.16$.
50. $1.37, 0.12$.
51. $2.34, -3.84$.

Pages 278–283.

1. 12, 13. **2.** 15, 16. **3.** 10, 21. **4.** 8, 42. **5.** 6, 8. **6.** $\frac{7}{12}$. **7.** 3, 4.
8. 5 in., 2 in.; or 7 in., 4 in. **9.** 500 ft. by 596 ft.
10. 6 in. square.
11. 2 ft.
12. 7.9 – ft.
13. 6.4 + rd.
14. 13.9 + rd.
15. 3.2 – rd.
16. 6 in., 8 in., 10 in.
17. 12 in., 16 in.
18. 5 hr.
19. 30 mi. per hour, 40 mi. per hour.
20. 36 mi. per hour.
21. 40 mi. per hour.
22. 25.
23. 80¢.
24. 85.
25. 160 acres.
26. 24 min., 18 min.
27. $15\frac{3}{8}$ min.
28. 15 hr.

29. 8 hr., 10 hr. **30.** 10 mi. per hour; 30 mi. per hour.

31. 9.9 − in., 6.1 + in. **34.** 129.4 + ft. **37.** 2.16 + in.

32. 8.1 − in. **35.** 20 in. square. **38.** 1.56 + in.

33. 9.7 + in. **36.** 13 in. by 22 in.

39. Hind, 12 ft., front, 10 ft. **42.** Bases, 8 in., 22 in.; altitude, 8 in.

40. 37½ in., 8½ in. **43.** 7.16 + in., 0.84 − in. **45.** 10 ft. **47.** 3.8 + ft.

41. 356.7 − ft. **44.** 18.62 −, 3.38 +. **46.** 4 ft.

Pages 284, 285.

1. $\pm\frac{1}{a}\sqrt{ab}$. **2.** $\pm a$. **3.** $\frac{n\pm\sqrt{n^2+16}}{2}$. **4.** $3r,\ -\frac{1}{2}r;\ \frac{1}{3}x,\ -2x$.

5. $2A,\ A;\ \frac{1}{2}W,\ W$. **6.** $-2m-n,\ n;\ -y-2m,\ y$.

7. $-r\pm\sqrt{r^2-s}$. **8.** $z,\ -\frac{1}{z};\ n,\ -\frac{1}{n}$. **9.** $v,\ -2$. **10.** $-a,\ -b$.

11. $m,\ -n$. **12.** $-1\pm\frac{1}{R}\sqrt{R^2-1}\,;\ \pm\sqrt{\frac{-1}{n^2+2n}}$.

13. $\frac{d-2a\pm\sqrt{d^2-4ad+4a^2+8dS}}{2d}$. **14.** $\frac{1\pm\sqrt{1+8S}}{2}$. **15.** $\pm\frac{1}{4}\sqrt{s}$.

16. 8.9 + sec. **17.** $\frac{-v\pm\sqrt{v^2+64s}}{32}$. **18.** 3 sec.

19. $\frac{-5L\pm\sqrt{57L^2+19200LHD}}{8L}$.

Page 285.

2. $\frac{-2\pm\sqrt{-11}}{3}$. **4.** $-1,\ 1\frac{4}{5}$. **7.** $\frac{9\pm\sqrt{501}}{30}$. **10.** $\frac{7\pm\sqrt{229}}{6}$.

5. $-1,\ \frac{11}{12}$.

3. $1,\ -\frac{4}{7}$. **6.** $\frac{1\pm\sqrt{29}}{6}$. **8.** $1,\ -\frac{5}{6}$. **11.** $\frac{2}{3},\ 1\frac{1}{2}$.

9. $3,\ \frac{2}{5}$.

Pages 287–289.

1. $a=1,\ b=2;\ a=2\frac{1}{5},\ b=-\frac{2}{5}$. **4.** $A=2, B=1;\ A=-1, B=-2$.

2. $x=3,\ y=-2;\ x=-3,\ y=2$. **5.** $u=2,\ v=5;\ u=-5,\ v=-2$.

3. $m=1,\ n=4;\ m=4,\ n=1$. **6.** $t=1,\ v=2;\ t=4,\ v=-4$.

7. $r_1=\frac{3+\sqrt{-15}}{2},\ r_2=\frac{-39+3\sqrt{-15}}{8};$

$r_1=\frac{3-\sqrt{-15}}{2},\ r_2=\frac{-39-3\sqrt{-15}}{8}.$

8. $p=\sqrt{2}, q=2\sqrt{2};\ p=-\sqrt{2}, q=-2\sqrt{2}.$

9. $R=7, r=0;\ R=1\frac{2}{5}, r=11\frac{1}{5}.$ **10.** $c=-2, d=3;\ c=6, d=-1.$

11. $M=1+\frac{1}{3}\sqrt{15}, N=1-\frac{1}{3}\sqrt{15};\ M=1-\frac{1}{3}\sqrt{15}, N=1+\frac{1}{3}\sqrt{15}.$

12. $x=-5, y=2;\ x=-10, y=-8.$ **17.** $a=2, b=2;\ a=2, b=2.$

13. $x=5, y=7;\ x=-4\frac{7}{13}, y=-7\frac{4}{13}.$ **18.** $v=4, w=7;\ v=-7, w=-4.$

14. $W=2, D=\frac{1}{2};\ W=\frac{1}{2}, D=2.$ **19.** $V=1, t=5;\ V=-3, t=-3.$

15. $h=-4, k=-3;\ h=\frac{2}{3}, k=-1\frac{4}{9}.$ **20.** $x=3, x'=1;\ x=1\frac{1}{5}, x'=-2\frac{3}{5}.$

16. $E=1, F=6;\ E=6, F=1.$

21. $y=9+\sqrt{89}, z=-4-\sqrt{89};\ y=9-\sqrt{89}, z=-4+\sqrt{89}.$

22. $m=-8, n=-3;\ m=9\frac{1}{4}, n=2\frac{3}{4}.$ **30.** $A=4, B=5;\ A=5, B=4.$

23. $R=10, S=3.$ **31.** 3 and 5, or −5 and −3.

24. $T=2, t=-6;\ T=2, t=-6.$ **32.** 12 ft. and 16 ft.

25. $a=3, b=2;\ a=2, b=3.$ **33.** 59.39+ rd. by 19.39+ rd.

26. $m=1\frac{1}{3}, n=\frac{2}{3};\ m=\frac{1}{2}, n=1\frac{1}{2}.$ **34.** 36 rd. by 80 rd.

27. $p=11, q=-8;\ p=8, q=-11.$ **35.** 12 in., 7 in.

28. $G=3, t=1;\ G=1, t=3.$ **36.** 15 ft., 8 ft.

29. $u=12, v=4;\ u=-1\frac{5}{7}, v=-5\frac{1}{7}.$ **37.** 3 ft. by 4 ft. by 12 ft.

38. 6 hr., 3 hr. **40.** 5 hr., 7 hr. **42.** $S=\pi D^2.$

39. 12 da., 6 da. **41.** $S=\pi\frac{1}{4}D^2.$ **43.** $S^2=\pi VH.$

Page 292.

7. $x=8, y=6;\ x=-6, y=-8.$ **8.** $x=4, y=10;\ x=-5, y=-8.$

9. $x=1.8-, y=1.1+;\ x=-2.8-, y=3.4-.$

10. $x=2, y=16;\ x=-7\frac{5}{23}, y=-11\frac{15}{23}.$

11. $x=14, y=5;\ x=-14\frac{4}{7}, y=-6\frac{3}{7}.$ **13.** Imaginary.

12. $x=20, y=10;\ x=\frac{10}{11}, y=-2\frac{8}{11}.$ **14.** Equal.

Pages 292, 293.

2. 2, −1.
3. 4, −2.
4. 5, −2.
5. −3, −4.
6. 3, −4.
7. 5, −3.
8. 2, $\frac{1}{2}$.
9. $\frac{1}{2}$, −3.
10. 1, $-\frac{1}{5}$.
11. $2\frac{1}{2}$, $-2\frac{1}{3}$.
12. 6, $-\frac{3}{4}$.
13. 2, $-1\frac{2}{5}$.

Pages 294, 295.

1. a^7.
2. N^{11}.
3. x^{20}.
4. t^{24}.
5. v^{10}.
6. A^{24}.
7. r^{24}.
8. y^{20}.
9. P^{15}.
10. d^{23}.
11. a^{10}.
12. V^{20}.
13. s^{16}.
14. Q^{15}.
15. k^{36}.
16. x^{6n}.
17. a^{5n}.
18. P^{5t-1}.
19. b^{13}.
20. V^{4n+m}.
21. Q^{2r}.

Page 295.

1. p^4.
2. A^7.
3. w^8.
4. R^9.
5. N^{18}.
6. t^8.
7. x^4.
8. Q^{16}.
9. v^4.
10. m^4.
11. B^9.
12. k^{16}.
13. W^{4n}.
14. R^{4a}.
15. x^{4n}.
16. A^{x+3}.
17. y^{4l+2}.
18. Y^{a+b}.
19. Z^2.
20. G^{10}.
21. K^8.

Page 296.

1. a^{12}.
2. x^{30}.
3. P^{12}.
4. r^{18}.
5. y^{30}.
6. t^{32}.
7. B^{84}.
8. k^{27}.
9. E^{100}.
10. m^{84}.
11. A^{64}.
12. s^{100}.
13. a^{xy}.
14. w^{2mn}.
15. x^{6mn}.
16. Q^{20t^2}.
17. y^{a^2-1}.
18. $R^{a^2-b^2}$.
19. U^{2mn-n^2}.
20. 2^8.

Page 297, § 161.

1. $8a^{15}$.
2. x^8y^8.
3. $32N^{15}M^{10}$.
4. $a^{12}b^{18}c^6$.
5. $P^{32}t^{16}v^{24}$.
6. $w^{35}v^{14}z^{49}$.
7. $625A^8B^{40}$.
8. $m^{50}n^{40}p^{30}$.
9. $81V^{24}P^8$.
10. $Q^{24}n^{72}$.
11. $\frac{a^{24}}{b^{30}}$.
12. $\frac{32E^{30}}{P^{20}}$.
13. $\frac{a^{16}b^{16}c^{16}}{x^{24}y^{24}z^{24}}$.
14. $\frac{a^{nm}}{t^{2nm}}$.
15. $\frac{x^{12a}}{y^{8b}}$.
16. $\frac{Nx^2M^{4x}}{W^{x^2+xy}}$.

Page 297, § 162.

1. a^6.
2. y^3.
3. N^3.
4. x^3.
5. z^3.
6. P^3.
7. t^5.
8. W^3.
9. n^5.
10. a^4.
11. v^2.
12. t^4.
13. R^4.
14. p^6.
15. B^3.
16. s^6.
17. W^7.
18. k^{12}.
19. G^4.
20. m^{25}.

Page 299.

1. n^{-5}.
2. x^{-13}.
3. w^5.
4. p^{-4}.
5. r^{-2}.
6. N^{17}.
7. y^3.
8. D^{-8}.
9. V^{-4}.
10. k^8.
11. T^2.
12. a.
13. $x^{\frac{3}{5}}$.
14. $P^{\frac{2}{3}}$.
15. $t^{\frac{1}{12}}$.
16. $K^{\frac{11}{15}}$.
17. $n^{\frac{1}{4}}$.
18. $V^{\frac{3}{8}}$.
19. $R^{\frac{5}{12}}$.
20. $b^{\frac{1}{4}}$.
21. $L^{\frac{1}{16}}$.
22. x^6.
23. a^{-16}.
24. W^{-5}.
25. t^{-14}.
26. m^{30}.
27. $a^{\frac{1}{4}}$.
28. $D^{\frac{1}{4}}$.
29. $k^{\frac{5}{12}}$.
30. $R^{\frac{1}{15}}$.
31. $V^{\frac{1}{6}}$.
32. x^{-3}.
33. t^{-4}.
34. V^{-4}.
35. $a^{\frac{2}{7}}$.
36. $m^{\frac{3}{8}}$.

Page 299 (Supp. Ex.)

1. 4.
2. 27.
3. 32.
4. $\frac{1}{2}$.
5. $\frac{1}{9}$.
6. $n^{\frac{1}{2}}$.
7. $A^{\frac{3}{4}}$.
8. $k^{\frac{1}{3}}$.
9. $\dfrac{1}{\sqrt[9]{y^2}}$.
10. $\frac{1}{8}$.
11. $a^{-3}+a^{-2}-10a^{-1}+8$.
12. $x^{-3}-1$.
13. $t^{-2}-t^{-1}+1$.
14. $m^{-1}+n^{-1}$.
15. $N^{-6}-2N^{-3}+1$.

Pages 300–304.

1. $53485\frac{3}{7}$.
2. 3549.6+.
3. $L=r\sqrt{2-\sqrt{2}}$; 9.2−
4. $x=\dfrac{dr'}{r-r'}$; 32.
5. $t=\pm\dfrac{2\pi}{F}\sqrt{Fmr}$; $m=\dfrac{Ft^2}{4\pi^2 r}$; $r=\dfrac{Ft^2}{4\pi^2 m}$.
6. $n=\pm\dfrac{1}{2Lw}\sqrt{Egw}$; $L=\pm\dfrac{1}{2nw}\sqrt{Egw}$; $w=\dfrac{Eg}{4n^2L^2}$.
7. $E=\dfrac{CC'r}{C-C'}$.
8. 21.86+.
9. $\dfrac{b^2+c^2-a^2}{2b}$.
10. $\dfrac{gs}{2s+n-1}$.
11. 26.81+ lb.
12. 28 in.
13. $4a(4a^2-3b+c)$.
14. $(2a+3b)^2$.
15. $(3x-2y)^2$.
16. $n(5m^2+n)^2$.
17. $16(2rt+sv)(2rt-sv)$.
18. $(4y^4+1)(2y^2+1)(2y^2-1)$.
19. $n(m^2+n^2)(m+n)(m-n)$.
20. $2y(2x-y)(4x^2+2xy+y^2)$.

21. $6m(m+x)^2$.

22. $(a+x)(d+y)$.

23. $5xb(x+2)(x-3)$.

24. $(r-3)(r-\frac{1}{2})$.

25. $(y+\frac{1}{2}z)(y+3z)$.

26. $\frac{2}{3}$.

27. $\frac{x^{m-1}}{ab+b^2x}$.

28. $2-\frac{2}{2x^2-x+1}$.

29. $\frac{m^3+mn^2+3}{m}$.

30. $\frac{b^2+2bc+c^2-a^2}{2bc}$.

31. $\frac{x^3-2x^2-3x}{x-2}$.

32. $\frac{a^3b-ab^3}{a^2-b^2}$, $\frac{a^2-2ab+b^2}{a^2-b^2}$, $\frac{a^2+2ab+b^2}{a^2-b^2}$.

33. $6a-\frac{22x}{45}$.

34. 0.

35. $\frac{2x-3}{2x^3+3x^2-2x-3}$.

36. $\frac{ny}{4m}$.

37. $x^2+1+\frac{1}{x^2}$.

38. $\frac{1-y}{x}$.

39. $x^2-\frac{1}{x^2}$.

40. $\frac{-cb(b-c)^2}{c^4+c^2b^2+b^4}$.

41. $\frac{4}{3x+3}$.

42. 2.

43. $\frac{abc-b^2c-ac+4ab}{b-c}$.

44. $\frac{ab+bc-b^2}{a}$.

45. $x=4$, $y=3$.

46. $x=8$, $y=2$.

47. $x=5$, $y=7$.

48. $a=12$, $b=8$.

49. $t=10$, $w=5$.

50. $t=11\frac{1}{27}$, $z=6\frac{17}{18}$.

51. $x=\frac{abcm+acdn}{bc+ad}$, $y=\frac{bcdn-abdm}{bc+ad}$.

52. $x=4$, $y=6$, $z=2$.

53. $x=8$, $y=7$, $z=6$.

54. 20%.

55. 52%.

56. $13\frac{1}{3}$ bu. of 60-cent, $26\frac{2}{3}$ bu. of 90-cent.

57. 8¢, 7¢, 4¢.

58. $1\frac{2}{13}$ gal.

59. $2a^2-4a+2$.

60. $m+1-\frac{1}{m}$.

61. 5.76.

62. 1.414.

63. $4ax\sqrt{6x}$.

64. $(x+y)\sqrt{x-y}$.

65. $\frac{b}{2(a+x)}\sqrt{a^2+ax}$.

66. $\frac{1}{x-1}\sqrt{x^2-1}$.

67. $\frac{3a^3}{2b}$.

68. a^2-4b.

69. 2.

70. $1\frac{1}{2}$, $-5\frac{1}{2}$.

71. 4, $-1\frac{3}{4}$.

72. 1.786, -1.119.

73. 6 yd. square;
7 yd. square.

74. $x=3$, $y=2$;
$x=15\frac{2}{3}$, $y=-4\frac{1}{3}$.

75. $x=5$, $y=3$;
$x=-9\frac{1}{4}$, $y=-6\frac{1}{2}$

76. 20 rd. square;
30 rd. square.

77. $a^2bc^{\frac{3}{2}}$.

78. $a^{33}b^{10}c^5$.

79. $x^{2n}y^{4m}$.

80. $\frac{1}{ab}$.

81. 3^{n+4}.

1-304

STONE–MILLIS ELEMENTARY ALGEBRA

ANSWERS

SECOND COURSE

Pages 2, 3.

1. By writing them side by side, with the sign ×, a dot, or no sign at all between them.

2. $5n$; xy; $2ab$.

3. By enclosing each in parentheses and writing these side by side, as in Ex. 1.

4. $(t-4)(2t+3)$.

5. aa; nnn; $(x-y)(x-y)(x-y)(x-y)$.

6. Exponent; base; a power.

7. Numerical coefficient.

8. 2; 12; 6.

9. No; $3a=12$, $a^3=64$; $3a=30$, $a^3=1000$.

10. 1, monomial; 2, binomial; 3, trinomial; 2, binomial; 4, polynomial; 1, monomial; 1, monomial; 2, binomial; 4, polynomial; 5, polynomial.

11. 81; 256.

12. 60.

13. 16.

14. 49.

15. 179.

16. 528.

17. 64.

18. 113.0976 cu. in.

19. 132 sq. in.

20. 7439.3088 cu. in.

Page 4.

1. 20.

2. 20.

3. 19.

4. 20.

5. 30.

6. 36.

7. 2300.

8. 2487.

9. 2092.

10. 25,004.

Page 5.

1. -8; $+15$; -6; $+2$; -2; -18.

2. -8. **3.** $+5$. **4.** -5. **5.** -2. **6.** $+6$. **7.** $+3$.

Pages 5, 6.

1. $+3a$.
2. $-7n$.
3. $-15W$.
4. $+12t^2$.
5. $+2a^2b^3$.
6. $-6N$.
7. $+9x^3$.
8. $-4ab$.
9. $+4Vt$.
10. $-7y^4$.
11. $-4k$.
13. $3a^2+2ab-2b^2$.
14. $6x-3$.
15. $-3A^3-5B^3+3A^2B$.
16. $-3y^3-5y^2-4y-10$.
17. $7a-5b-c$.
18. $3w^2+2wv-4v^2$.
19. 160.
20. 520.
21. 1440.
22. 4800.
23. 1440.
24. 6500.
25. 1860.
26. 314.16.
27. 62.832.

Page 7.

1. $2a-8b-4c$.
2. $3a^2+b^2$.
3. $3x^2-5x+4$.
4. $2a^3+4a^2b-2ab^2$.
5. $-3a^2+10a+3$.
6. $28+2t+t^2$.
7. $3x^2+2xy+3y^2$.
8. $a^3+3a^2+2-5ab+b^2$.
9. $21+12v+2v^2$.
10. $2a$.
11. $4x^2+2x+5$.
12. $-3a^2+2a$.
13. $2x^2-2xy+5y^2$.

Pages 7, 8.

1. -2; $+4$; $+7$; -6; $+4$; $+7$.
2. -27; $+12$; -6; -15; -3; -4.
3. $+6$; $+36$; $+6$; $+40$; $+14$; $+7$.
4. $+2$; -9; -6.
5. $+18$; $+7$; $+4$; $+19$; $+14$; $+24$.
6. −6°.
7. 110.
8. 63.
9. 56.
10. −5 subtracted from −6; −1.
11. 12; 6; −10.

Pages 8, 9.

1. $+12a$; $-9V$; $-4x$; $-4t^2$; $-29P^3$.
2. $-12ab$; $+2y^2$; $-5R$; $+4Ww$; $-5\pi R^2$.
3. $+n^2$; $-13n^2$; $-9n^2$; $-7n^2$; 0.

4. $10x$; $-14v$; $-8\,S$.
5. $-2x^2-11x+4$.
6. $-2a+4b-c$.
7. $4a^2+4a+1$.
8. $-6m+4n$.
9. $8x^3-4x^2+8x+5$.
10. $3r^2$.
11. $-3A^2B+3AB^2$.
12. $-2x^3-3x^2+4x-1$.
13. $4a-3a^2$; $5a^2-4a-2$.
14. $y-6z$ subtracted from $3y-z$; $2y+5z$.
15. $2T^2+6T-10$.
16. $2ab-2b^2$.

Page 10.

1. $3a$.
2. $2x-y$.
3. $-2n$.
4. $3t-5a$.
5. $5-x-x^2$.
6. M.
7. $6-2v$.
8. $2b^2-2ab$.
9. $-3W$.
10. $[4a-6b][-2a+2b]$.
11. $[P+Q-R][P-Q+R]$.
12. $[x+5y+a][x-5y-3a]$.
13. $(a^2-b^2)-(x^2+2xy+y^2)$.
14. $(2ab-a^2)-(m^2-n^2)$.
15. $(a^2-ab+b^2)-(x^2+2xy-y^2)$.
16. $(a^3-b^3)-(x^3-3x^2y+3xy^2-y^3)$.
17. $(5b-2a)-(3r+4s-t)$.
18. $(a^4-b^4)-(w^4-w^2v^2+v^4)$.

Page 12.

1. 720.
2. 480.
3. 840.
4. 840.
5. 3400.
6. 9600.
7. 700.
8. 90.
9. 1600.
10. 252.
11. $3\frac{1}{8}$.
12. 17,500.

Pages 12, 13.

1. +12; +15; −32; −14; +27; −35.
2. +8; −56; −30; +36.
3. −42; +40; +36; −135.
4. −6; −16; +60.
5. +63; +64; −125.
6. +1; +360.
7. +4; +16; +9; −1; +8.
8. −1; +36; +100; +36.
9. −8; +20; +52.
10. −65.

Pages 14, 15.

1. a^7; a^{10}; x^{10}; n^{16}; P^{11}.

2. y^6; t^{12}; D^{12}; m^{18}.

3. $15a^5$; $30V^6$; $18z^6$; $28b^{10}$.

4. $+12a^8$; $-40x^5$; $-12N^5$.

5. $-24a^4b^6$; $+18m^7n^7$; $+48x^6y^{10}$.

6. a^2-ab; $2m^2n+3mn^2$; $-4x^3y+12xy^3$.

7. $36n^3-60n^2$; $24A^6-32A^5+8A^4$; $x^4-x^3y+x^2y^2$.

8. $-15t^6+35t^5-10t^4$; $-3\pi R^4+6\pi R^3T-3\pi R^2T^2$.

9. $x^2-4x-21$; w^4-8w^2+15; $N^3-10N^2+28N-24$.

10. a^4-1; y^6-1; t^8-2t^4+1.

11. a^4+a^2+1.

12. $8s^4+26s^3-21s^2+26s-15$.

13. $-12x^5-3x^4+38x^3+23x^2-16x-40$.

14. $-4D^4-16D^3+19D^2+12D-12$.

15. n^5-1.

16. $p^3+6p^2+11p+6$.

17. A^4-16.

18. $81y^4-625$.

19. $v^{12}-x^{12}$.

20. $4a^2+a$.

21. $14x-47$.

22. $-xy-18x^2+35y^2$.

23. $2t^2-10$.

24. $2n^4-2n^3-4n^2-13n+5$.

25. $-2Q^2-22Q+10$.

26. $-2x^2y^2-2y^4$.

27. t^6; v^{20}; x^8.

28. $-125x^6y^9$; $36w^8v^{14}$; $-8H^{18}t^{15}$.

29. x^2+6x+9; $a^2-2ab+b^2$; $4a^2-12ab+9b^2$.

30. $9m^2+30mn+25n^2$; $a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$.

Pages 16, 17.

1. 2^6.

2. 10^4.

3. 5^4.

4. 30^4.

5. 20^6.

6. $8a^6b^3$.

7. $9a^8b^4$.

8. $1000a^9b^{15}$.

9. $400x^2y^{12}$.

10. $27,000x^6y^{15}$.

11. $a^{2n}b^{3n}$.

12. $x^my^{2m}z^{3m}$.

13. $2^na^{2n}b^n$.

14. $a^{ncd}b^{mcd}$.

15. $64a^{18}b^6$.

16. n^2-16.

17. x^2-100.

18. R^2-1.

19. w^2-36.

20. $9-t^2$.

21. $4a^2-9b^2$.

22. $36r^2-s^2$.

23. $49m^2-4n^2$.

24. $x^2+7x+10$.

25. $T^2-11T+28$.

26. $V^2+5V-36$.

27. $y^2-13y+42$.

28. $4a^2+16a+15$.

29. $25x^2-20x-21$.

30. $144P^2+48P-45$.

31. $64a^2+72a+14$.

32. $x^2+2xy+y^2$.

33. $4n^2+20n+25$.

34. $R^2-14R+49$.

35. $25d^2-120d+144$.

36. $9x^2+24xy+16y^2$.

37. $225m^2-60mn+4n^2$.

38. $1-8t^2+16t^4$.

39. $81+36V+4V^2$.

40. $x^2+y^2+z^2+2xy-2xz-2yz$.

41. $16m^2+9n^2+p^2-24mn+8mp-6np$.

42. $1-4x+6x^2-4x^3+x^4$.

43. $16a^2+4b^2+25c^2+16ab-40ac-20bc$.

44. $a^4+a^3b-a^2b^2-ab^3+b^4$.

45. $x^2+y^2+z^2+w^2+2xy-2xz-2xw-2yz-2yw+2zw$.

46. $81+108t-64t^2$.

47. $9K^2-49L^2$.

48. $x^2y^2+2xy-143$.

49. $64M^4+16M^2-3$.

50. $256v^2t^2-s^4$.

51. $196A^6-1$.

52. 1681; 4896; 9991; 4225; 6804.

53. 11,009; 8464; 3584; 1575; 6241.

Pages 17, 18.

1. $+2$; $+6$; -2; -8; $+3$; -7.

2. $+6$; -7; -8.

3. a^5; t^5; V^6; R^{12}; n^8.

4. $8x^2$; $4P^4$; $16y^{12}$; $7T^3$.

5. $13a^5$; $-4V^4$.

6. $-5x^2y^3$; a^9.

7. $4x^2-2$; $-3m^2+2mn-4n^2$.

8. $-2ab+6b^2+3a^2$.

9. $-3r^3-4s^3+5r^2s$.

10. $8W^3+7W^2V-9WV^2+3V^3$.

11. $7x^2y^2-6y^4-8x^4$.

12. $M^2-\frac{1}{2}MN+N^2$.

Page 19.

1. $n+5$.

2. $x-4$.

3. $6+a$.

4. $t-6$.

5. $y+4$.

6. $3-v$.

7. $1-5N$.

8. $R+9$.

9. a^2+a+1.

10. $9-3m+m^2$.

11. $1+x+x^2+x^3$.

12. $s-t$.

13. A^2-AB+B^2.

14. $a^2-2ab-5a-10b+25+4b^2$.

Page 20.

1. x^2+5x+3, rem. 3.
2. $2x^3+8x^2+23x+95$, rem. 375.
3. a^3+2a^2+a+9, rem. 17.
4. $3n^2-2n+3$, rem. 6.
5. $y^3-9y^2+18y-34$, rem. 56.
6. t^2-2t+8, rem. 64.
7. $V^4+3V^3+9V^2+30V+91$, rem. 266.
8. $2r^2-5r+15$, rem. -52.
9. $A^3+A^2-4A+17$, rem. -76.
10. $n^3-6n^2+39n-234$, rem. 1395.

Page 21.

1. B^n.
2. $ttt\ldots\ldots$ to a factors.
4. $(a-b+c)x$.
5. $(m+n-p)r^2$.
6. $(2a-3b+c)y$.
7. $(4a+9n-5b)x^2y$.
8. $(a+b+c)x+(b+c+a)y+(c+a+b)z$.
9. $5m-3n+4p$.
10. $(A-C)x-(B+D)y$.
11. $2ax-5by+(4c+a)z$.
12. -7.
13. $[5x+y-4z]\ [5x-3y+4z]$.
14. x^{3n}.
15. $a^{n+4}b^{n+2}$.
16. $-6R^{2a+2}$.
17. $A^{2x}-B^{2x}$.
18. $a^{4n}-b^{4n}$.
19. $T^{3a}-t^{3b}$.
20. x^{3n}.
21. x^{mn}.
22. $x^{na}y^{nb}$.
23. $2^c a^{nc} b^c$.
24. M^{3a}; B^{2x+2}.
25. $12x^{3m-n}y^{n-3m}$.
26. $8W^{a-4}V^2$.
27. $1-3N+2N^2$.
28. x^n+y^n.
29. $A^{3n}+A^{2n}B^n+A^nB^{2n}+B^{3n}$.
30. $3an^2$.

Pages 22, 23.

1. $6a^3(a-3)$.
2. $3mn(5m^2+4mn+3n^2)$.
3. $2x^3(2x^3-7xy^2+5y^3)$.
4. $2\pi R(R+2H)$.
5. $7t^2(t^2-4tv-2v^2)$.
6. $-A(A^2+A+1)$.
7. $-1(a^2+2ab+b^2)$.
8. $x^3y^3z^4(xz^2+x^2y+y^2z)$.
9. $4\pi(R^2+r^2+Rr)$.
10. $8V_1(V_1-2V_2+V_3)$.
11. $3m^2n(m^2-2mn+n^2)$.
12. $x^n(1+a)$.
13. $a^n(b^n-1)$.
14. $t^2v^2(v^n+t^n)$.
15. $P^n(P+Q)$.
16. $5a^{n-1}y^{n-1}(a+2y)$.

Page 24.

1. $(3A+4)(3A-4)$.
2. $(2t+5)(2t-5)$.
3. $(1+7n)(1-7n)$.
4. $(5x+8y)(5x-8y)$.
5. $(9V+1)(9V-1)$.
6. $(2P+3Q)(2P-3Q)$.
7. $(5R+4)(5R-4)$.
8. $(6v+7w)(6v-7w)$.
9. $(a-2b)(a^2+2ab+4b^2)$.
10. $(M-3)(M^2+3M+9)$.
11. $(1-4k)(1+4k+16k^2)$.
12. $(x-5y)(x^2+5xy+25y^2)$.
13. $(6-T)(36+6T+T^2)$.
14. $(m+3n)(m^2-3mn+9n^2)$.
15. $(A+1)(A^2-A+1)$.
16. $(2r+1)(4r^2-2r+1)$.
17. $(1+4P)(1-4P+16P^2)$.
18. $(5p+2q)(25p^2-10pq+4q^2)$.
19. $(2W-1)(4W^2+2W+1)$.
20. $(3+s)(9-3s+s^2)$.
21. $3(a+2b)(a-2b)$.
22. $5(1-2N)(1+2N+4N^2)$.
23. $6(2x+y)(4x^2-2xy+y^2)$.
24. $(x-1)(x+1)(x^2+1)$.
25. $(y+z)(y-z)(y^2-yz+z^2)(y^2+yz+z^2)$.
26. $(A-B)(A+B)(A^2+B^2)(A^4+B^4)$.
27. $(1+t)(1-t)(1+t+t^2)(1-t+t^2)$.
28. $(m^2+n^2)(m^4-m^2n^2+n^4)$.
29. $(x+y)(x-y)(x^2+y^2)(x^2-xy+y^2)(x^2+xy+y^2)(x^4-x^2y^2+y^4)$.
30. $a(m+n)(m-n)(m^2+n^2)$.
31. $p^3q^3(p+q)(p^2-pq+q^2)$.
32. $\pi R(R+H)(R-H)$.
33. $8vt(v-2t)(v^2+2vt+4t^2)$.
34. $(a+b+c)(a+b-c)$.
35. $(m-n+x)(m-n-x)$.
36. $(p+q-r)(p-q+r)$.
37. $(v+t+w+u)(v+t-w-u)$.
38. $(M-N+P-Q)(M-N-P+Q)$.
39. $(2A+B+C-3D)(2A+B-C+3D)$.

Page 26.

1. $(x-5)^2$.
2. $(2a+1)^2$.
3. $(3t-4v)^2$.
4. $(1+7P)^2$.
5. $(R-6r)^2$.
6. $(n+5)(n+2)$.
7. $(x-9)(x-2)$.
8. $(A-8)(A+7)$.
9. $(w+7v)(w-5v)$.
10. $(T-10U)(T+3U)$.
11. $(4a-3)(2a-1)$.
12. $(2V-5)(V+6)$.
13. $(5M-4)(3M+2)$.
14. $(4R+3r)(3R+4r)$.
15. $(2x+7y)(5x-3y)$.
16. $(8p+3q)^2$.
17. $(4W-9)(W+3)$.
18. $(L+7v)(L-2v)$.
19. $m(n-9)(n+6)$.
20. $5n(x+y^2)^2$.

21. $4a(F+5)(F-3)$.
22. $2n(2k-5m)(k+2m)$.
23. $-B(B+1)^2$.
24. $h^2(3m-2)(2m+7)$.
25. $(5r_1-2r_2)(r_1-2r_2)$.
26. $6(6+D)(1-D)$.
27. $t(T-15v)^2$.
28. $-4(1+3d^2)^2$.
29. $2\pi(2R-7)(R-2)$.
30. $3(5V_1+V_2)(3V_1+4V_2)$.
31. $(x+b-2)(x+b-3)$.
32. $(m-n+w+v)^2$.
33. $(4A+4B-9C)(A+B+3C)$.

Pages 27, 28.

1. $(a-b+c)(a-b-c)$.
2. $(x+y+1)(x-y-1)$.
3. $(a-b)(n-m)$.
4. $(y+x)(y-z)$.
5. $(x+y)(P-Q)$.
6. $(5A+B-C)(5A-B+C)$.
7. $(m-n+v+t)(m-n-v-t)$.
8. $(B+A+2C)(B-A-2C)$.
9. $(s-t)(P-5)$.
10. $(n+1)(a-b)$.
11. $(V-1)(V-v)$.
12. $(R-r)(R+r+1)$.
13. $(1+2y+z)(1+2y-z)$.
14. $(1-3b+m-2n)(1-3b-m+2n)$.
15. $(H-4t)(H+4t+v)$.
16. $(2+d)(a-b+c)$.
17. $(a+b)(m^2+m+1)$.
18. $(x-z+y-w)(x-z-y+w)$.

Page 28.

1. $(n^2+n+1)(n^2-n+1)$.
2. $(1+a+2a^2)(1-a+2a^2)$.
3. $(x^2+x+1)(x^2-x+1)(x^4-x^2+1)$.
4. $(t^2+2tv+3v^2)(t^2-2tv+3v^2)$.
5. $(v^2+3v+3)(v^2-3v+3)$.
6. $(2a^2+2ab+5b^2)(2a^2-2ab+5b^2)$.
7. $(2M^2+4MN+5N^2)(2M^2-4MN+5N^2)$.
8. $(R^2+3Rr-3r^2)(R^2-3Rr-3r^2)$.
9. $(6A^2+4AT-5T^2)(6A^2-4AT-5T^2)$.
10. $(7y^4+3y^2z+2z^2)(7y^4-3y^2z+2z^2)$.
11. $(2W^4+6W^2w^3+5w^6)(2W^4-6W^2w^3+5w^6)$.
12. $(8a^2b^2+2ab+1)(8a^2b^2-2ab+1)$.
13. $(m^4+4m^2xy^2-3x^2y^4)(m^4-4m^2xy^2-3x^2y^4)$.
14. $2\pi(2R^2+5Rr-5r^2)(2R^2-5Rr-5r^2)$.
15. $4t(v^6+3v^3t+4t^2)(v^6-3v^3t+4t^2)$.
16. $3a(5a^2+2ab^2c+2b^4c^2)(5a^2-2ab^2c+2b^4c^2)$.

Pages 30, 31.

1. $(x-1)(2x+1)$.
2. $(x+1)(3x+2)$.
3. $(2y-1)(y-2)$.
4. $(4n-1)(n+2)$.
5. $(t-3)(2t-3)$.
6. $(3A+5)(A+3)$.
7. $(V-5)(3V+2)$.
8. $(x+2)(x-2)(x^2+1)$.
9. $(a+1)(a+4)(a-5)$.
10. $(y-1)(y-2)(y-3)$.
11. $(b+2)(b^2-b-1)$.
12. $(P-1)(P+3)(P+5)$.
13. $(n+1)(n^3-2)$.
14. $(v-4)(v+6)(2v-1)$.
15. $(T-1)(3T^2+T+2)$.
16. $(a+2)(a^2+a+1)$.
17. $(R+3)(R^2-R-1)$.
18. $(s+3)(s+4)(s^2+1)$.
19. $(z-1)(z+1)(z^4+1)$.
20. $(a-b)(a+b)(a+2b)$.
21. $(x-y)(2x-y)$.
22. $(x+y)(3x-y)$.
23. $(m+y)(m-y)(m+2y)$.
24. $(s-2y)(s+3y)(s-3y)$.
25. $(t-d)(t+2d)(t+3d)$.
26. $(A-B)^3$.
27. $(N+a)(N+2a)(N-3a)$.
28. In $a^{20}-b^{20}$, let $a=b$.
29. In $a^{15}+b^{15}$, let $a=-b$.

Page 32.

1. $(a-b)(a^6+a^5b+a^4b^2+a^3b^3+a^2b^4+ab^5+b^6)$.
2. $(x+1)(x^6-x^5+x^4-x^3+x^2-x+1)$.
3. $(a-2b)(a^4+2a^3b+4a^2b^2+8ab^3+16b^4)$.
4. $(1-t)(1+t+t^2+t^3+t^4)$.
5. $(1-n)(1+n+n^2+n^3+n^4+n^5+n^6)$.
6. $(a-b)(a^{10}+a^9b+\ldots\ldots+ab^9+b^{10})$.
7. $(a+b)(a^{10}-a^9b+\ldots\ldots-ab^9+b^{10})$.
8. $(R+r)(R^{12}-R^{11}r+\ldots\ldots-Rr^{11}+r^{12})$.
9. $(R-r)(R^{12}+R^{11}r+\ldots\ldots+Rr^{11}+r^{12})$.
10. $(2x+3y)(16x^4-24x^3y+36x^2y^2-54xy^3+81y^4)$.
11. $2(1-2B)(1+2B+4B^2+8B^3+16B^4+32B^5+64B^6)$.
12. $(m^2+n)(m^8-m^6n+m^4n^2-m^2n^3+n^4)$.
13. $(x^2+y^2)(x^4-x^2y^2+y^4)$.
14. $(1+a^2)(1-a^2+a^4-a^6+a^8)$.
15. $(x^4+y^4)(x^8-x^4y^4+y^8)$.
16. $(n^2+1)(n^{12}-n^{10}+n^8-n^6+n^4-n^2+1)$.

17. $(v^2+w^2)(v^4-v^2w^2+w^4)(v^{12}-v^6w^6+w^{12})$.

18. $5(1+P^2)(1-P^2+P^4-P^6+P^8)$. **19.** $4(1+N^2)(1-N^2+N^4)$.

20. $(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)(a^{10}-a^5b^5+b^{10})$.

21. $(1+y)(1-y+y^2-y^3+y^4)(1-y^5+y^{10}-y^{15}+y^{20})$.

22. $(a+1)(a-1)(a^2+a+1)(a^2-a+1)$.

23. $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)(x^2+y^2)(x^4-x^2y^2+y^4)$.

24. $(m+n)(m-n)(m^4+m^3n+m^2n^2+mn^3+n^4)(m^4-m^3n+m^2n^2-mn^3+n^4)$.

25. $(1-t)(1+t)(1+t^2)(1+t^4)(1+t^8)$. **26.** $(16-a^3)(16+a^3)$.

27. $(a+1)(a-1)(a^6+a^5+a^4+a^3+a^2+a+1)(a^6-a^5+a^4-a^3+a^2-a+1)$.

28. $(p-q)(p+q)(p^2+q^2)(p^4+q^4)$.

29. $(w^2-v)(w^2+v)(w^4+w^2v+v^2)(w^4-w^2v+v^2)$.

30. $(1-P)(1+P+P^2)(1+P^3+P^6)$.

Pages 33, 34.

1. $(2N+1)(N-1)$.
2. $a(x^2+8a^2)(x^2-8a^2)$.
3. $ab(n+1)(3n-1)$.
4. $(2P+13Q)(2P+3Q)$.
5. $t(t-b)(5t+6b)$.
6. $(2V-W)(V-W)$.
7. $(x+1)(x-1)(x^2+4)$.
8. $(x-z)(y-z)$.
9. $(n+3)(n^2+6)$.
10. $(2a-2b+y+v)(2a-2b-y-v)$.
11. $(3R-4r)(3R-5r)$.
12. $(B^2+BC+C^2)(B^2-BC+C^2)$.
13. $(x^2+4xy-y^2)(x^2-4xy-y^2)$.
14. $(T^3+7)(T^3-6)$.
15. $(h^2+7k^2)(h^2-5k^2)$.
16. $y(y-3)(1+y)(1-y)$.
17. $(1-xt)(1+xt+bt)$.
18. $(x+y+z)(x+y-z)(x-y+z)(x-y-z)$.
19. $(V-v)(V+v)^3$.
20. $a(y+10)(y-3)$.
21. $(c-b)(a+d)$.
22. $(p+3)(p-2)(p^2-p+6)$.
23. $(8t+9)(9t-5)$.
24. $(a+1)(a-1)(b+1)(b-1)$.
25. $(2z-3t)(z+4t)$.
26. $-a^2(N-3a)^2$.
27. $(1+2m^2)(1-2m^2)(1+4m^4)$.
28. $(a-b)(a+b-3)$.
29. $n(c+n^2)(b+an)$.
30. $(a^2-b^2-c^2)^2$.
31. $(x-4a^4)(x-a^2b^2)$.
32. $(n+2x)(n-x-y)$.
33. $(2w^2+3wv+3v^2)(2w^2-3wv+3v^2)$.
34. $(A+C)(A-C)(B+C)$.
35. $(K-3)(K^2+2K+2)$.
36. $(n+3)(n^2-2n-1)$.

37. $(D-1)(D-3)(D+5)$.
38. $2(6+F)(36-6F+F^2)$.
39. $(1-x+a-b)(1-x-a+b)$.
40. $(P^2+P+1)(P^2-P+1)$.
41. $(ax+1)(x+b)$.
42. $(2t-b)(t-a)$.
43. $(2H+b)(H+2a)$.
44. $(z-1)(az+b)$.
45. $(x^2y^2+7a)(x^2y^2+2a)$.
46. $2(11+v)(10-v)$.
47. $5c(2c-3)^2$.
48. $(1+2Qn^2)(1-2Qn^2+4Q^2n^4-8Q^3n^6+16Q^4n^8)$.
49. $(1-d)(1+d+d^2)(1+d^3+d^6)$.
50. $(y-1)(y-3)(y-7)$.
51. $(L-1)(L-2)(L-3)$.
52. $(5x-y+3z)(5x-y-3z)$.
53. $(3a-3b-2)(a-b-4)$.
54. $(7a+7b+3c)(a+b-2c)$.
55. $(x+3)(x-4)(x^2-5)$.
56. $(a+1)(a-1)(a-A)$.
57. $b(m+2)(m-2)(2m+3)$.
58. $a(a+b)(a^2-ab+b^2)(a^6-a^3b^3+b^6)$.
59. $(v-t)(v+t)(v^2+t^2)(v^4+t^4)(v^8+t^8)$.
60. $(P^2+P-6Q)(P^2-P+6Q)$.
61. $(1-n)(1+n^2)(1-n^2+n^4)$.
62. $g^3(289g^4-83g^2+1)$.
63. $(x+y)(x^2-xy+y^2+1)$.
64. $(M-N)(1+M^2+MN+N^2)$.
65. $(1-a)(1+a+a^2)(1+a^3+a^6+a^9+a^{12}+a^{15}+a^{18})$.
66. $(x+2+2a-y)(x+2-2a+y)$.

Page 34.

1. 6.
2. 12.
3. 72.
4. $6a^3b^3$.
5. $2ma^2$.
6. $x-1$.
7. $V-1$.
8. $a+b$.
9. x^2+y.
10. $7N^2+5N+5$.
11. $p+q$.
12. $1+x+x^2$.
13. $R-5$.
14. $2v+2t$.
15. P^2+5P.
16. $m-2$.

Page 35.

1. 1512.
2. 1800.
3. 2016.
4. $144a^{10}b^5$.
5. $225\pi R^2r(R-r)(2R+r)^3$.
6. $(x+1)(x-1)^2$.
7. $2x^2y(x+y)(x^3-y^3)$.
8. $(N+1)(N-1)(N+2)(N^2+1)$.
9. $(m-n)(m+n)(m^2+mn+n^2)(m^2+n^2)$.

10. $1-V^8$.

11. $6a^2(a-1)(a+3)(a^2+1)$.

12. $(P^2+Q^2)^2(P^2-Q^2)^2$.

13. $(w+v)^2(w-v)^2$.

14. $(y+1)(y-1)(y+2)(y-2)(y+3)(y-3)$.

15. $(3T+2)(3T-2)^2(9T^2+6T+4)$.

16. $(6M^3-7M^2N-2MN^2)(3M^2+MN-4N^2)$.

Pages 35, 36.

1. $(1+x)(1-x)^2$.

2. $(a-b)(2a-2b+1)(2a-2b-1)$.

3. $(4c-1)(c^2-2)$.

4. $(x-y)(x+y)(x^2+y^2)(x^4+y^4)(x^8+y^8)$.

5. $32(2^{x-1}-1)(2^{4x-4}+2^{3x-3}+2^{2x-2}+2^{x-1}+1)$.

6. $(xy-8)(x^2y^2+8xy+64)$.

7. $a^{2n}(a^n-b^n)$.

8. $(x^n-2)(x^n+1)$.

9. $(t^{2n}+7v^n)(t^{2n}-4v^n)$.

10. $(p^n+q^n)(p^n-q^n)$.

11. $(A^n+B^n)(A^n-B^n)(A^{2n}+B^{2n})$.

12. $x(x^{2n}-y^n)$.

13. $(V^t-1)(V^{2t}+V^t+1)$.

14. $n(k^{5n}-n^{6n-1})$.

15. $(a^n-b^n)(a^n+b^n-c^n)$.

16. $(3b^n+5c^n)(2b^n-3c^n)$.

17. $(x^{n-2}+y^{n+2})(x^{n-2}-y^{n+2})$.

18. $(R^n+r)(\pi R^n+1)$.

19. $(V^{2t}+V^t+1)(V^{2t}-V^t+1)$.

20. $(N^a+1)(N^{4a}-N^{3a}+N^{2a}-N^a+1)$.

21. $(2^{2n}+1)^2$. 22. $(a^2-b^2+3ab)(a^2-b^2-3ab)$. 23. $2(2^{n-1}+1)(2^n+1)$.

24. $(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)(a^{10}-a^5b^5+b^{10})$.

25. $(3^n-5)(3^n+1)$.

26. $(2^a+2^{2b})(2^a-2^{2b})$.

27. $(2^a-3^b)(2^{2a}+2^a3^b+3^{2b})$.

28. $(6+a)(3-2a)$.

29. D^m-1.

30. $a^{2n}+b^n$.

31. $2V^t-3$.

32. $x+3$.

33. $2a-3$.

34. $a-5$.

35. $x^{12}-y^{12}$.

36. $(x+2)(x-2)(x-5)(x^2+4)$.

37. $(9by^3-3by^2-42by)(6b^2-26b+28)$.

38. $(a^n-b^n)(a^n+b^n)(a^{2n}+b^{2n})(a^{2n}+a^nb^n+b^{2n})$.

39. $(2P^x+1)^2(2P^x-1)^2$.

40. $(x-y)^2(a^{n+1}-b^{n-1})$.

Page 38.

1. $-\frac{4}{7}$.

2. $-\frac{6}{5}$.

3. $\frac{2}{5}$.

4. $-\frac{8}{9}$.

5. $\dfrac{2x^2}{3y^2}$.

6. $-\dfrac{4ab}{5c^3}$.

7. $-\dfrac{1}{t^2}$.

8. $\dfrac{\pi r}{R}$.

9. $\frac{-1}{m-n}$. **10.** $\frac{mn}{m-n}$. **11.** $\frac{b-a}{m-n}$.

12. $\frac{3}{a-b}+\frac{4}{a-b}$. **15.** $\frac{6\pi r}{r^2-s^2}+\frac{r^2}{r^2-s^2}+\frac{r^2+s^2}{r^2-s^2}$.

13. $\frac{x}{x-1}-\frac{1}{x-1}+\frac{x+1}{x-1}$. **16.** $\frac{-1}{(a-b)(a-c)}$.

14. $\frac{1+2m}{m^2-1}+\frac{3m}{m^2-1}+\frac{m-2}{m^2-1}$. **17.** $\frac{a^3}{(a-b)(a-c)(a-d)}$.

18. $-\frac{a+b+c+d}{(a-b)(a-c)(a-d)}$.

Page 39.

1. $\frac{3}{4}$. **3.** $\frac{6}{7}$. **5.** $\frac{5}{7}$. **7.** $\frac{xy}{z^2}$. **9.** $\frac{2a}{3b}$.

2. $\frac{1}{3}$. **4.** $\frac{8}{9}$. **6.** $\frac{1}{14}$. **8.** $\frac{1}{2A^2}$. **10.** $\frac{1}{2R^2}$.

11. $\frac{x+1}{x^2+x+1}$. **14.** $\frac{t-7}{t+6}$. **17.** $\frac{w-v}{wv}$.

12. $\frac{a^2-ab+b^2}{a^2}$. **15.** $\frac{5p+q}{4p+5q}$. **18.** $\frac{F^4-F^3+F^2-F+1}{(F+1)^4}$.

13. $\frac{m^2-n^2}{m^2+n^2}$. **16.** $\frac{1-N+N^2}{N^2-5N}$. **19.** $\frac{1}{A^2-AB+B^2}$.

20. $\frac{a+x}{a-y}$.

Page 40.

1. $\frac{10}{12}, \frac{9}{12}, \frac{6}{12}$. **3.** $\frac{12}{42}, \frac{21}{42}, \frac{28}{42}$. **5.** $\frac{30}{225}, \frac{72}{225}, \frac{35}{225}$. **7.** $\frac{42}{84}, \frac{64}{84}, \frac{75}{84}$.

2. $\frac{8}{24}, \frac{6}{24}, \frac{9}{24}$. **4.** $\frac{20}{48}, \frac{9}{48}, \frac{14}{48}$. **6.** $\frac{27}{216}, \frac{30}{216}, \frac{160}{216}$. **8.** $\frac{405}{720}, \frac{160}{720}, \frac{192}{720}$.

9. $\frac{80a}{60a^3}, \frac{12a^2}{60a^3}, \frac{35}{60a^3}$. **12.** $\frac{1+x}{1-x^2}, \frac{1-x}{1-x^2}, \frac{x}{1-x^2}$.

10. $\frac{a^3c^2}{a^2b^2c^2}, \frac{a^2b^3}{a^2b^2c^2}, \frac{b^2c^3}{a^2b^2c^2}$. **13.** $\frac{a(a-b)(a^2+b^2)}{a^4-b^4}, \frac{b(a+b)(a^2+b^2)}{a^4-b^4}, \frac{ab(a+b)(a-b)}{a^4-b^4}$.

11. $\frac{m^2}{mnp}, \frac{p^2}{mnp}, \frac{n^2}{mnp}$. **14.** $\frac{M^3-MN^2}{M^2N-N^3}, \frac{-M^3+M^2N}{M^2N-N^3}, \frac{MN^2}{M^2N-N^3}$.

15. $\frac{t^2-2t}{(t+3)(t-3)(t-2)}, \frac{t^2+2t-3}{(t+3)(t-3)(t-2)}$.

16. $\frac{R^2-R-6}{(R-7)(R-2)(R+2)}, \frac{-3R+21}{(R-7)(R-2)(R+2)}.$

17. $\frac{m^2n+mn^2}{mn(m^3-n^3)}, \frac{2m^2n+2mn^2+2n^3}{mn(m^3-n^3)}.$

18. $\frac{6x+3}{(x+1)(x+2)(2x+1)}, \frac{5x+5}{(x+1)(x+2)(2x+1)}, \frac{7x+14}{(x+1)(x+2)(2x+1)}.$

19. $\frac{b-c}{(a-b)(a-c)(b-c)}, \frac{c-a}{(a-b)(a-c)(b-c)}, \frac{a-b}{(a-b)(a-c)(b-c)}.$

20. $\frac{y^2-z^2}{(x-y)(x-z)(y-z)}, \frac{z^2-x^2}{(x-y)(x-z)(y-z)}, \frac{x^2-y^2}{(x-y)(x-z)(y-z)}.$

Pages 41, 42.

1. $\frac{31}{20}.$
2. $\frac{17}{48}.$
3. $\frac{97}{168}.$
4. $\frac{257}{240}.$
5. $\frac{6+a}{2a^2}.$
6. $\frac{x-2}{2x^2}.$
7. $\frac{a^2+b^2+c^2}{abc}.$
8. $\frac{4N}{N^2-1}.$
9. $\frac{1+2t}{1-t^2}.$
10. $\frac{7s+3}{s^3-s}.$
11. $\frac{2r^2+14r-6}{(r+2)(r+7)(r+8)}.$
12. $\frac{4RH}{H^2-R^2}.$
13. $\frac{V^2+18}{V^2-9}.$
14. $\frac{2a^2+ab+b^2}{a^3-b^3}.$
15. $\frac{1}{(a-b)(a-c)}.$
16. 0.
17. $\frac{M^2P-M^2N+MN^2-PN^2+P^3-NP^2}{(M-N)(M-P)(P-N)}.$
18. $\frac{24r^3-8rR^2}{(r^2-R^2)(9r^2-R^2)}.$
19. $\frac{18n^2-360}{(n^2-16)(n^2-25)}.$
20. $\frac{4bc^2-4bd^2+8a^2d-8b^2d}{(a^2-b^2)(c^2-d^2)}.$
21. $\frac{16x^4-1476}{(x^4-81)(x^4-1)}.$
22. $\frac{t^2-1}{t}.$
23. $\frac{3V+6}{V+3}.$
24. $\frac{ab}{a-b}.$
25. $\frac{1}{1+R}.$
26. $\frac{m^2+n^2}{m+n}.$
27. $\frac{5+2y}{4-y}.$

28. $\frac{N^2+MN+M^2}{N^2}$.

29. $\frac{x^3}{x-y}$.

30. 0.

31. $\frac{2D^4-D^2+1}{D^2-D+1}$.

Pages 43, 44.

1. $\frac{3}{10}$.
2. $\frac{4}{7}$.
3. 4.
4. 8.
5. $\frac{1}{12}$.
6. $\frac{17}{32}$.
7. $\frac{19}{12}$.
8. $\frac{5}{8}$.
9. $\frac{ad}{b}$.
10. 1.
11. m^3n.
12. x^3-x^2+x-1.
13. $\frac{v^2-2t^2}{v^3+3v^2t+3vt^2+2t^3}$.
14. $\frac{n+1}{n+5}$.
15. $\frac{RH}{R^2-2RH+H^2}$.
16. b^2-2b+1.
17. $\frac{V^2+3VT+9T^2}{V^3}$.
18. $\frac{1}{A-1}$.
19. $a+b$.
20. $\frac{1-N^4}{N^4}$.
21. $\frac{1}{F^3-F^2}$.
22. $\frac{a+b}{a^2}$.
23. $\frac{mn^2-m^3}{n^3}$.
24. $\frac{h}{h-k}$.
25. $\frac{-2P^3+2P^2+P-1}{P^2+P^3}$.
26. $2n$.
27. B.
28. x^2-y^2.
29. $2K^2-2$.
30. $-v^2-2$.
31. $4M^2-11M+2$.

Page 45.

1. $\frac{7}{8}$.
2. $\frac{25}{24}$.
3. 20.
4. $\frac{1}{32}$.
5. 27.
6. $\frac{22}{5}$.
7. $\frac{8}{45}$.
8. $\frac{4}{15}$.
9. $\frac{2m^2}{15pn}$.
10. $\frac{2R}{3}$.
11. x^2-x.
12. $\frac{6-V-V^2}{2}$.
13. $\frac{a^2-ab+b^2}{a^2}$.
14. $\frac{t-1}{t}$.
15. $\frac{3m^3-m^2n-mn^2-4n^3}{12m^2+11mn-5n^2}$.
16. $\frac{P^3+P^2Q+PQ^2}{Q^2}$.
17. $\frac{b}{a^2-ab}$.
18. $\frac{x^2+2}{x^2+x-3}$.
19. $\frac{M^3-M^2N+MN^2}{M-N}$.
20. $\frac{t^2-tk}{t^2+tk+k^2}$.

21. $\dfrac{r_1r_2r_3}{r_1r_2+r_1r_3+r_2r_3}$.

Pages 46, 47.

1. $\frac{14}{9}$.
2. $\frac{48}{5}$.
3. $\frac{4}{5}$.
4. $\dfrac{N^2}{M^2}$.
5. $\dfrac{x^2}{y^2}$.
6. $\dfrac{x^2}{1+x}$.
7. $\dfrac{A-B}{A+B}$.
8. $\dfrac{n-m}{n+m}$.
9. $\dfrac{b-c}{c}$.
10. $\dfrac{-b}{a^2+2ab+b^2}$.
11. $t+s$.
12. $\dfrac{a^2+b^2}{a^2-b^2}$.
13. $\dfrac{mn}{m^2-mn+n^2}$.
14. $\dfrac{2}{x+1}$.
15. $\dfrac{a^2-a+1}{2a-1}$.
16. $\dfrac{3x+2}{4x+3}$.
17. $\dfrac{nE}{nR+r}$.
18. 2.98–.

Pages 47, 48.

1. $\dfrac{2}{t}$.
2. $\dfrac{3V}{2+2V}$.
3. -1.
4. $\dfrac{(x+y+z)^2}{x^2+xy-xz}$.
5. $\dfrac{b^4+5a^2b+5ab^2-25a^4}{5ab^2}$.
6. 0.
7. 1.
8. $\dfrac{a^3-5a^2n-5an^2+n^3+a^3n-2a^2n^2+an^3}{an(a-n)(a+n)}$.
9. $w+v$.
10. $\dfrac{a^2-x^2}{a^2+x^2}$.
11. 1.
12. $\dfrac{-1}{x}$.
13. $\dfrac{1}{a}$.
14. x.
15. $\dfrac{1}{m^2-n^2}$.
16. $\dfrac{a^6+b^6}{a^6+2a^3b^3+b^6}$.
17. $\dfrac{ab}{a^2+b^2}$.
18. 1.

Page 51.

1. $n=3$.
2. $t=10$.
3. $V=6$.
4. $x=1$.
5. $P=-2$.
6. $k=2$.
7. $N=8$.
8. $a=24$.
9. $x=\frac{12}{11}$.
10. $H=-\frac{12}{5}$.
11. $m=4\frac{68}{163}$.

Page 53.

1. $n=6$.
2. $V=1$.
3. $x=3$.
4. $A=\frac{1}{15}$.
5. $R=\frac{1}{4}$.
6. $t=1$.
7. $N=\frac{1}{2}$.
8. $y=-27$.
9. $P=-3$.
10. $a=1\frac{5}{8}$.
11. $r=-8$.
12. $x=6$.
13. $K=9$.
14. $b=10$.
15. $y=2\frac{10}{23}$.
16. $B=-\frac{1}{4}$.

Pages 54, 55.

1. $a=4$.
2. $x=2$.
3. $m=8$.
4. $t=5$.
5. $b=2$.
6. $V=-\frac{4}{5}$.
7. $R=1$.
8. $t=4$.
9. $a=-3$.
10. $k=-3\frac{9}{10}$.
11. $x=2\frac{1}{2}$.
12. $n=11$.
13. $W=-13$.
14. $x=0$.

Pages 55, 56.

1. $x=\dfrac{m^2-n^2}{5m}$.

2. $n=\dfrac{2P}{1-P}$.

3. $V=\dfrac{1+d}{2d}$.

4. $x=\dfrac{na-y}{n}$.

5. $A=\dfrac{BC}{B-C+BC}$.

6. $t=\dfrac{12a+12b}{3a+b}$.

7. $b=\dfrac{2A-hb'}{h}$.

8. $R=\dfrac{S-\pi Lr}{\pi L}$; $r=\dfrac{S-\pi LR}{\pi L}$; $L=\dfrac{S}{\pi(R+r)}$.

9. $L=\dfrac{S-\pi(R^2+r^2)}{\pi(R+r)}$.

10. $H=\dfrac{3V}{\pi(R^2+r^2+Rr)}$.

11. $F=\dfrac{Dd+df+Df}{f+D}$; $f=\dfrac{Dd-DF}{F-d-D}$; $D=\dfrac{Ff-df}{f-F+d}$; $d=\dfrac{Ff-Df+DF}{D+f}$.

12. $R=\dfrac{r_1r_2r_3r_4r_5}{r_2r_3r_4r_5+r_1r_3r_4r_5+r_1r_2r_4r_5+r_1r_2r_3r_5+r_1r_2r_3r_4}$; etc.

13. $K=\dfrac{H}{ND(D-1)(R+2)}$; $N=\dfrac{H}{KD(D-1)(R+2)}$;

$R=\dfrac{H-2KND(D-1)}{KND(D-1)}$.

14. $H=\dfrac{P(460+T)(460+t)}{7.6t-7.9T-138}$; $T=\dfrac{H(7.6t-138)-460P(460+t)}{460P+Pt+7.9H}$;

$$t=\frac{138H+211600P+460PT+7.9HT}{7.6H-PT-460P}.$$

15. $V=\dfrac{V'P'(273+t)}{P(273+t')}$; $V'=\dfrac{VP(273+t')}{P'(273+t)}$; $P=\dfrac{V'P'(273+t)}{V(273+t')}$;

$P'=\dfrac{VP(273+t')}{V'(273+t)}$; $t=\dfrac{273VP+VPt'-273V'P'}{V'P'}$:

$t'=\dfrac{273V'P'+V'P't-273VP}{VP}$.

Pages 57–61.

1. 24, 26. **5.** 18 nickels; 15 dimes.

2. $37\frac{1}{2}$ lb. of 20¢ grade; $62\frac{1}{2}$ lb. of 28¢ grade. **6.** \$3500 at 6%; \$6500 at 5%.

7. \$16,000 at 4%; \$9000 at $5\frac{1}{2}$%.

3. 3 lb. of 24¢ grade; 6 lb. of 30¢ grade. **8.** \$4000 at 4%; \$8000 at 8%.

9. $5\frac{5}{11}$ gal. cream; $54\frac{6}{11}$ gal. milk.

4. $6\frac{2}{3}$ lb. of 50¢ grade; $3\frac{1}{3}$ lb. of 80¢ grade. **10.** $17\frac{21}{47}$ gal. cream; $2\frac{26}{47}$ gal. milk.

11. 25 gal. **12.** $3\frac{3}{4}$ qt.

13. 2 lb. **16.** \$7.20. **19.** 6 hr.; 36 mi.

14. 30 lb. tin; 20 lb. zinc. **17.** \$2.50. **20.** 4 hr.

15. 24 yr. **18.** $26\frac{2}{3}$ in.; $21\frac{1}{3}$ in. **21.** 1650 ft.

22. $10\frac{10}{11}$ min. after 2. **24.** $10\frac{10}{11}$ min. after 8.

23. $49\frac{1}{11}$ min. after 3. **25.** $54\frac{6}{11}$ min. after 7.

26. $65\frac{5}{11}$ min. **27.** $586\frac{17}{28}$ days. **28.** Dec. 29, 1913. **29.** $115\frac{265}{277}$ days.

30. $398\frac{2281}{3968}$ days; April 3, 1910. **31.** $3\frac{3}{5}$ days. **32.** $5\frac{5}{13}$ days.

33. $13\frac{1}{3}$ hr. **35.** 25 mi. per hour. **37.** $1\frac{19}{21}$ mi. per hour.

34. $8\frac{2}{11}$ hr. **36.** 15 mi. per hour. **38.** $\dfrac{pqr}{pq+q-p}$.

Pages 62, 63.

1. $n=1$. **2.** $A=3$. **3.** $x=3\frac{1}{2}$. **4.** $x=7$.

5. $t=4$. **6.** $x=\dfrac{bc}{c^2-b}$. **7.** $n=\dfrac{5b}{3}$.

8. $V=\dfrac{2TE-2RTE}{3pR-5p}$; $p=\dfrac{2TE-2RTE}{3RV-5V}$; $E=\dfrac{5pV-3pRV}{2RT-2T}$;

$$T=\frac{5pV-3pRV}{2RE-2E}.$$

9. 7 hr. **10.** 82. **11.** 30 mi. per hour. **12.** 4.

13. 1:18 P.M. the next day; 1129.6 mi.

15. $\frac{19}{37}$. **17.** $\frac{4}{33}$. **19.** $\frac{41}{333}$. **21.** $\frac{16}{37}$. **23.** $\frac{2057}{3333}$.

16. $\frac{4}{55}$. **18.** $\frac{2}{33}$. **20.** $\frac{9}{11}$. **22.** $\frac{58}{111}$. **24.** $1\frac{2}{3}$.

Page 66.

12. $4\frac{4}{9}$° C.; $-11\frac{1}{9}$° C.; $-7\frac{7}{9}$° C.; $-23\frac{1}{3}$° C.; $-28\frac{8}{9}$° C.; $-20\frac{5}{9}$° C.; $93\frac{1}{5}$° F.; 59° F.; 104° F.; $46\frac{2}{5}$° F.; 14° F.

Page 67.

1. $x=6$, $y=2$. **4.** $x=2$, $y=3$. **7.** $x=9$, $y=2$. **10.** Parallel.

2. $x=4\frac{6}{13}$, $y=1\frac{5}{13}$. **5.** $x=3$, $y=-5$. **8.** $x=5$, $y=7$.

3. $x=3$, $y=2$. **6.** $x=2$, $y=3$. **9.** $x=17$, $y=13$.

Pages 69, 70.

1. $m=5$, $n=7$. **10.** $a=2$, $b=1$. **19.** $D=1\frac{57}{59}$, $T=\frac{5}{59}$.

2. $A=4$, $B=3$. **11.** $E=7$, $F=9$. **20.** $k=\frac{1}{3}$, $s=\frac{1}{4}$.

3. $x=5$, $y=1$. **12.** $m=15$, $n=8$. **21.** $x=-1$, $y=3$.

4. $v=6\frac{7}{13}$, $t=-1\frac{2}{13}$. **13.** $a=17$, $b=13$. **22.** $a=20$, $b=32$.

5. $R=-2$, $r=-3$. **14.** $w=9$, $v=2$. **23.** $a=-7\frac{1}{5}$, $b=-17\frac{3}{5}$.

6. $V=5$, $t=6$. **15.** $r=-11$, $r'=7$. **24.** $m=10$, $n=24$.

7. $p=13$, $q=17$. **16.** $F=1$, $C=-1\frac{1}{2}$. **25.** $x=a-b$, $y=b-a$.

8. $M=4$, $N=-5$. **17.** $A=12$, $B=-3$. **26.** $x=a$, $y=b$.

9. $x=2\frac{12}{17}$, $y=7\frac{2}{17}$. **18.** $y=5$, $z=7$. **27.** $x=a$, $y=-\frac{b}{a}$.

28. $x=\frac{m^2+n^2}{m+2n}$, $y=\frac{2m-n}{m+2n}$. **29.** $x=\frac{p^2}{p^2+q^2}$, $y=\frac{-1}{p^2+q^2}$.

30. $x=\frac{ce-bf}{ae-bd}$, $y=\frac{af-cd}{ae-bd}$.

31. $x=\frac{1}{13}$, $y=\frac{1}{17}$. **33.** $A=\frac{1}{3}$, $B=\frac{1}{7}$. **35.** $v=\frac{1}{4}$, $t=-\frac{1}{5}$.

32. $a=-\frac{1}{2}$, $b=-\frac{1}{3}$. **34.** $m=-1$, $n=\frac{1}{3}$. **36.** $M=\frac{1}{5}$, $N=\frac{1}{2}$.

Page 72.

1. $a=1$, $b=2$, $c=3$. **3.** $M=0$, $N=-4\frac{21}{89}$, $P=-\frac{80}{89}$.

2. $x=2$, $y=3$, $z=4$. **4.** $r=7\frac{8}{13}$, $s=3\frac{5}{13}$, $t=0$.

5. $A=2$, $B=0$, $C=0$.
6. $x=-7$, $y=8$, $z=1$.
7. $a=4$, $b=5$, $c=6$.
8. $x=1$, $y=2$, $z=3$, $w=4$.
9. $p=2\frac{10}{13}$, $q=\frac{2}{13}$, $r=-1\frac{11}{13}$, $s=5\frac{10}{13}$.
10. $E=\frac{1}{2}$, $G=0$, $H=4$, $K=-2$.
11. $a=\frac{1}{3}$, $b=\frac{1}{2}$, $c=1$.
12. $x=\frac{1}{12}$, $y=\frac{1}{7}$, $z=\frac{1}{5}$.
13. $x=a+b$, $y=a-b$, $z=2a$.
14. See § 60 for solution.

Page 74.

1. $x=5$, $y=2$.
2. All three meet at one point.
3. By multiplication by 3 and transposition.
4. They are identical.
6. They are parallel.

Pages 74–78.

1. A, $2400; B, $1200.
2. $6400 at 5%; $3600 at 8%.
3. $15,000 in bonds; $9000 in stock.
4. 30 lb. of 20¢ grade; 20 lb. of 30¢ grade.
5. 12 lb. of 40¢ grade; 8 lb. of 65¢ grade.
6. 35 5/9 gal. cream; 4 4/9 gal. milk.
7. 97 23/41 gal. milk; 2 18/41 gal. cream.
8. 46 2/13 qt. mixture; 13 11/13 qt. alcohol.
9. 90%; 80%.
10. Gold, 11 oz.; silver, 7 oz.
11. Gold, 15 1/8 lb.; silver, 4 7/8 lb.
12. Tin, 30 lb.; zinc, 15 lb.
13. Fahrenheit, 176°; Centigrade, 80°.
14. Fahrenheit, 662°; Centigrade, 350°.
15. 126 lb.; 90 lb.
16. 33 1/3 lb.; 26 2/3 lb.
17. 15 mi. per hour; 30 mi. per hour.
18. Passenger, 66 ft. per second; freight, 44 ft. per second.
19. Man, 40 yr.; son, 10 yr.
20. 20 hr.; 16 hr.
21. 30 hr.; 24 hr.
22. 9 in.; 12 in.; 15 in.
23. 12 in.; 16 in.; 22 in.
24. 4 in.; 8 in.; 12 in.
25. Sides are divided into 12 in. and 16 in., 12 in. and 24 in., 16 in. and 24 in.
26. 8 hr.; 9 hr.; 12 hr.

Pages 78, 79.

1. 26.
2. -35.
3. -1.
4. 36.
5. -10.
6. -10.
7. 16.
8. 10.
9. -30.
10. $ad-bc$.
11. $2x$.
12. a^2-1.

Page 80.

1. $x=1,\ y=2.$
2. $M=2,\ N=3.$
3. $p=3,\ q=1.$
4. $a=1,\ b=4.$
5. $R=5,\ T=4.$
6. $v=\frac{86}{89},\ t=1\frac{58}{89}.$
7. $x=\frac{29}{33},\ y=1\frac{5}{33}.$
8. $h=\frac{7}{9},\ k=1\frac{8}{9}.$
9. $P=1\frac{3}{11},\ Q=1\frac{6}{11}.$
10. $m=4,\ n=4.$
11. $a=-\frac{1}{12},\ V=-\frac{17}{24}.$
12. $m=\frac{1}{6},\ n=2\frac{2}{3}.$

Page 81.

1. $-22.$
2. $-107.$
3. 177.
4. 4.
5. 0.
6. 70.
7. 10.
8. 250.
9. 268.
10. $a-2b+c.$
11. $x^2+2yz-2xz-y^2.$
12. $1-n^2.$

Page 83.

1. $x=4,\ y=1,\ z=2,$
2. $A=5,\ B=2,\ C=1.$
3. $p=1,\ q=-1,\ r=4.$
4. $M=-2,\ N=3,\ P=-1.$
5. $a=10,\ b=-6,\ c=1.$
6. $r=\frac{22}{23},\ s=1\frac{57}{115},\ t=\frac{51}{115}$
7. $x=3,\ y=-4\frac{1}{2},\ z=5\frac{1}{2}.$
8. $m=\frac{9}{25},\ n=\frac{18}{25},\ k=\frac{28}{25}.$
9. $A=6,\ B=4,\ C=2.$
10. Indeterminate.

Pages 83, 84.

1. $a=1\frac{121}{139},\ b=7\frac{9}{73}.$
2. $m=\frac{1}{5},\ n=\frac{1}{2}.$
3. $x=2,\ y=3.$
4. $a=5,\ b=7.$
5. $m=\frac{1}{2},\ n=\frac{1}{2}.$
6. $A=7,\ B=9,\ C=6.$
7. $x=2,\ y=1,\ z=1.$
8. $\dfrac{P}{Q}=\dfrac{ab}{a'b'}.$
9. $V=\frac{1}{3}H(B+b+\sqrt{Bb}).$
10. $P_1=\dfrac{wd_2}{d_1+d_2},\ P_2=\dfrac{wd_1}{d_1+d_2}.$
11. $x=\dfrac{b+c}{2},\ y=\dfrac{a-c}{2},\ z=\dfrac{a-b}{2}.$

12. $x=\dfrac{bdr+cpf-bqf}{acf+bde},\quad y=\dfrac{pde-adr+aqf}{acf+bde},\quad z=\dfrac{acr-cpe+bqe}{acf+bde}.$

13. $x=\frac{2}{5},\ y=\frac{2}{3},\ z=2.$
14. $x=\dfrac{2}{a+b-c},\ y=\dfrac{2}{a-b+c},\ z=\dfrac{2}{-a+b+c}.$

Page 87.

1. 5.
2. 9.
3. n^3.
4. P^5.
5. $4ab^2$.
6. $6V^2t^3$.
7. $10m^3n^4$.
8. $11AB^6$.
9. $2p^3qr^2$.
10. $3x^2y^3z^4$.
11. 3.
12. 4.
13. A^2.
14. s^4.
15. $2xy^2$.
16. $3P^2Q^3$.
17. $5a^4b^5$.
18. $4W^3V^6$.
19. $2xy^2z^3$.
20. $3m^4np^7$.
21. 2.
22. 3.
23. a^2b^3.
24. $4xy^4$.
25. $5M^2N^3P$.
26. 2.
27. 3.
28. xy^3.
29. $2A^3B^4$.
30. $p^5q^3r^6s^4$.

Pages 88, 89.

1. $A-3$.
2. $2t+3$.
3. $1-5V$.
4. $7m^2+2f$.
5. $6a-4b$.
6. $8x+y$.
7. n^2-3n+2.
8. $1+5k+2k^2$.
9. t^3-5t-2.
10. $5R^3-4R^2+3$.
11. a^2-ab+b^2.
12. $6a+b-2c$.
13. $x-5y+3z$.
14. v^3-2v^2+5v-2.
15. $1-2x-2x^2-4x^3$.
16. $1+4y-8y^2+32y^3$.
17. $1-\frac{1}{2}a-\frac{1}{8}a^2-\frac{1}{16}a^3$.
18. $x-3$.
19. $m-2n$.

Pages 90, 91.

1. 47.
2. 92.
3. 362.
4. 806.
5. 1635.
6. 8.3.
7. 1.27.
8. 0.43.
9. 0.086.
10. 3.464.
11. 2.645.
12. 4.472.
13. 0.447.
14. 1.268.
15. 0.223.
16. 80 inches.
17. 11.49 − inches.
18. 21.95 + inches.
19. 4.37 + inches.
20. 29.39 + inches.
21. 10.39 + inches.
22. 94.10 + sq. in.
23. 322.49 + sq. in.

Page 92.

1. 3.2.
2. 3.9.
3. 4.5.
4. 5.3.
5. 6.3.
6. 7.2.
7. 7.7.
8. 2.5.
9. 1.5.
10. 3.5.
12. 2.3; 2.7; 3.6; 4.6; 5.3.

Pages 95, 96.

1. $2\sqrt{2}$.
2. $3\sqrt{3}$.
3. $2\sqrt{3}$.
4. $6\sqrt{2}$.
5. $2\sqrt[3]{2}$.
6. $2\sqrt[3]{3}$.
7. $2\sqrt[3]{9}$.
8. $5\sqrt[3]{2}$.
9. $2\sqrt[4]{2}$.
10. $3\sqrt[4]{2}$.
11. $2\sqrt[5]{2}$.
12. $2\sqrt[5]{7}$.
13. $\frac{1}{3}\sqrt{3}$.
14. $\frac{1}{5}\sqrt{10}$.
15. $\frac{2}{9}\sqrt{6}$.
16. $\frac{1}{2}\sqrt[3]{4}$.
17. $\frac{1}{2}\sqrt[3]{6}$.
18. $\frac{1}{7}\sqrt[3]{245}$.
19. $\frac{1}{2}\sqrt{2}$.
20. $\frac{1}{2}\sqrt[5]{6}$.
21. $ab\sqrt{a}$.
22. $m^2\sqrt{3mn}$.
23. $2xy^3\sqrt{2y}$.
24. $v\sqrt[3]{5vt}$.
25. $2Rr\sqrt[3]{2r}$.
26. $2B\sqrt[4]{2A^2B}$.
27. $4x^3y\sqrt{5}$.
28. $\sqrt{ab}$.
29. $b\sqrt[3]{a^2b}$.
30. $xy\sqrt{2x}$.
31. $wv^2\sqrt{3w}$.
32. $Q\sqrt{2PQ}$.
33. $z\sqrt[12]{16x^9y^6}$.
34. $\frac{1}{b}\sqrt{ab}$.
35. $\frac{1}{y^2}\sqrt{xy}$.
36. $\frac{1}{2n}\sqrt{6mn}$.
37. $\frac{1}{2x}\sqrt{2x}$.
38. $\frac{1}{5c^2}\sqrt[3]{25a^2bc^2}$.
39. $\frac{2x}{3y}\sqrt[3]{2x^2y}$.
40. $\frac{v}{7t}\sqrt{35vt}$.
41. $\frac{1}{a-b}\sqrt{a^2-b^2}$.
42. $\frac{m}{m+n}\sqrt{mn+n^2}$.
43. $\frac{1}{2A}\sqrt{B^2-4AC}$.
44. $\frac{1}{x^2y^2}\sqrt{x^5y+xy^5}$.
45. $\frac{9}{2x}\sqrt[3]{x^2y}$.
46. $\frac{a^3b}{6}\sqrt[5]{9ab^2}$.
47. $\frac{1}{K^3}\sqrt{K-K^2}$.
48. $2\sqrt{1-2\sqrt{3}}$.
49. $R\sqrt{1-3\sqrt{5}}$.
50. $(a-b)\sqrt{a+b}$.

Page 97.

1. $5\sqrt{2}$.
2. $2\sqrt{3}$.
3. $\sqrt{5}$.
4. $14\sqrt{6}$.
5. $\sqrt[3]{2}$.
6. $24\sqrt[3]{3}$.
7. $\frac{3}{4}\sqrt{2}$.
8. $\frac{1}{12}\sqrt{6}$.
9. $\frac{5}{6}\sqrt[3]{6}$.
10. $\frac{7}{2}\sqrt[3]{12}$.
11. $7\sqrt{2}$.
12. $3\sqrt{3}$.
13. $4\sqrt{n}$.
14. $20\sqrt[3]{t^2}$.
15. $a\sqrt{a}$.
16. $6m\sqrt{m-n}$.
17. $7m\sqrt{m-n}$.
18. $3A\sqrt[3]{A+B}$.
19. $\frac{2}{x}\sqrt{5xy}$.
20. $10\sqrt{p^2-q^2}$.

Page 98.

1. $\sqrt[12]{n^8}$, $\sqrt[12]{n^9}$, $\sqrt[12]{n^{10}}$. **6.** $\sqrt[6]{8}$, $\sqrt[6]{9}$.
2. $\sqrt[20]{V^{10}}$, $\sqrt[20]{V^{10}}$, $\sqrt[20]{V^{12}}$. **7.** $\sqrt[12]{256}$, $\sqrt[12]{216}$, $\sqrt[12]{64}$.
3. $\sqrt[6]{8x^3}$, $\sqrt[6]{9x^2}$, $\sqrt[6]{4x^4}$. **8.** $\sqrt[20]{729N^{18}}$, $\sqrt[20]{32N^{10}}$.
4. $\sqrt[12]{729t^{18}}$, $\sqrt[12]{16t^8}$, $\sqrt[12]{t^3}$. **9.** $\sqrt[12]{\frac{1}{64}}$, $\sqrt[12]{\frac{1}{81}}$, $\sqrt[12]{\frac{1}{64}}$.
5. $\sqrt[20]{a^{50}b^{30}}$, $\sqrt[20]{a^{16}b^4}$, $\sqrt[20]{a^5b^{15}}$. **10.** $\sqrt[12]{625y^8z^4}$, $\sqrt[12]{\frac{1}{64}y^6z^{12}}$, $\sqrt[12]{\frac{1}{27}y^9}$.
11. $\sqrt[3]{16}$. **12.** $\sqrt{5}$. **13.** $\sqrt{3}$. **14.** $\sqrt[4]{48}$.

Pages 99, 100.

1. $\sqrt{35}$. **3.** $8\sqrt{2}$. **5.** $7\sqrt[3]{3}$. **7.** $\sqrt[6]{360}$. **9.** $\sqrt{\frac{1}{2}}$.
2. 6. **4.** $3\sqrt[3]{2}$. **6.** $2\sqrt[4]{5}$. **8.** $\sqrt{\frac{1}{6}}$. **10.** 1.
11. $\frac{1}{8}$. **17.** $y^2\sqrt[6]{360}$. **23.** $28+17\sqrt{6}$.
12. $\sqrt[6]{72}$. **18.** $V\sqrt[12]{648V^7}$. **24.** $-1-\sqrt{10}$.
13. $x\sqrt[20]{x^3}$. **19.** $\sqrt{a^2-ab}$. **25.** $2+2\sqrt{3}$.
14. $n\sqrt[6]{200n}$. **20.** $\sqrt{m^2-n^2}$. **26.** $2\sqrt{6}$.
15. $t\sqrt[12]{1728t^4}$. **21.** $(x+y)\sqrt{x-y}$. **27.** $a+2b+3\sqrt{ab}$.
16. $2\sqrt[3]{3}$. **22.** 1.
28. $m\sqrt{mn}-n\sqrt{mn}-m\sqrt{n}+n\sqrt{m}$.
29. $x+\sqrt{xy}+y$. **30.** $n+14+8\sqrt{n-1}$. **31.** $a+b$.
32. $p+q+r+2\sqrt{pq}+2\sqrt{pr}+2\sqrt{qr}$. **33.** $\frac{1}{4}R^2\sqrt{2}$.
34. $\frac{1}{2}R\sqrt{2+\sqrt{2}}$. **37.** $\sqrt{\frac{1}{2}}$. **40.** $\sqrt{a^3b}$.
35. $\sqrt{12}$. **38.** $\sqrt[3]{16}$. **41.** $\sqrt[4]{2t^{13}}$.
36. $\sqrt{175}$. **39.** $\sqrt[3]{\frac{1}{9}}$. **42.** $\sqrt[3]{64m^3n^5}$.
43. $\sqrt{\frac{a}{b}}$. **44.** $\sqrt[3]{\frac{2v}{t^5}}$. **45.** $\sqrt[4]{\frac{c^2}{k}}$.

Pages 102, 103.

1. $\frac{2}{3}\sqrt{3}$. **3.** $\frac{4}{3}\sqrt{6}$. **5.** $\frac{8}{7}\sqrt{21}$. **7.** $\frac{1}{2}\sqrt{2}$. **9.** $2\sqrt[3]{9}$.
2. $\frac{1}{5}\sqrt{10}$. **4.** $2\sqrt{2}$. **6.** $\frac{2}{5}\sqrt{5}$. **8.** $\frac{5}{2}\sqrt[3]{4}$. **10.** $\frac{1}{2}\sqrt[3]{4}$.
11. $\frac{1}{n}\sqrt{mn}$. **14.** $\frac{1}{2y}\sqrt[3]{16y^2}$. **17.** $-\frac{5}{2}-\frac{5}{2}\sqrt{5}$.
12. $5\sqrt{2r}$. **15.** $\sqrt[6]{x}$. **18.** $2-\sqrt{2}$.
13. $3\sqrt{5V}$. **16.** $18\sqrt{3}-18$. **19.** $2+\sqrt{3}$.

20. $\frac{4}{5}+\frac{2}{5}\sqrt{10}-\frac{1}{5}\sqrt{6}-\frac{1}{5}\sqrt{15}$.

21. $\frac{14}{13}\sqrt{42}-\frac{87}{13}$.

22. $\left(\frac{1}{m-n}\right)(\sqrt{m}+\sqrt{n})$.

23. $\sqrt{a}-\sqrt{b}$.

24. $\sqrt{x^3}+\sqrt{y^3}$.

25. $\frac{1}{2}\sqrt{3}-\frac{1}{4}\sqrt{2}-\frac{1}{4}\sqrt{6}$.

26. $\frac{5}{59}\sqrt{7}-\frac{25}{59}\sqrt{5}+\frac{45}{59}\sqrt{3}+\frac{10}{59}\sqrt{105}$.

27. $\frac{1}{2}\sqrt{10}-\frac{1}{2}\sqrt{6}$.

28. $\dfrac{p^2-pq-pr-p\sqrt{pr}-q\sqrt{pr}+r\sqrt{pr}-p\sqrt{pq}+q\sqrt{pq}-r\sqrt{pq}+2p\sqrt{qr}}{p^2+q^2+r^2-2pq-2pr-2qr}$.

Page 104.

1. $3\sqrt{3}$.
2. 25.
3. $16\sqrt{2}$.
4. $16\sqrt[3]{4}$.
5. $405\sqrt[3]{5}$.
6. $x\sqrt[3]{x}$.
7. $N^2\sqrt[4]{N}$.
8. $t^7\sqrt[4]{t}$.
9. $8a^4b^4\sqrt{ab}$.
10. $36x^5$.
11. $10000m^{12}n^2\sqrt[5]{m^4n^2}$.
12. $243p^{17}q^{14}$.
13. $16(a^2-b^2)^2$.
14. $\sqrt[6]{5}$.
15. $\sqrt{2}$.
16. $\sqrt[3]{5}$.
17. 2.
18. $\sqrt[3]{2}$.
19. $\sqrt[4]{x}$.
20. $\sqrt[3]{5n^2}$.
21. $y\sqrt[4]{2x^3}$.
22. $7r^3s^4\sqrt{2}$.
23. $\sqrt[3]{a^2-b^2}$.
24. $\sqrt[p]{a}$.

Pages 105, 106.

1. $6i$.
2. $8i$.
3. $9i$.
4. $10i$.
5. $25i$.
6. $\sqrt{6}\sqrt{-1}$.
7. $2\sqrt{3}\sqrt{-1}$.
8. $2\sqrt{5}\sqrt{-1}$.
9. $\frac{1}{2}i$.
10. $\frac{1}{3}i$.
11. a^2bi.
12. $4mn^2i$.
13. $(a+b)i$.
14. $(3x-4y)i$.
15. $7i$.
16. $18i$.
17. $2i$.
18. $16i$.
19. $(\frac{5}{6}\sqrt{6}+1)i$.
20. $3a^2i$.
21. $3x+y\sqrt{-1}$.
22. $16+2i$.
23. $2m-5ni$.
24. $-4-23i$.
25. $(a-c)-(b+d)i$.

Pages 107, 108.

1. i.
2. 1.
3. $-i$.
4. i.
5. 1.
6. -1.
7. $-i$.
8. i.
9. -10.
10. -12.

11. -42.
12. -72.
13. -10.
14. $-ab$.
15. $-2\sqrt{15}$.
16. -30.
17. $-14i$.
18. $-18i$.
19. $-\sqrt{70}\sqrt{-1}$.
20. $-n^4\sqrt{n}\sqrt{-1}$.
21. $420i$.
22. -1080.
23. 3.
24. -13.
25. 3.
26. 1.
27. -169.
28. $b-a$.
29. $-x^3-y^3+2xy\sqrt{xy}$.
30. a^2+b^2.

Pages 108, 109.

1. $-5i$.
2. $-\frac{3}{2}i$.
3. $-5i$.
4. $-7i$.
5. $-\sqrt{n}\sqrt{-1}$.
6. 2.
7. $\frac{4}{5}$.
8. 2.
9. $\frac{1}{3}i$.
10. $\frac{5}{6}a$.
11. $\frac{5}{7}v$.
12. $\frac{3}{2x}$.
13. $\frac{1}{5}-\frac{3}{5}i$.
14. $\frac{1}{3}+\frac{1}{3}\sqrt{-3}+\frac{1}{3}\sqrt{-2}-\frac{1}{3}\sqrt{6}$.
15. $\frac{2}{7}-\frac{3}{7}\sqrt{-5}$.
16. $\frac{2}{9}-\frac{4}{9}\sqrt{3}+\frac{1}{9}\sqrt{-6}+\frac{4}{9}\sqrt{-2}$.
17. $\frac{1}{9}-\frac{2}{45}\sqrt{-5}$.
18. $\sqrt{10}+2\sqrt{15}+3\sqrt{6}+3$.
19. $\frac{x^2+2\sqrt{-y}-y}{x^2+y}$.
20. $\frac{a^2-b^2}{a^2+b^2}+\frac{2ab}{a^2+b^2}i$.

Pages 110, 111.

1. 0.632.
2. 0.654.
3. 0.674.
4. 0.606.
5. 0.912.
6. 0.547.
7. 0.806.
8. a^2b^3.
9. $x^3\sqrt{y}$.
10. A^2B.
11. $x^2\sqrt{x}$.
12. $R\sqrt[n]{1-3\sqrt{5}}$.
13. $\sqrt[n]{a^2}-2\sqrt[n]{ab}+\sqrt[n]{b^2}$.
14. Yes.
15. Yes.
16. Yes.
17. Yes.
19. $\sqrt{7}-\sqrt{3}$.
20. $\sqrt{5}+\sqrt{6}$.
21. $\sqrt{5}+3$.
22. $\sqrt{12}-1$.
23. $2+\sqrt{2}$.
24. $4-\sqrt{10}$.
25. $\sqrt{7}+\sqrt{11}$.
26. $\frac{6+3\sqrt{-3}}{7}$.
27. $\frac{\sqrt{3}-\sqrt{-1}}{2}$.
28. $\frac{1-4\sqrt{-3}}{7}$.
29. $\frac{3\sqrt{-3}+3\sqrt{2}}{-5}$.
30. $\frac{x^2-y^2+2xy\sqrt{-1}}{x^2+y^2}$.
31. $-\sqrt{-1}$.
32. $\frac{7}{4}+\frac{1}{4}\sqrt{-5}$.
33. $114+7\sqrt{-21}$.

Page 112.

1. $2x^2-10=0$; pure.
2. $4n^2+10n-2=0$; complete.
3. $12t^2-7t+21=0$; complete.
4. $3V^2-10V+12=0$; complete.
5. $4a^2-7a+6=0$; complete.
6. $y^2+y+1=0$; complete.
7. $2r^2-5r-3=0$; complete.
8. $5P^2-3=0$; pure.
9. $a^2+1=0$; pure.
10. $r^2-4=0$; pure.
11. $5T^2=0$; pure.

Pages 113, 114.

1. $n=1$ or 4.
2. $k=4$ or 8.
3. $P=2$ or -8.
4. $A=-1$ or $1\frac{1}{2}$.
5. $v=2$ or $-\frac{3}{4}$.
6. $t=\frac{1}{3}$ or $-3\frac{1}{2}$.
7. $k=-\frac{2}{3}$ or $3\frac{1}{2}$.
8. $N=-\frac{1}{4}$ or $1\frac{3}{5}$.
9. $x=5$ or -3.
10. $p=5$ or $-4\frac{1}{6}$.
11. $d=3$ or $-1\frac{2}{3}$.
12. $x=4$ or -1.
13. $n=5$ or $-5\frac{1}{4}$.
14. $R=6$ or $-1\frac{1}{5}$.
15. $y=4$ or $-\frac{1}{6}$.
16. $h=-1$ or -9.
17. $D=6$ or -1.
18. $a=\frac{1}{5}$ or $8\frac{1}{2}$.
19. $x=2$ or -2.
20. $g=4$ or 5.

Pages 114, 115.

1. $y=\pm 2$.
2. $F=\pm 5$.
3. $k=\pm 3$.
4. $A=\pm\frac{3}{4}$.
5. $p=\pm\frac{1}{3}\sqrt{21}$.
6. $T=\pm\frac{1}{4}\sqrt{-2}$.
7. $n=\pm\frac{1}{5}\sqrt{-30}$.
8. $G=\pm\frac{1}{2}\sqrt{-1}$.
9. $b=\pm\sqrt{3}$.
10. $R=\pm\frac{4}{5}\sqrt{-5}$.
11. $x=\pm 3$.
12. $N=\pm\frac{1}{2}\sqrt{58}$.
13. $a=\pm 4$.
14. $z=\pm 2\sqrt{5}$.
15. $A=\pm\sqrt{-23}$.
16. $n=\pm\frac{1}{2}\sqrt{13}$.
17. $t=\pm 1$.
18. $a=\pm 3\sqrt{-1}$.
19. $K=\pm 7$.
20. $m=\pm 6$.
21. $B=\pm\frac{1}{6}\sqrt{39}$.
22. $x=\pm 2.45$.
23. $M=\pm 1.32$.
24. $t=\pm 0.77$.
25. $t=\pm 0.45$.
26. $D=\pm 3.06$.
27. $s=\pm 1.41$.
28. $H=\pm 6.56$.

Pages 116, 117.

1. $n=2$ or -12.
2. $A=-3$ or $\frac{1}{4}$.
3. $x=-4$ or $1\frac{2}{3}$.
4. $v=8$ or $\frac{1}{6}$.
5. $T=-3$ or $\frac{5}{12}$.
6. $y=-2$ or $-\frac{2}{3}$.
7. $N=\frac{2}{3}$ or $-\frac{1}{2}$.
8. $k=1\frac{1}{2}$ or $-\frac{4}{5}$.
9. $D=\frac{6}{7}\pm\frac{1}{7}\sqrt{57}$.
10. $s=3$ or $-\frac{1}{4}$.
11. $z=2\frac{1}{2}$ or $\frac{2}{5}$.
12. $H=\frac{7}{12}\pm\frac{1}{12}\sqrt{13}$.
13. $t=1\frac{1}{2}$ or $\frac{2}{3}$.
14. $R=-1$ or $\frac{1}{2}$.
15. $V=7$ or $-1\frac{1}{3}$.
16. $m=\frac{1}{3}\pm\sqrt{3}$.
17. $G=1$ or $-\frac{5}{6}$.
18. $E=\frac{1}{3}\pm\frac{2}{3}\sqrt{-5}$.
19. $x=\frac{5}{4}\pm\frac{1}{4}\sqrt{3}$.
20. $n=2$ or $-1\frac{1}{3}$.
21. $v=-1\pm\frac{1}{2}\sqrt{-10}$.
22. $a=-15\pm\sqrt{298}$.
23. $m=-7$ or $-1\frac{1}{6}$.
24. $d=5$ or $1\frac{1}{2}$.
25. $y=1$ or $2\frac{2}{5}$.
26. $N=\frac{21}{5}\pm\frac{1}{5}\sqrt{21}$.
27. $t=4$ or $3\frac{3}{8}$.
28. $x=-2$ or $-1\frac{20}{29}$.
29. $a=-7.46$ or -0.54.
30. $V=-4.79$ or -0.21.
31. $n=1.28$ or -0.78.
32. $D=0.70$ or -1.20.

Pages 118, 119.

1. $Q=3$ or 5.
2. $z=9$ or -7.
3. $a=-4$ or -12.
4. $x=-1$ or $2\frac{1}{2}$.
5. $v=-6$ or $-\frac{1}{6}$.
6. $N=\frac{2}{3}$ or $-3\frac{1}{2}$.
7. $b=\frac{2}{3}$ or $-\frac{4}{5}$.
8. $y=1\frac{1}{3}$ or $\frac{5}{7}$.
9. $t=\frac{7}{2}\pm\frac{1}{2}\sqrt{41}$.
10. $B=-\frac{3}{16}\pm\frac{1}{16}\sqrt{161}$.
11. $n=3\pm\frac{1}{2}\sqrt{22}$.
12. $V=\frac{1}{40}\pm\frac{1}{40}\sqrt{-1039}$.
13. $P=2$ or $-1\frac{1}{3}$.
14. $k=\frac{1}{3}$ or -4.
15. $R=\frac{1}{18}\pm\frac{1}{18}\sqrt{217}$.
16. $x=\frac{3}{2}\pm\frac{1}{2}\sqrt{73}$.
17. $x=-8\pm8\sqrt{2}$.
18. $p=\frac{13}{6}\pm\frac{1}{6}\sqrt{13}$.
19. $n=-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$.
20. $n=\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$.
21. $v=0$ or 2.
22. $y=2\pm\sqrt{5}$.
23. $g=\frac{3}{4}\pm\frac{1}{4}\sqrt{-7}$.
24. $R=3$ or $-\frac{1}{3}$.
25. $F=3$ or $-6\frac{1}{2}$.
26. $b=6$ or -1.
27. $H=-\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$.
28. $n=0$ or -5.
29. $c=3$ or $\frac{3}{5}$.
30. $r=3$ or $\frac{3}{4}$.
31. $V=4$ or $-\frac{4}{7}$.
32. $n=3$ or -5.
33. $A=\frac{1}{2}$ or $-4\frac{3}{8}$.
34. $Q=4$ or -1.
35. $s=13$ or $-4\frac{1}{3}$.
36. $f=-5$ or -1.
37. $S=1.28$ or -0.78.
38. $v=0.86$ or -0.46.
39. $n=0.30$ or -1.13.
40. $x=5.16$ or -1.16.
41. $T=1.19$ or -1.69.

Pages 119, 120.

1. $v=\pm a.$
2. $x=2n$ or $-n$; $n=\frac{1}{2}x$ or $-x.$
3. $x=\frac{1}{2}a\pm\frac{1}{2}\sqrt{a^2-4b}.$
4. $v=-s\pm\sqrt{s^2-t}.$
5. $m=a$ or $b.$
6. $t=2m$ or $-n.$
7. $y=b$ or $b+2.$
8. $P=-1$ or $\frac{1}{n-1}.$
9. $n=2a\pm a\sqrt{2}.$
10. $R=-2m$ or $\frac{m-n}{2n}.$
11. $t=\frac{ac-bd\pm\sqrt{(a^2-b^2)(c^2-d^2)}}{ad-bc}.$
12. $r=\frac{-(mn+mp+np)\pm\sqrt{m^2n^2+m^2p^2+n^2p^2-m^2np-mn^2p-mnp^2}}{m+n+p}.$
13. $V=0$ or $n^3-n.$
14. $N=\frac{k^2\pm\sqrt{k^4+d^2f^2}}{f}.$
15. $E=\pm\sqrt{\frac{JWhR}{t}}.$
16. $C=\pm 5\sqrt{\frac{H}{6Rt}}.$
17. $L=\pm\frac{1}{2n}\sqrt{\frac{Eg}{w}}.$
18. $D=\pm d\sqrt{\frac{f}{F}}.$
19. $D=\frac{KN(R+2)\pm\sqrt{K^2N^2R^2+4K^2N^2R+4K^2N^2+4KNRH+8KNH}}{2KN(R+2)}.$
20. $t=\frac{-513\pm\sqrt{263169+4H}}{2}.$
21. $r=\pm\sqrt{\frac{\pi r'^2+A}{\pi}}$; $r'=\pm\sqrt{\frac{\pi r^2-A}{\pi}}.$
22. $R=\frac{-Hr\pm\sqrt{12HV-3H^2r^2}}{2H}$; $r=\frac{-HR\pm\sqrt{12HV-3H^2R^2}}{2H}.$

Pages 121, 124.

1. 12⅓; 17⅔.
2. 10 in.; 8 in.
3. 16 in.; 12 in.
4. 20 in.; 24 in.
5. 768 sq. in.
6. 20 in.
7. 20 in. by 40 in.
8. 6 in.; 4 in.
9. 30 rd.; 60 rd.
10. 24 ft.; 18 ft.
11. $PC=12.2-.$
12. 3 mi. 4712 ft.
13. 5.414 in.; 2.586 in.
14. 0.546 ft.
15. 7.64 rd.
16. 45½ ft.
17. 12 in.
18. 1200 sq. rd.
19. 1500 bu.
20. 140 acres.
21. 14.
22. 100 ft.
23. 35 ft.
24. 6 in.; 8 in.

25. 24 min.; 18 min.
26. $15\frac{5}{8}$ min.
27. 45 mi. per hour; 30 mi. per hour.
28. $2\frac{1}{2}$ hr.
29. 12.
30. 1 mi. per hour; 2 mi. per hour.
31. 64 sq. in.
32. 4 in.
33. 14 in.
34. 582.74– sq. rd.
35. 1 ft.

Page 125.

1. Real, unequal, and free of surds.
2. Real, unequal, and involve surds.
3. Imaginary.
4. Real, unequal, and free of surds.
5. Imaginary.
6. Imaginary.
7. Imaginary.
8. Real, unequal, and free of surds.
9. Real and equal.
10. Imaginary.
11. Real, unequal, and involve surds.
12. Real and equal.
13. Real, unequal, and involve surds.
14. Real, unequal, and involve surds.
16. $c=-2\frac{1}{4}$.
17. $c=\pm 20$.
18. $c=\pm 16$.
19. $c=\frac{1}{3}$.
20. $c=8$.
21. $c=5\frac{1}{8}$.
22. $4mn=25$.
23. $n^2=16m$.
24. $m^2=4n$.
25. $m^2=8n$.

Pages 126, 127.

1. $x^2-8x+12=0$.
2. $x^2-10x+9=0$.
3. $x^2+8x+15=0$.
4. $x^2-6x-16=0$.
5. $x^2+x-42=0$.
6. $x^2+14x+24=0$.
7. $x^2-5x-50=0$.
8. $x^2+23x+60=0$.
9. $x^2+2x-99=0$.
10. $6x^2-5x+1=0$.
11. $25x^2-15x-4=0$.
12. $24x^2-x-3=0$.
13. $24x^2+26x+5=0$.
14. $x^2-4ax+3a^2=0$.
15. $x^2+2nx-5mx-10mn=0$.
16. $x^2-2x-4=0$.
17. $x^2+8x+13=0$.
18. $x^2-2x+7=0$.
19. $x^2+6x+13=0$.
20. $n=2$ or -3.
21. $t=\frac{5}{2}\pm\frac{1}{2}\sqrt{-3}$.
22. $a=1\frac{2}{3}$ or -1.
23. $V=\frac{1}{10}\pm\frac{1}{10}\sqrt{-19}$.
24. $N=4$ or -1.
25. $r=\frac{5}{4}\pm\frac{5}{4}\sqrt{-7}$.
26. $A=-\frac{1}{2}\pm\frac{1}{2}\sqrt{37}$.
27. $x=\frac{1}{2}a\pm\frac{1}{2}\sqrt{a^2-12b}$.

Pages 128, 129.

1. $x=5$ or 7.

2. $x=8$ or -9.

3. $x=-6$ or -8.

4. $x=8$ or $3\frac{1}{2}$.

5. $x=2$ or $-6\frac{2}{3}$.

6. $x=-5$ or $4\frac{2}{3}$.

7. Real, unequal, and free of surds. Graph meets x-axis at two distinct points.

8. Same as Ex. 7.

9. Real and equal. Graph touches x-axis at one point.

10. Imaginary. Graph does not meet x-axis.

11. Same as Ex. 10.

12. Same as Ex. 9.

Pages 129, 130.

1. Real and free of surds.

2. Real and free of surds.

3. Imaginary.

4. Imaginary.

5. Real and free of surds.

6. Real, involving surds.

7. Imaginary.

8. Real, involving surds.

9. Imaginary.

10. Real and free of surds.

12. $2\left(S-\frac{1}{4}-\frac{1}{4}\sqrt{-15}\right)\left(S-\frac{1}{4}+\frac{1}{4}\sqrt{-15}\right)$.

13. $5\left(P+\frac{4}{5}-\frac{2}{5}\sqrt{-1}\right)\left(P+\frac{4}{5}+\frac{2}{5}\sqrt{-1}\right)$.

14. $3\left(b-\frac{1}{6}-\frac{1}{6}\sqrt{13}\right)\left(b-\frac{1}{6}+\frac{1}{6}\sqrt{13}\right)$.

15. $9\left(d+\frac{2}{9}-\frac{1}{9}\sqrt{-5}\right)\left(d+\frac{2}{9}+\frac{1}{9}\sqrt{-5}\right)$.

16. $(n-11)(n+13)$.

17. $(2y-1)(5y+8)$.

18. $7\left(A-\frac{1}{14}-\frac{1}{14}\sqrt{-447}\right)\left(A-\frac{1}{14}+\frac{1}{14}\sqrt{-447}\right)$.

19. $\left(T+1-\sqrt{-14}\right)\left(T+1+\sqrt{-14}\right)$.

Pages 132, 133.

1. $n=1,\ -\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$.

2. $a=1,\ -1,\ \frac{1}{2}\pm\frac{1}{2}\sqrt{-3},$
$-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$.

3. $V=3,\ -\frac{3}{2}\pm\frac{3}{2}\sqrt{-3}$.

4. $x=1,\ -1,\ \pm\sqrt{-1}$.

5. $y=3,\ -3,\ \pm3\sqrt{-1}$.

6. $t=1,\ -1,\ 2,\ -2$.

7. $k=2,\ -2,\ \pm2\sqrt{-1}$.

8. $N=\pm\sqrt{5},\ \pm\sqrt{-3}$.

9. $s=1\frac{1}{2},\ -\frac{3}{4}\pm\frac{3}{4}\sqrt{-3}$.

10. $A=-\frac{5}{4},\ \frac{5}{8}\pm\frac{5}{8}\sqrt{-3}$.

11. $g=2,\ -2,\ \pm\frac{1}{2}\sqrt{-6}$.

12. $m=1,\ -2,\ 1\pm\sqrt{-8},$
$-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$.

13. $E=\pm\frac{1}{2}\sqrt{6},\ \pm\frac{1}{2}\sqrt{14}$.

14. $c=\pm\frac{2}{3}\sqrt{-3},\ \pm\frac{1}{3}\sqrt{-6}$.

15. $x=1,\ -1,\ -2$.

16. $z=1,\ 2,\ -2$.

17. $T=3,\ \pm\sqrt{-1}$.

18. $a=-5,\ +2,\ -2$.

19. $R=-4,\ -1,\ \frac{1}{2}\pm\frac{1}{2}\sqrt{-3}.$

20. $h=1,\ \pm\sqrt{-1},\ -\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}.$

21. $P=0,\ -2,\ \frac{5}{6}\pm\frac{1}{6}\sqrt{-119}.$

22. $n=0,\ 1,\ 1,\ 1.$

23. $4,\ -2,\ \pm2\sqrt{-3}.$

24. $3,\ -3,\ \pm3\sqrt{-1}.$

25. $1,\ -1,\ \frac{1}{2}\pm\frac{1}{2}\sqrt{-3},\ -\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}.$

26. $2,\ -2,\ 1\pm\sqrt{-3},\ -1\pm\sqrt{-3}.$

Pages 134, 135.

1. $x=\pm\sqrt{6},\ \pm2\sqrt{-2}.$

2. $a=\pm\frac{1}{2}\sqrt{6},\ \pm\frac{1}{3}\sqrt{-15}.$

3. $V=\pm\frac{1}{3}\sqrt{6},\ \pm\frac{1}{4}\sqrt{-10}.$

4. $n=-1,\ -2,\ 1\pm\sqrt{-3},\ \frac{1}{2}\pm\frac{1}{2}\sqrt{-3}.$

5. $p=2,\ -1\frac{1}{2},\ -1\pm\sqrt{-3},$
$\frac{3}{4}\pm\frac{3}{4}\sqrt{-3}.$

6. $y=1,\ 1,\ -1,\ -1,\ \pm\sqrt{-1},$
$\pm\sqrt{-1}.$

7. $x=2,\ -2,\ 3,\ -3.$

8. $t=-1,\ -1,\ 1,\ -3.$

9. $n=-1,\ 2,\ -2,\ 3.$

10. $P=-\frac{7}{4}\pm\frac{1}{4}\sqrt{89},\ -\frac{7}{4}\pm\frac{1}{4}\sqrt{33}.$

11. $x=-\frac{1}{2}\pm\frac{1}{2}\sqrt{5},\ -\frac{1}{4}\pm\frac{1}{4}\sqrt{17}.$

12. $m=\frac{1}{4}\pm\frac{1}{4}\sqrt{-7},\ \frac{1}{4}\pm\frac{1}{4}\sqrt{-7}.$

13. $d=1\pm\sqrt{2},\ 1\pm\sqrt{2}.$

14. $A=\pm1,\ \pm\sqrt{-3}.$

15. $c=1,\ -2,\ -\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}.$

16. $H=0,\ 0,\ -3,\ -3.$

17. $n=\frac{5}{2}\pm\frac{1}{2}\sqrt{17},\ \frac{5}{2}\pm\frac{1}{2}\sqrt{23}.$

18. $v=-\frac{3}{2}\pm\frac{1}{2}\sqrt{5},\ \frac{3}{2}\pm\sqrt{3}.$

19. $R=\frac{1}{2}\pm\frac{3}{2}\sqrt{-3},\ \frac{1}{2}\pm\frac{1}{2}\sqrt{-7}.$

20. $b=\frac{3}{2}\pm\frac{1}{2}\sqrt{-11},\ \frac{3}{2}\pm\frac{1}{2}\sqrt{17}.$

21. $W=0,\ 0,\ -5,\ -5.$

22. $x=2,\ 4,\ -\frac{7}{2}\pm\frac{1}{2}\sqrt{17}.$

23. $n=3\pm\sqrt{14},\ -\frac{5}{2}\pm\frac{3}{2}\sqrt{5}.$

24. $a=1,\ 1,\ -\frac{3}{2}\pm\frac{1}{2}\sqrt{5}.$

25. $y=1,\ -1,\ -\frac{3}{2}\pm\frac{1}{2}\sqrt{13}.$

26. $D=1,\ 1,\ \frac{1}{2}\pm\frac{1}{2}\sqrt{-3}.$

Page 136.

2. $x=3,\ 5,\ 7.$

3. $x=-1,\ 6,\ -5.$

4. $x=2,\ -2,\ 5,\ -5.$

5. $x=-2,\ 4,\ 8,\ -8.$

6. One real; two imaginary.

7. All real; two equal.

Pages 140, 141.

1. $x=3.$

2. $y=5.$

3. $t=61.$

4. $x=14.$

5. $n=9.$

6. $a=-5.$

7. $v=14.$

8. $m=4.$

9. $R=7.$

10. $k=7$.
11. $D=-2$.
12. $x=-8, 40$.
13. $M=-4$.
14. $t=5, \frac{5}{49}$.
15. $A=144$.
16. $d=16$.
17. $s=9$.
18. $n=-2\frac{2}{5}$.
19. $n=2, 3$.
20. Impossible.
21. $h=4$.
22. Impossible.
23. Impossible.
24. Impossible.
25. $a=1$.
26. Impossible.
27. $p=8$.
28. $a=-\frac{10}{13}$.
29. $x=\frac{13}{6}\pm\frac{1}{6}\sqrt{13}$.
30. $t=0, -1$.
31. $y=0, \pm\sqrt{-1}$.
32. $x=4$.
33. $x=1\frac{1}{3}$.
34. $n=6\frac{3}{4}$.
35. $a=\frac{1}{3}$.
36. $s=9$.
37. $k=1$.
38. $R=1$.
39. $n=\frac{4}{5}$.
40. $a=16$.
41. $A=0, -12$.
42. $N=2$.
43. $r=3$.
44. $x=3, -1\frac{2}{3}$.
45. $n=0, -3$.
46. $t=3, -\frac{1}{2}$.
47. $r=5, -4\frac{2}{3}$.
48. $a=4, 20$.
49. Impossible.
50. $l=\frac{t^2g}{\pi^2}$.
51. $s=\frac{v^2}{2g}$.
52. $t=\frac{n^2T}{N^2}$; $T=\frac{N^2t}{n^2}$.
53. $s=ut-\frac{1}{2}gt^2$.

Page 142.

1. $x=1$.
2. $x=5$.
3. $x=6$.
4. $x=4$.

Pages 144, 145.

1. $r=2, s=1$; $r=-1, s=-2$.
2. $h=4, k=2$; $h=2, k=4$.
3. $a=5, b=1$; $a=-7, b=-5$.
4. $x=1, y=3$; $x=2\frac{1}{2}, y=-3$.
5. $m=7, n=2$; $m=2, n=7$.
6. $x=4, y=2$; $x=3\frac{1}{7}, y=-\frac{4}{7}$.
7. $r=10, R=-1$; $r=-4, R=2\frac{1}{2}$.
8. $t=-\frac{1}{2}\pm\frac{1}{6}\sqrt{-951}$,
   $v=\frac{3}{4}\pm\frac{1}{4}\sqrt{-951}$.
9. $N=2, P=\frac{1}{2}$; $N=\frac{1}{2}, P=2$.
10. $a=5, b=-5$; $a=0, b=-10$.
11. $T=15, V=10$;
    $T=-16\frac{1}{3}, V=-10\frac{7}{9}$.
12. $x=-4, y=-3$; $x=\frac{2}{3}, y=-1\frac{4}{9}$.
13. $m=2+8\sqrt{-1}, n=3-8\sqrt{-1}$;
    $m=2-8\sqrt{-1}, n=3+8\sqrt{-1}$.
14. $k=7, n=4$; $k=-4, n=-7$.
15. $x=2, y=2$; $x=2, y=2$.
16. $A=1\frac{1}{3}, B=\frac{2}{3}$; $A=\frac{1}{2}, B=1\frac{1}{2}$.
17. $p=8, q=-11$; $p=11, q=-8$.
18. $t=4, u=12$; $t=-5\frac{1}{7}, u=-1\frac{5}{7}$.
19. $P=4, Q=5$; $P=5, Q=4$.

20. $x=\frac{1}{2}a+\frac{1}{2}\sqrt{5a^2+4b^2}$,
$y=-\frac{1}{2}a+\frac{1}{2}\sqrt{5a^2+4b^2}$;
$x=\frac{1}{2}a-\frac{1}{2}\sqrt{5a^2+4b^2}$,
$y=-\frac{1}{2}a-\frac{1}{2}\sqrt{5a^2+4b^2}$.

21. $x=\frac{1}{2}m+\frac{1}{2}n$, $y=\frac{1}{2}m-\frac{1}{2}n$;
$x=\frac{1}{2}m-\frac{1}{2}n$, $y=\frac{1}{2}m+\frac{1}{2}n$.

22. $x=\frac{1}{2}h+\frac{1}{2}\sqrt{h^2+4t^2}$,
$y=-\frac{1}{2}h+\frac{1}{2}\sqrt{h^2+4t^2}$;
$x=\frac{1}{2}h-\frac{1}{2}\sqrt{h^2+4t^2}$,
$y=-\frac{1}{2}h-\frac{1}{2}\sqrt{h^2+4t^2}$.

23. $x=\frac{1}{2}a+\frac{1}{2}\sqrt{2b-a^2}$,
$y=\frac{1}{2}a-\frac{1}{2}\sqrt{2b-a^2}$;
$x=\frac{1}{2}a-\frac{1}{2}\sqrt{2b-a^2}$,
$y=\frac{1}{2}a+\frac{1}{2}\sqrt{2b-a^2}$.

24. $c^2=4\pi S$.

Pages 146, 147.

1. $x=4, k=1$; $x=4, k=-1$; $x=-4, k=1$; $x=-4, k=-1$.
2. $A=\sqrt{2}$, $B=\sqrt{3}$; $A=\sqrt{2}$, $B=-\sqrt{3}$; $A=-\sqrt{2}$, $B=\sqrt{3}$; $A=-\sqrt{2}$, $B=-\sqrt{3}$.
3. $r=4, s=3$; $r=4, s=-3$; $r=-4, s=3$; $r=-4, s=-3$.
4. $m=\frac{8}{5}\sqrt{105}$, $n=\frac{8}{5}\sqrt{-5}$; $m=\frac{8}{5}\sqrt{105}$, $n=-\frac{8}{5}\sqrt{-5}$; $m=-\frac{8}{5}\sqrt{105}$, $n=\frac{8}{5}\sqrt{-5}$; $m=-\frac{8}{5}\sqrt{105}$, $n=-\frac{8}{5}\sqrt{-5}$.
5. $W=\frac{6}{13}\sqrt{65}$, $V=\frac{2}{13}\sqrt{91}$; $W=\frac{6}{13}\sqrt{65}$, $V=-\frac{2}{13}\sqrt{91}$; $W=-\frac{6}{13}\sqrt{65}$, $V=\frac{2}{13}\sqrt{91}$; $W=-\frac{6}{13}\sqrt{65}$, $V=-\frac{2}{13}\sqrt{91}$.
6. $x=5, y=3$; $x=5, y=-3$; $x=-5, y=3$; $x=-5, y=-3$.
7. $v=6, t=1$; $v=6, t=-1$; $v=-6, t=1$; $v=-6, t=-1$.
8. $R=5, D=2$; $R=5, D=-2$; $R=-5, D=2$; $R=-5, D=-2$.
9. $a=\frac{1}{2}\sqrt{2}$, $b=\frac{1}{3}\sqrt{2}$; $a=\frac{1}{2}\sqrt{2}$, $b=-\frac{1}{3}\sqrt{2}$; $a=-\frac{1}{2}\sqrt{2}$, $b=\frac{1}{3}\sqrt{2}$; $a=-\frac{1}{2}\sqrt{2}$, $b=-\frac{1}{3}\sqrt{2}$.
10. $P=2, Q=3$; $P=2, Q=-3$; $P=-2, Q=3$; $P=-2, Q=-3$.
11. $x=\frac{2}{3}, y=5$; $x=\frac{2}{3}, y=-5$; $x=-\frac{2}{3}, y=5$; $x=-\frac{2}{3}, y=-5$.
12. $E=6, R=\frac{4}{5}$; $E=6, R=-\frac{4}{5}$; $E=-6, R=\frac{4}{5}$; $E=-6, R=-\frac{4}{5}$.
13. $C=6, y=9$; $C=6, y=-9$; $C=-6, y=9$; $C=-6, y=-9$.
14. $r=4, r'=2$; $r=4, r'=-2$; $r=-4, r'=2$; $r=-4, r'=-2$.
15. $x=\frac{1}{2}, y=\frac{1}{4}$; $x=\frac{1}{2}, y=-\frac{1}{4}$; $x=-\frac{1}{2}, y=\frac{1}{4}$; $x=-\frac{1}{2}, y=-\frac{1}{4}$.

Pages 148, 149.

1. $x=2$, $y=1$; $x=-2$, $y=-1$; $x=\frac{3}{2}\sqrt{2}$, $y=\frac{1}{2}\sqrt{2}$; $x=-\frac{3}{2}\sqrt{2}$, $y=-\frac{1}{2}\sqrt{2}$.

2. $a=6$, $b=3$; $a=-6$, $b=-3$; $a=\frac{9}{4}\sqrt{6}$, $b=\frac{3}{4}\sqrt{6}$; $a=-\frac{9}{4}\sqrt{6}$, $b=-\frac{3}{4}\sqrt{6}$.

3. $m=2$, $n=-6$; $m=-1\frac{7}{9}$, $n=5\frac{1}{3}$; $m=-1+\sqrt{33}$, $n=-1+\sqrt{33}$;
$m=-1-\sqrt{33}$, $n=-1-\sqrt{33}$.

4. $v=-5+\sqrt{91}$, $t=\frac{5}{2}-\frac{1}{2}\sqrt{91}$; $v=-5-\sqrt{91}$, $t=\frac{5}{2}+\frac{1}{2}\sqrt{91}$; $v=$
$-\frac{19}{8}+\frac{1}{8}\sqrt{3001}$, $t=\frac{19}{40}-\frac{1}{40}\sqrt{3001}$; $v=-\frac{19}{8}-\frac{1}{8}\sqrt{3001}$, $t=\frac{19}{40}+\frac{1}{40}\sqrt{3001}$.

5. $x=4$, $y=4$; $x=-4$, $y=-4$; $x=4\sqrt{3}$, $y=-4\sqrt{3}$; $x=-4\sqrt{3}$, $y=4\sqrt{3}$.

6. $V=0$, $g=\sqrt{19}$; $V=0$, $g=-\sqrt{19}$; $V=3$, $g=-2$; $V=-3$, $g=2$.

7. $P=\sqrt{5}$, $Q=0$; $P=-\sqrt{5}$, $Q=0$; $P=5$, $Q=-3$; $P=-5$, $Q=3$.

8. $a=0$, $b=3\sqrt{3}$; $a=0$, $b=-3\sqrt{3}$; $a=3$, $b=-3$; $a=-3$, $b=3$.

Page 150.

1. $x=3$, $y=4$; $x=-3$, $y=-4$; $x=\frac{7}{3}\sqrt{3}$, $y=\frac{2}{3}\sqrt{3}$; $x=-\frac{7}{3}\sqrt{3}$, $y=-\frac{2}{3}\sqrt{3}$.

2. $m=2$, $n=4$; $m=-2$, $n=-4$; $m=\sqrt{2}$, $n=3\sqrt{2}$; $m=-\sqrt{2}$, $n=-3\sqrt{2}$.

3. $A=1$, $B=2$; $A=-1$, $B=-2$; $A=\sqrt{3}$, $B=0$; $A=-\sqrt{3}$, $B=0$.

4. $h=3$, $k=1$; $h=-3$, $k=-1$; $h=\frac{4}{5}\sqrt{10}$, $k=-\frac{8}{5}\sqrt{10}$; $h=-\frac{4}{5}\sqrt{10}$,
$k=\frac{8}{5}\sqrt{10}$.

5. $v=2$, $w=-2$; $v=-2$, $w=2$; $v=\frac{4}{7}\sqrt{7}$, $w=-\frac{6}{7}\sqrt{7}$; $v=-\frac{4}{7}\sqrt{7}$, $w=\frac{6}{7}\sqrt{7}$.

6. $R=5$, $t=-2$; $R=-5$, $t=2$; $R=\frac{17}{5}\sqrt{-5}$, $t=\frac{7}{5}\sqrt{-5}$; $R=-\frac{17}{5}\sqrt{-5}$,
$t=-\frac{7}{5}\sqrt{-5}$.

7. $x=1$, $y=5$; $x=-1$, $y=-5$; $x=14$, $y=-8$; $x=-14$, $y=8$.

8. $M=2$, $N=5$; $M=-2$, $N=-5$; $M=4\sqrt{3}$, $N=-\sqrt{3}$; $M=-4\sqrt{3}$,
$N=\sqrt{3}$.

9. $a=2$, $b=5$; $a=-2$, $b=-5$; $a=4\sqrt{2}$, $b=3\sqrt{2}$; $a=-4\sqrt{2}$, $b=-3\sqrt{2}$.

10. $p=8$, $q=-5$; $p=-3$, $q=-5$; $p=3$, $q=5$; $p=-8$, $q=5$.

11. $F=5$, $C=4$; $F=-5$, $C=-4$; $F=\sqrt{3}$, $C=3\sqrt{3}$; $F=-\sqrt{3}$, $C=-3\sqrt{3}$.

12. $v=1$, $g=5$; $v=-1$, $g=-5$; $v=\frac{3}{5}\sqrt{-10}$, $g=-\frac{8}{5}\sqrt{-10}$; $v=-\frac{3}{5}\sqrt{-10}$,
$g=\frac{8}{5}\sqrt{-10}$.

Page 152.

1. $x=-1, y=2;\ x=2, y=-1.$ **2.** $p=-2, q=5;\ p=5, q=-2.$

3. $a=6, b=1;\ a=-6, b=-1;\ a=1, b=6;\ a=-1, b=-6.$

4. $m=1, n=6;\ m=6, n=1.$

5. $P=3, Q=2;\ P=2, Q=3;\ P=-2, Q=-3;\ P=-3, Q=-2.$

6. $w=6, v=2;\ w=-2, v=-6.$ **7.** $x=5, y=-3;\ x=3, y=-5.$

8. $x=5, y=-1;\ x=-5, y=1;\ x=1, y=-5;\ x=-1, y=5.$

9. $A=1, B=-3;\ A=-1, B=3;\ A=3, B=-1;\ A=-3, B=1.$

10. $m=3, n=2;\ m=-2, n=-3.$ **11.** $s=3, t=1;\ s=1, t=3.$

12. $x=\frac{1}{2}\sqrt{5}+\frac{1}{2}, y=\frac{1}{2}\sqrt{5}-\frac{1}{2};\ x=\frac{1}{2}\sqrt{5}-\frac{1}{2}, y=\frac{1}{2}\sqrt{5}+\frac{1}{2};\ x=-\frac{1}{2}\sqrt{5}+\frac{1}{2},$
$y=-\frac{1}{2}\sqrt{5}-\frac{1}{2};\ x=-\frac{1}{2}\sqrt{5}-\frac{1}{2}, y=-\frac{1}{2}\sqrt{5}+\frac{1}{2}.$

13. $a=4, b=5;\ a=5, b=4;\ a=\sqrt{-14}-5, b=-\sqrt{-14}-5;$
$a=-\sqrt{-14}-5, b=\sqrt{-14}-5.$

14. $W=6, V=9;\ W=-9, V=-6.$

15. $r=1, r'=5;\ r=-1, r'=-5;\ r=-5, r'=-1;\ r=5, r'=1.$

16. $x=5, y=3;\ x=-5, y=-3;\ x=3, y=5;\ x=-3, y=-5.$

Page 154.

1. $m=2, n=3;\ m=3, n=2.$

2. $v=3, t=0;\ v=0, t=-3.$

3. $a=5, b=-2;\ a=2, b=-5.$

4. $x=4, y=1;\ x=1, y=4.$

5. $P=1, Q=3;\ P=3, Q=1.$

6. $a=1, b=3;\ a=3, b=1.$

7. $w=4, v=2;\ w=-2, v=-4.$

8. $M=3, N=2;\ M=-2, N=-3.$

9. $x=3, y=3;\ x=-3, y=-3;\ x=3, y=3;\ x=-3, y=-3.$

10. $r=3, s=1;\ r=-3, s=-1;\ r=1, s=3;\ r=-1, s=-3.$

11. $a=2, b=3;\ a=3, b=2.$

12. $m=0, n=0;\ m=\frac{1}{3}\sqrt{6}, n=-\frac{2}{3}\sqrt{6};\ m=-\frac{1}{3}\sqrt{6}, n=\frac{2}{3}\sqrt{6}.$

13. $v=0, t=1;\ v=1, t=0;\ v=-2, t=3.$

14. $x=-7, y=7;\ x=3, y=2;\ x=1, y=3.$

15. $m=0, n=0;\ m=5, n=1;\ m=-5, n=-1.$

16. $a=2, b=7;\ a=\frac{1}{2}, b=1\frac{3}{4};\ a=-\frac{1}{2}+\frac{1}{2}\sqrt{-3}, b=0;\ a=-\frac{1}{2}-\frac{1}{2}\sqrt{-3}, b=0.$

Pages 155, 156.

1. $m=1,\ n=2;\ m=2,\ n=1;\ m=\frac{3}{2}+\frac{1}{2}\sqrt{-55},\ n=\frac{3}{2}-\frac{1}{2}\sqrt{-55};$
$m=\frac{3}{2}-\frac{1}{2}\sqrt{-55},\ n=\frac{3}{2}+\frac{1}{2}\sqrt{-55}.$

2. $w=4,\ v=1;\ w=-1,\ v=-4;\ w=\frac{3}{2}+\frac{1}{2}\sqrt{-79},\ v=-\frac{3}{2}+\frac{1}{2}\sqrt{-79};$
$w=\frac{3}{2}-\frac{1}{2}\sqrt{-79},\ v=-\frac{3}{2}-\frac{1}{2}\sqrt{-79}.$

3. $A=1,\ B=3;\ A=3,\ B=1;\ A=2+5\sqrt{-1},\ B=2-5\sqrt{-1};$
$A=2-5\sqrt{-1},\ B=2+5\sqrt{-1}.$

4. $x=\frac{5}{2}+\frac{1}{2}\sqrt{-75+4\sqrt{353}},\ y=\frac{5}{2}-\frac{1}{2}\sqrt{-75+4\sqrt{353}};$
$x=\frac{5}{2}+\frac{1}{2}\sqrt{-75-4\sqrt{353}},\ y=\frac{5}{2}-\frac{1}{2}\sqrt{-75-4\sqrt{353}};$
$x=\frac{5}{2}-\frac{1}{2}\sqrt{-75+4\sqrt{353}},\ y=\frac{5}{2}+\frac{1}{2}\sqrt{-75+4\sqrt{353}},$
$x=\frac{5}{2}-\frac{1}{2}\sqrt{-75-4\sqrt{353}},\ y=\frac{5}{2}+\frac{1}{2}\sqrt{-75-4\sqrt{353}}.$

5. $n=3,\ m=2;\ n=2,\ m=3;\ n=6,\ m=-1;\ n=-1,\ m=6;$
$n=\frac{5}{2}+\frac{1}{2}\sqrt{21},\ m=\frac{5}{2}-\frac{1}{2}\sqrt{21};\ n=\frac{5}{2}-\frac{1}{2}\sqrt{21},\ m=\frac{5}{2}+\frac{1}{2}\sqrt{21};$
$n=\frac{5}{2}+\frac{1}{2}\sqrt{29},\ m=\frac{5}{2}-\frac{1}{2}\sqrt{29};\ n=\frac{5}{2}-\frac{1}{2}\sqrt{29},\ m=\frac{5}{2}+\frac{1}{2}\sqrt{29}.$

6. $a=1,\ b=1;\ a=-1,\ b=-1;\ a=\frac{1}{2}\sqrt{-3},\ b=\frac{1}{2}\sqrt{-3};\ a=-\frac{1}{2}\sqrt{-3},$
$b=-\frac{1}{2}\sqrt{-3}.$

7. $h=2,\ k=2;\ h=-6,\ k=6;\ h=2\sqrt{-3},\ k=3-\sqrt{-3};\ h=-2\sqrt{-3},\ k=3+\sqrt{-3};\ h=\frac{12}{7}-\frac{6}{7}\sqrt{-3},\ k=\frac{15}{7}+\frac{8}{7}\sqrt{-3};\ h=\frac{12}{7}+\frac{6}{7}\sqrt{-3},\ k=\frac{15}{7}-\frac{8}{7}\sqrt{-3}.$

8. $x=3,\ y=3;\ x=-3,\ y=-3;\ x=3\sqrt{-3},\ y=-3\sqrt{-3};\ x=-3\sqrt{-3},$
$y=3\sqrt{-3};$ and eight other complicated imaginary solutions.

9. $p=0,\ q=0.$

Pages 161, 162.

1. $x=6,\ y=3;\ x=-1\frac{1}{5},\ y=6\frac{3}{5}.$ 2. $x=9.1,\ y=4.1;\ x=-4.1,\ y=-9.1.$

3. $x=5.7,\ y=4.7;\ x=5.7,\ y=-4.7;\ x=-7.9,\ y=2.8;\ x=-7.9,\ y=-2.8.$

4. $x=4.5,\ y=4.1;\ x=4.5,\ y=-4.1;\ x=-3.5,\ y=2.9;\ x=-3.5,\ y=-2.9.$

5. $x=5.4,\ y=5.4;\ x=5.4,\ y=-5.4;\ x=-5.4,\ y=5.4;\ x=-5.4,\ y=-5.4.$

6. $x=6.96,\ y=5.70;\ x=6.96,\ y=-5.70;\ x=-6.96,\ y=5.70;\ x=-6.96,$
$y=-5.70.$

7. $x=4.5, y=4; x=-4.5, y=4; x=3.6, y=-3; x=-3.6, y=-3.$

8. $x=7.5, y=6.6; x=7.5, y=-6.6; x=-7.5, y=6.6; x=-7.5, y=-6.6.$

Pages 162, 165.

1. 4; 9.

2. 12.8+ ft.; 7.8+ ft.

3. 50 rd.; 40 rd.

4. 30 ft.; 40 ft.

5. 20 yd.; 30 yd.

6. 18 ft.; 24 ft.

7. 12 in.; 16 in.; 20 in.

8. 16 ft.; 20 ft.

9. 6 in. by 6 in., or 9 in. by 4 in.

10. 2 in. and 1 in., or 3 in. and 2 in.

11. 6; 8.

12. $96.

13. 40 ft.; 50 ft.

14. 25 ft.; 40 ft.

15. 10 da.; $3.25.

16. 6 men; 8 da.

17. Apples $32, potatoes $22; or apples $31.05, potatoes $22.95.

18. $250; 6%.

19. 8.4−.

20. $9\frac{3}{5}$; 96.

21. 24, $200; 25, $192.

22. 130 mi.; 117 mi.

23. A, $2\frac{1}{2}$ mi. per hour; B, 3 mi. per hour.

24. 48 rd.; 32 rd.

25. 20; 40.

26. 4 yd.; 5 yd.

Pages 165, 166.

1. $m=5, n=-1; m=-1, n=5; m=-1, n=-4; m=-4, n=-1.$

2. $x=0, y=0; x=5, y=-5; x=5, y=4; x=1\frac{1}{2}, y=\frac{1}{2}.$

3. $A=3, B=1; A=-1, B=-3; A=1, B=3; A=-3, B=-1.$

4. $p=-\frac{1}{4}, q=796.$

5. $x=3, y=6; x=6, y=3; x=-\frac{19}{2}+\frac{1}{2}\sqrt{513}, y=-\frac{19}{2}-\frac{1}{2}\sqrt{513};$
$x=-\frac{19}{2}-\frac{1}{2}\sqrt{513}, y=-\frac{19}{2}+\frac{1}{2}\sqrt{513}.$

6. $r=5, s=-1; r=1, s=-5; r=1, s=4; r=-4, s=-1.$

7. $M=4, N=5; M=5, N=4; M=-\frac{9}{2}-\frac{1}{2}\sqrt{161}, N=-\frac{9}{2}+\frac{1}{2}\sqrt{161};$
$M=-\frac{9}{2}+\frac{1}{2}\sqrt{161}, N=-\frac{9}{2}-\frac{1}{2}\sqrt{161}.$

8. $x=0, y=0; x=4, y=1; x=1, y=-2; x=-\frac{1}{2}, y=-\frac{1}{2}.$

9. $w=2, v=1; w=1, v=2; w=-1, v=-2; w=-2, v=-1;$
$w=\frac{1}{2}\sqrt{-1}+\frac{1}{2}\sqrt{11}, v=\frac{1}{2}\sqrt{-1}-\frac{1}{2}\sqrt{11}; w=\frac{1}{2}\sqrt{-1}-\frac{1}{2}\sqrt{11},$
$v=\frac{1}{2}\sqrt{-1}+\frac{1}{2}\sqrt{11}; w=-\frac{1}{2}\sqrt{-1}+\frac{1}{2}\sqrt{11}, v=-\frac{1}{2}\sqrt{-1}-\frac{1}{2}\sqrt{11};$
$w=-\frac{1}{2}\sqrt{-1}-\frac{1}{2}\sqrt{11}, v=-\frac{1}{2}\sqrt{-1}+\frac{1}{2}\sqrt{11}.$

10. $d=\sqrt{18}$, $t=\frac{1}{3}\sqrt{33}$; $d=-\sqrt{13}$, $t=\frac{1}{3}\sqrt{33}$; $d=\sqrt{13}$, $t=-\frac{1}{3}\sqrt{33}$; $d=-\sqrt{18}$, $t=-\frac{1}{3}\sqrt{33}$.

11. $P=2$, $Q=3$. The other three solutions may be obtained approximately graphically.

12. $a=0$, $b=0$; $a=-\frac{1}{2}$, $b=-\frac{1}{2}$.

13. $h=0$, $k=0$; $h=14$, $k=14$; $h=1-\sqrt{13}$, $k=1+\sqrt{13}$; $h=1+\sqrt{13}$, $k=1-\sqrt{13}$.

14. $x=25$; $y=9$; $x=9$, $y=25$; $x=-81+8\sqrt{-97}$, $y=-81-8\sqrt{-97}$; $x=-81-8\sqrt{-97}$, $y=-81+8\sqrt{-97}$.

15. $m=27$, $n=8$; $m=-8$, $n=-27$. **16.** Impossible. **17.** $x=10$, $y=6$.

18. $w=8$, $v=2$; $w=2$, $v=8$. **19.** $x=\frac{1}{3}$, $y=\frac{1}{2}$; $x=-\frac{1}{2}$, $y=-\frac{1}{3}$.

Pages 167, 168.

1. $\frac{3}{4}$.
2. $\frac{3}{100}$; 0.03.
3. $\frac{1}{25000}$; 0.00004.
4. 3.1416, approx.; no; it cannot be expressed by two whole numbers.
5. $\frac{36}{5}$; 7.2.
6. $\frac{1}{m^2-mn+n^2}$.
7. $\frac{4}{5}$.
8. $\frac{vv'}{tt'}$.
9. Ratio $=\frac{35}{36}$.
12. Yes.
13. Ratio $=\sqrt{2}$.
14. $\frac{5}{8}$.
15. \$960; \$1440.

Pages 169, 172.

1. 3.
2. $\frac{2}{3}$.
3. $3\frac{3}{8}$.
4. $13\frac{1}{3}$.
5. 21.
6. $10\frac{1}{2}$.
7. $3\frac{3}{5}$.
8. $110\frac{1}{4}$.
9. $\frac{xz}{y}$.
10. $\frac{v8}{2t}$.
11. k^2.
12. $\frac{a^2}{bc}$.
13. 0.
14. $\frac{r^2}{r'-r}$.
15. 8.
16. $2\frac{2}{3}$.
17. 10.
18. $\frac{1}{16}$.
19. $2x^3$.
20. m^2-n^2.
21. 48.
22. $\frac{1}{8}$.
23. $\frac{1}{27}$.
24. wv^2.
25. $\frac{q^2}{p}$.
26. $(a+b)(a^2-b^2)$.
27. ± 4.
28. ± 15.
29. ± 18.
30. $\pm\sqrt{35}$.
31. $\pm\sqrt{ab}$.
32. $\pm(V-V^2)$.
33. 7.
34. 15,000.
35. $4\frac{2}{3}$ inches.
36. $26\frac{2}{3}$ in.; $33\frac{1}{3}$ in.

37. $42\frac{6}{17}$ ft. **41.** 11.1+ ft.; 6.9− ft. **45.** $42\frac{6}{17}$ lb.
38. $10\frac{2}{7}$ in.; $7\frac{5}{7}$ in. **42.** $5\frac{1}{3}$ ft. **46.** 240 lb.
39. $8\frac{2}{5}$ in.; $5\frac{3}{5}$ in. **43.** 58 ft. **47.** 54 lb.
40. 9.9− in.; 6.1+ in. **44.** 80 lb. **48.** 90 lb.

Pages 178, 181.

1. $\frac{v}{t}=k;\ \frac{v'}{t'}=\frac{v''}{t''}.$ **2.** $v \propto \frac{1}{p};\ vp=k;\ v'p'=v''p''.$
3. 45. **4.** $\frac{1}{15}$. **5.** 900. **6.** 56 lb. **7.** 90. **8.** 72.
9. $157\frac{1}{2}$ lb.; 90 lb.; 45 lb. **16.** 75 candle-power.
10. 96 lb.; $37\frac{1}{2}$ lb.; $12\frac{12}{49}$ lb. **17.** 2 ft.
11. 256 ft.; 1600 ft.; 57,600 ft. **18.** $\frac{9}{10}$ sec.; $\frac{4}{5}$ sec.; $\frac{2}{5}$ sec.
12. 2400 lb. **19.** 400 cm.; 1600 cm.; 10,000 cm.
13. 4320 lb. **20.** $355\frac{5}{9}$ gal.
14. $\frac{1}{2}$ candle-power. **21.** 88 days.
15. 4 candle-power. **22.** 4333 days.

Page 183.

1. Circumference. **2.** 1. **3.** $\frac{1}{3}$. **4.** 0; 0; 6. **5.** 0. **6.** 0.

Page 187.

1. $\frac{1}{2}$. **2.** $\frac{1}{12}$. **3.** −4. **4.** $\frac{5}{7}$.
5. Inconsistent. **7.** Inconsistent. **9.** Indeterminate.
6. Inconsistent. **8.** Indeterminate. **10.** Indeterminate.

Pages 187, 189.

5. $\pm(m^3-n^3)$. **13.** 0. **22.** 2.
6. $2\sqrt{3}$. **14.** 0. **23.** $\frac{1}{2}$.
7. $F=kw$. **15.** ∞. **24.** $\frac{1}{3}$.
8. $F=\frac{kmm'}{d^2}$. **16.** ∞. **25.** 1.
9. $F=\frac{kwv^2}{l}$. **17.** 1. **26.** $\frac{1}{2}$.
10. $Hd^2=k$. **18.** 0. **27.** 1.
11. $R=\frac{kl}{A}$. **19.** $\frac{1}{4}$. **28.** 6.
12. 0. **21.** 1.

Page 190.

1. 5, 6, 7, 8.
2. 10, 12, 14, 16.
3. 11, 8, 5, 2.
4. 16, 32, 64, 128.
5. $\frac{1}{81}, \frac{1}{243}, \frac{1}{729}, \frac{1}{2187}$.
6. 25, 36, 49, 64.
7. −12, −17, −22, −27.
8. $\frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}$.
9. 41, 122, 365, 1094.
10. $16n^4, 32n^5, 64n^6, 128n^7$.
11. $\sqrt{14}, \sqrt{17}, \sqrt{20}, \sqrt{23}$.
12. $-32a^3t^{10}, 16a^4t^8, -8a^5t^6, 4a^6t^4$.

Pages 192, 193.

1. 146.
2. 61.
3. 61.
4. −102.
5. $15\frac{1}{2}$.
6. 257.
7. 16th.
8. 31st.
9. 37th.
10. 3.
11. 5.
12. 8.
13. 57.
14. 2; 3.
15. $-7\frac{1}{3}$; $3\frac{2}{3}$.
16. 49; 81.
17. $14\frac{1}{2}$; $26\frac{1}{2}$.
18. 200.
19. 199.
20. 496.
21. $73n$.
22. $1+45x$.
23. $a+(n-2)d$.
24. $a+(n-4)d$.
25. 624 ft.; 1904 ft.
26. $at-\frac{1}{2}a$.
27. 14 sec.

Pages 194, 195.

1. 5, 9, 13, 17, 21, 25.
2. 12, $8\frac{1}{2}$, 5, $1\frac{1}{2}$, −2.
3. 3, 7, 11, 15, 19, 23, 27.
4. −6, −1, 4, 9, 14, 19, 24, 29.
5. $\frac{1}{2}$, −1, $-2\frac{1}{2}$, −4, $-5\frac{1}{2}$, −7, $-8\frac{1}{2}$, −10.
6. 1, $1\frac{4}{13}$, $1\frac{8}{13}$, $1\frac{12}{13}$, $2\frac{3}{13}$, $2\frac{7}{13}$, $2\frac{11}{13}$, $3\frac{2}{13}$, $3\frac{6}{13}$, etc.
7. 3, $5\frac{1}{7}$, $7\frac{2}{7}$, $9\frac{3}{7}$, $11\frac{4}{7}$, $13\frac{5}{7}$, $15\frac{6}{7}$, 18, $20\frac{1}{7}$, etc.
8. $\frac{1}{2}, \frac{4}{9}, \frac{7}{18}, \frac{1}{3}$.
9. $n, \frac{31}{36}n, \frac{13}{18}n, \frac{7}{12}n, \frac{4}{9}n, \frac{11}{36}n, \frac{1}{6}n$.
10. 12; 40; 25; 2; −2; $\frac{13}{16}$; $\frac{1}{12}$; $\frac{3}{2}\sqrt{3}$; x.
11. a^2+b^2.
12. 48 ft. per second.
13. 176 ft. per second.
14. 32 mi. per hour; $\frac{5}{8}$ hr.
15. $12\frac{4}{5}$ in.; $23\frac{3}{5}$ in.; $34\frac{2}{5}$ in.; $45\frac{1}{5}$ in.
16. 3 oz.; 5 oz.; 7 oz.; 9 oz.; 11 oz.; 13 oz.
17. 4°.

Pages 197, 199.

1. 1575.
2. 608.
3. 231.
4. −945.
5. $117\frac{1}{2}$.
6. −2385.
7. 5050.
8. 10,100.
9. 10,000.
10. 2475.
11. 7350.
12. 7500.
14. n^2+n.
15. 80.
16. 5 or 6.

17. 24.
18. 20.
19. $n=13$; $S=286$.
20. $n=24$; $l=117$.
21. $a=42$; $l=-26$.
22. $-8, -4, 0, 4, 8$.
23. 156.
24. 300.
25. 2550 yd.
26. 150 in.
27. 590 ft.; 30 sec.
28. 576 ft.; 1600 ft.; 57,600 ft.
29. 16,896 ft.
30. 10 sec.
31. $10,600; $7300.
32. $12,250.
33. 144; 121; 100; 650.
34. 4994 – in.
35. 2, 6, 10.
36. 1, 5, 9.

Pages 201, 202.

1. 128.
2. 100,000,000.
3. $\frac{10}{81}$.
4. 1458.
5. 1024.
6. $-\frac{2}{9}$.
7. 16,384.
8. 2.
9. 6.
10. $\frac{1}{1024}$.
11. $\pm 12\sqrt{2}$.
12. $\frac{27}{4}$.
13. 9, 18, 36, 72, 144; or $-9, 18, -36, 72, -144$.
14. $\frac{1}{128}, \frac{1}{32}, \frac{1}{8}, \frac{1}{2}, 2, 8$.
15. 16.
16. $\frac{27\sqrt{2}}{4}$.
17. $\frac{n^2}{m}, \frac{n^3}{m^2}$.
18. $11,809.80.
19. $119.41.
20. $579.64.
21. 259,374.
22. $\frac{15}{16384}$ ft.

Pages 203, 204.

1. 7, 14, 28, 56.
2. 2, 6, 18, 54.
3. $16, 8\sqrt[4]{54}, 4\sqrt{54}, 2\sqrt[4]{2916}, 54$.
4. 1, 4, 16, 64, 256, 1024.
5. 80, 40, 20, 10, 5.
6. $24, 16, \frac{32}{3}, \frac{64}{9}, \frac{128}{27}, \frac{256}{81}$.
7. $-9, -18, -36, -72, -144$.
8. $1, \sqrt[3]{2}, \sqrt[3]{4}, 2$.
9. $m, \sqrt[3]{m^2n}, \sqrt[3]{mn^2}, n$.
10. ± 2; ± 12; ± 10; $\pm 2\sqrt{3}$; ± 2.
11. $\pm(a^2-ab)$. **12.** 15; $4\sqrt{3}$. **13.** 4; 16. **14.** 10; 18. **15.** 11.088; 23.088.

Pages 205, 206.

1. 6138. **2.** 364. **3.** 86. **4.** $15\frac{15}{16}$. **5.** $63\frac{63}{64}$.

6. 5555.
7. $\frac{1-x^{10}}{1-x}$.
8. $49\frac{355}{384}$.
9. $\frac{6n-384n^7}{1+2n}$.
10. $24+26\sqrt{3}$.
11. $2\frac{85}{128}$.
12. 3.
13. $\frac{2}{3}$; $\frac{4}{81}$.
14. 8.
15. 6.
16. $149\frac{23}{25}$ ft.
17. $152\frac{228}{375}$ cu. in.
18. 4 yr.
19. $\frac{81}{32}$.
20. $1.125509 –.
21. $1159.27 +.
22. $119.41 –.
23. $3736.29.
24. $18,681.45.

Pages 208, 209.

1. 16. **2.** 18. **3.** $22\frac{1}{2}$. **4.** $3\frac{8}{11}$. **5.** $4\frac{1}{2}$. **6.** $\frac{2}{3}$. **7.** $1\frac{1}{4}$. **8.** $4+2\sqrt{2}$.

9. $\frac{1}{1-n}$. **10.** $\frac{n^2}{n^2-1}$. **11.** 60. **12.** $-\frac{1}{10}$.

13. $\frac{3}{2}, \frac{3}{8}, \frac{3}{32}, \frac{3}{128}, \frac{3}{512}$; or $\frac{1}{2}, \frac{3}{8}, \frac{9}{32}, \frac{27}{128}, \frac{81}{512}$. **14.** $\frac{2}{27}$ or $4\frac{20}{27}$.

15. $\frac{2}{3}$. **17.** $\frac{18}{33}$. **19.** $\frac{50}{333}$. **21.** $4\frac{4}{11}$.

16. $\frac{3}{11}$. **18.** $\frac{5}{11}$. **20.** $1\frac{1}{3}$. **22.** $3\frac{238}{330}$.

23. $\frac{51}{1100}$. **24.** 12π in. **25.** $\frac{16}{3}\pi$ sq. in.

Pages 209, 210.

2. $\frac{1}{81}$. **3.** $\frac{1}{49}$. **4.** $\frac{2}{25}$. **5.** $\frac{1}{39}$. **6.** $\frac{2}{111}$.

7. $-\frac{2}{3}, -2, 2, \frac{2}{3}, \frac{2}{5}$. **10.** $\frac{1}{4}, \frac{18}{61}, \frac{9}{25}, \frac{6}{13}, \frac{9}{14}, \frac{18}{17}, 3$.

8. $\frac{1}{9}$; 3. **13.** ± 12.

9. $-\frac{6}{5}, -2, -6, 6, 2, \frac{6}{5}$. **14.** 0.1 and 159.9, approx.

Page 213.

1. n^{10}.
2. V^{12}.
3. k^9.
4. R^{15}.
5. 2^6.
6. 5^7.
7. 3^9.
8. $(-1)^5$.
9. $(-2)^6$.
10. $(-a)^8$.
11. x^{2n}.
12. t^{n+2}.
13. P^{2m}.
14. a^7.
15. y^4.
16. D^8.
17. r.
18. mv^2.
19. $2t$.
20. 2^3.
21. 3^2.
22. $(-1)^2$.
23. $(-n)^4$.
24. x^{20}.
25. N^{18}.
26. s^{48}.
27. k^{100}.
28. 2^6.
29. 3^8.
30. $(-1)^{18}$.
31. $(-b)^{20}$.
32. x^{n^2-n}.
33. m^4n^4.
34. $a^{12}b^{18}$.
35. $x^{15}y^3$.
36. $v^{36}t^{16}$.
37. $2^3\times 5^3$.
38. $7^2\times 3^2$.
39. $a^{5n}b^{10n}$.
40. $\frac{3^3}{4^3}$.
41. $\frac{2^4}{3^4}$.
42. $\frac{4^4}{5^4}$.
43. $+\frac{1^6}{2^6}$.
44. $-\frac{2^3}{3^3}$.
45. $\frac{a^4}{b^4}$.
46. $\frac{8m^6}{27n^{15}}$.
47. $\frac{v^{15}t^{25}}{32a^{20}s^{10}}$.
48. a^3.
49. n^3.
50. x^2y^4.
51. $2r^4$.
52. a^2b^3.
53. a^2.
54. t^2.
55. m^6.
56. $1024a^{20}b^4$.

Page 214.

1. 1.
2. 1.
3. 1.
4. 1.
5. 1.
6. 81.
7. n^5.
8. 1.

Pages 215, 216.

1. $\frac{1}{4}$.
2. $\frac{1}{32}$.
3. $\frac{1}{81}$.
4. $\frac{1}{7}$.
5. $\frac{1}{1000000}$.
6. 1.
7. 4.
8. $1,000,000$.
9. 4.
10. $\frac{3}{2}$.
11. 125.
12. $\frac{64}{9}$.
13. $\frac{625}{16}$.
14. $\frac{1}{16}$.
15. $\frac{9}{160000}$.
16. 324.
17. $\frac{1}{n^4}$.
18. $\frac{a^2}{b^3}$.
19. $\frac{\pi R^2}{r^2}$.
20. $\frac{n^4}{m^2}$.
21. a^5b^5.
22. $\frac{1}{ab^2c^3}$.
23. $\frac{2}{-x^2y^2}$.
24. $\frac{-5N^3}{-3P^4}$.
25. $\frac{1}{x^2y^4z^6}$.
26. $\frac{x^3}{y^3}$.
27. $\frac{b}{-6a^6}$.
28. $\frac{1}{5V^2}$.
29. $\frac{m}{n}+\frac{n}{m}$.
30. $\frac{b+a}{b-a}$.
31. $\frac{x^2y^2-1}{x^2y^2+1}$.
32. $\frac{a-b}{a+b}$.
33. N^6.
34. a^2b^{-2}.
35. $\frac{2}{3}xy^{-1}$.
36. $-\frac{1}{5}n^{-1}$.
37. $v^{-2}t^2$.
38. $x^{-1}+y^{-1}$.
39. $ab^{-1}+a^{-1}b$.
40. $2mn$.
41. $(a^2+ab+b^2)^{-1}$.

Page 217.

1. 2.
2. 3.
3. 9.
4. 8.
5. 125.
6. 8.
7. 27.
8. 25.
9. 8.
10. 32.
11. 243.
12. 256.
13. $\frac{1}{2}$.
14. $\frac{1}{8}$.
15. $\frac{1}{9}$.
16. $\frac{1}{64}$.
17. $\frac{9}{4}$.
18. $\frac{256}{625}$.
19. $\frac{16}{9}$.
20. $\frac{243}{32}$.
21. $\frac{4}{27}$.
22. $\frac{8}{3125}$.
23. $\frac{9}{5}$.
24. 24.
25. $\frac{1}{2}$.
26. $\sqrt[3]{a^2b}$.
27. $\frac{\sqrt{q}}{\sqrt{p}}$.
28. $\frac{1}{\sqrt[12]{M^8V^9}}$.
29. $\frac{4}{\sqrt{x^5}}$.
30. $-9\sqrt[4]{t^3}$.
31. $\frac{2\sqrt{m^3}}{\sqrt[3]{n^2}}$.
32. $\frac{9\sqrt[5]{t^2}}{\sqrt[5]{V^4}}$.
33. $\frac{z}{\sqrt{xy}}$.
34. $\frac{\sqrt[6]{p^4q^9}}{m^2}$.
35. $\frac{\sqrt[4]{(a+b)^3}}{\sqrt[4]{(a-b)^3}}$.
36. $h-k$.
37. $\sqrt{mn}$.
38. $v^{\frac{2}{5}}$.
39. $n^{\frac{7}{3}}$.
40. $a^{\frac{3}{2}}$.
41. $w^{\frac{1}{2}}t^{\frac{1}{3}}$.
42. $a^{\frac{1}{2}}b^{\frac{3}{4}}m^{\frac{1}{4}}n^{\frac{1}{4}}$.
43. $3^{\frac{1}{2}}x^{\frac{1}{2}}y$.
44. $b^{\frac{3}{4}}c^{\frac{1}{2}}$.
45. $a^{\frac{1}{m}}$.
46. $x^{\frac{m}{2}}$.
47. $(a-b)^{\frac{1}{2}}$.
48. $(1+x^2)^{\frac{1}{3}}$.
49. $x^{\frac{n}{m}}x^{\frac{m}{n}}$.
50. $(1-x)^{-\frac{1}{2}}$.
51. $(1+x^2)^{-\frac{2}{3}}$.
52. $(a+b)^{\frac{3}{2}}(a-b)^{-\frac{3}{2}}$.

Page 219.

1. A^2.
2. $4n$.
3. 1.
4. $12t^4$.
5. a.
6. $15p^{-\frac{1}{6}}$.
7. $h^{\frac{1}{5}}k^{\frac{1}{6}}$.
8. $V^{-\frac{1}{12}}t^{-\frac{1}{20}}$
9. $y^{\frac{5}{4}}z^{\frac{7}{12}}$.
10. x^2.
11. $a^{-2}n^{-5}$.
12. w^3v^{-2}.
13. $p^{-4}q^{-4}r^{-4}$.
14. $N^{\frac{3}{5}}$.
15. $K^{\frac{1}{3}}$.
16. ab.
17. $x^{-\frac{1}{2}}yz^{-\frac{5}{4}}$.
18. $2P^{-\frac{5}{4}}V^{-\frac{1}{7}}$.
19. a^4b^6.
20. $x^{-20}y^5$.
21. $\pi^{-4}R^{-8}$.
22. $m^{-1}n^{-3}$.
23. $2^{\frac{2}{3}}H^{\frac{1}{2}}n^{-\frac{2}{3}}$.
24. $\frac{1}{125}M^{\frac{1}{2}}A^{-1}$.
25. $\frac{1}{8}xy^{\frac{3}{8}}z^{\frac{3}{2}}$.
26. $3^{-\frac{1}{3}}a^{-1}b^{\frac{1}{2}}c^{\frac{3}{4}}$.
27. $\frac{1}{9}p^6d^{\frac{1}{2}}$.
28. $\frac{1}{8}rs^{-\frac{6}{11}}$.
29. $a^{\frac{5}{6}}$.
30. $t^{\frac{25}{12}}$.
31. $n^{\frac{5}{4}}$.
32. k.
33. a^2x.
34. $a^{\frac{5}{6}}$.
35. $w^{-\frac{5}{12}}v^{-\frac{1}{6}}$.
36. $n^{\frac{1}{3}}m^{-\frac{1}{6}}$.
37. $k^{-\frac{7}{6}}n^{-\frac{4}{3}}$.
38. $a^{-\frac{11}{3}}b^4$.
39. $t^{-\frac{112}{39}}$.
40. $n^{\frac{2}{3}}$.
41. $x^{-\frac{1}{4}}$.
42. $x-y$.
43. W^3-V^{-3}.
44. $m^{\frac{1}{2}}-n^{\frac{1}{2}}$.
45. $R^{\frac{2}{3}}-R^{\frac{1}{3}}r^{\frac{1}{3}}+r^{\frac{2}{3}}$.
46. $x^{\frac{4}{3}}+x^{\frac{2}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}}$.
47. $x+x^{\frac{1}{2}}-2$.
48. $9m^{-\frac{4}{5}}-16n^5$.
49. $a^{\frac{1}{2}}+2b^{\frac{1}{2}}$.
50. $t^{-2}+t^{-1}+1$.
51. $1-y^{-\frac{1}{3}}+y^{-\frac{2}{3}}$.
52. $n^{-2}-3n^{-1}+9$.
53. $k^{-3}+k^{-2}+k^{-1}+1$.
54. $v^{\frac{8}{5}}-v^{\frac{6}{5}}t^{\frac{2}{5}}+v^{\frac{4}{5}}t^{\frac{4}{5}}-v^{\frac{2}{5}}t^{\frac{6}{5}}+t^{\frac{8}{5}}$.
55. $M^{\frac{4}{3}}+2M^{\frac{2}{3}}N^{-\frac{3}{4}}+N^{-\frac{3}{2}}$.
56. $a-2a^{\frac{1}{2}}b^{-\frac{1}{2}}+b^{-1}$.
57. $1-2x^{-\frac{1}{4}}+x^{-\frac{1}{2}}$.
58. $E^{-\frac{1}{2}}+2E^{-\frac{1}{4}}F^{-\frac{1}{4}}+F^{-\frac{1}{2}}$.
59. $e^{2x}+E^{-2x}+6-4e^x-4e^{-x}$.
60. $1-2x^{-1}+x^{-2}+2x^{-3}-2x^{-4}+x^{-6}$.
61. $n^{\frac{1}{2}}+2n^{\frac{1}{4}}+3+2n^{-\frac{1}{4}}+n^{-\frac{1}{2}}$.
62. $h^{\frac{3}{4}}+h^{\frac{1}{2}}k^{\frac{1}{4}}+h^{\frac{1}{4}}k^{\frac{1}{2}}+k^{\frac{3}{4}}$.
63. $A^{-n}-2A^{-\frac{1}{2}n}+1$.
64. $a-b$.
65. $x^{\frac{8}{3}}+x^{\frac{4}{3}}y^{\frac{4}{3}}+y^{\frac{8}{3}}$.
66. t^2-1.
67. n^4-1+n^{-4}.
68. $p^{\frac{4}{5}}+4p^{\frac{2}{5}}q^{\frac{2}{5}}+16q^{\frac{4}{5}}$.
69. m^2-n^2.
70. $E+F$.
71. $x^{\frac{1}{2}}+1+x^{-\frac{1}{2}}$.
72. $\frac{2}{\sqrt[3]{n^2}}-3$.

Pages 220, 221.

1. $a^{-2n}+a^{-n}+1$. 2. x^n+y^n.

3. $t^{-4n}-2t^{-3n}+3t^{-2n}-4t^{-n}+5-4t^n+3t^{2n}-2t^{3n}+t^{4n}$.

4. $m^{\frac{1}{3}}+n^{\frac{1}{3}}$. 9. $3x^{\frac{1}{2}}-2+x^{-\frac{1}{2}}$.

5. $(4+n^{-1})(4-n^{-1})$. 10. $m-3m^{\frac{1}{2}}n^{\frac{1}{3}}+3n^{\frac{2}{3}}$.

6. $(R+R^{-1})^2$. 11. $A+4-2A^{-1}$.

7. $(3-N^{-1})(9+3N^{-1}+N^{-2})$. 12. $h^2+h^{-1}k+hk^{-1}$.

8. $(k^{-1}-3)(k^{-1}+2)$. 13. $r^{\frac{2}{5}}+2r^{\frac{1}{5}}+r^{-\frac{1}{5}}$.

14. $a^{-\frac{2}{3}}-a^{-\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$.

Page 224.

1. $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$.
2. $a^7-7a^6b+21a^5b^2-35a^4b^3+35a^3b^4-21a^2b^5+7ab^6-b^7$.
3. $m^8+8m^7n+28m^6n^2+56m^5n^3+70m^4n^4+56m^3n^5+28m^2n^6+8mn^7+n^8$.
4. $m^8-8m^7n+28m^6n^2-56m^5n^3+70m^4n^4-56m^3n^5+28m^2n^6-8mn^7+n^8$.
5. $x^9+9x^8y+36x^7y^2+84x^6y^3+126x^5y^4+126x^4y^5+84x^3y^6+36x^2y^7+9xy^8$
   $+y^9$.
6. $v^{10}-10v^9t+45v^8t^2-120v^7t^3+210v^6t^4-252v^5t^5+210v^4t^6-120v^3t^7+45v^2t^8$
   $-10vt^9+t^{10}$.
7. $h^{12}+12h^{11}k+66h^{10}k^2+220h^9k^3+495h^8k^4+792h^7k^5+924h^6k^6+792h^5k^7$
   $+495h^4k^8+220h^3k^9+66h^2k^{10}+12hk^{11}+k^{12}$.
8. $P^{13}-13P^{12}Q+78P^{11}Q^2-286P^{10}Q^3+715P^9Q^4-1287P^8Q^5+1716P^7Q^6$
   $-1716P^6Q^7+1287P^5Q^8-715P^4Q^9+286P^3Q^{10}-78P^2Q^{11}+13PQ^{12}-Q^{13}$.
9. $1+4x+6x^2+4x^3+x^4$.
10. $1-5n+10n^2-10n^3+5n^4-n^5$.
11. $a^6+6a^5+15a^4+20a^3+15a^2+6a+1$.
12. $32x^5+80x^4y+80x^3y^2+40x^2y^3+10xy^4+y^5$.
13. $a^{16}-8a^{14}b^2+28a^{12}b^4-56a^{10}b^6+70a^8b^8-56a^6b^{10}+28a^4b^{12}-8a^2b^{14}+b^{16}$.
14. $M^6-18M^5N^2+135M^4N^4-540M^3N^6+1215M^2N^8-1458MN^{10}+729N^{12}$.
15. $64V^6+576V^5t+2160V^4t^2+4320V^3t^3+4860V^2t^4+2916Vt^5+729t^6$.
16. $1-8x^3+24x^6-32x^9+16x^{12}$.

17. $128p^{14}+448p^{12}q+672p^{10}q^2+560p^8q^3+280p^6q^4+84p^4q^5+14p^2q^6+q^7$.

18. $1-15V^3+90V^6-270V^9+405V^{12}-243V^{15}$.

19. $y^{30}+10y^{27}z^3+45y^{24}z^6+120y^{21}z^9+210y^{18}z^{12}+252y^{15}z^{15}+210y^{12}z^{18}+120y^9z^{21}$ $+45y^6z^{24}+10y^3z^{27}+z^{30}$.

20. $1+16b^4+112b^8+448b^{12}+1120b^{16}+1792b^{20}+1792b^{24}+1024b^{28}+256b^{32}$.

21. $1-3n+\frac{15}{4}n^2-\frac{5}{2}n^3+\frac{15}{16}n^4-\frac{3}{16}n^5+\frac{1}{64}n^6$.

22. $\frac{1}{64}E^{12}+\frac{1}{16}E^{10}F^2+\frac{5}{48}E^8F^4+\frac{5}{54}E^6F^6+\frac{5}{108}E^4F^8+\frac{1}{81}E^2F^{10}+\frac{1}{729}F^{12}$.

23. $\frac{32}{243}a^{20}-\frac{80}{81}a^{16}+\frac{80}{27}a^{12}-\frac{40}{9}a^8+\frac{10}{3}a^4-1$.

24. $1296R^8+432R^6+54R^4+3R^2+\frac{1}{16}$.

25. $a^{18}+18a^{17}b+153a^{16}b^2+816a^{15}b^3+\ldots\ldots.$

26. $a^{20}-20a^{19}b+190a^{18}b^2-1140a^{17}b^3+\ldots\ldots.$

27. $1+40N+780N^2+9880N^3+\ldots\ldots.$

28. $t^{30}-30t^{29}+435t^{28}-4060t^{27}+\ldots\ldots.$

29. $x^{20}+40x^{19}+760x^{18}+9120x^{17}+\ldots\ldots.$

30. $M^{48}-48M^{46}N+1104M^{44}N^2-16192M^{42}N^3+\ldots\ldots.$

31. $w^{44}-66w^{42}v^2+2079w^{40}v^4-41580w^{38}v^6+\ldots\ldots.$

32. $(\frac{1}{2}x)^{30}+30(\frac{1}{2}x)^{29}(2y)+435(\frac{1}{2}x)^{28}(2y)^2+4060(\frac{1}{2}x)^{27}(2y)^3+\ldots\ldots.$

33. $n^{40}-80n^{38}+3120n^{36}-79040n^{34}+\ldots\ldots.$

34. $\frac{65536a^{16}}{b^{16}}+\frac{131072a^{13}}{b^{13}}+\frac{122880a^{10}}{b^{10}}+\frac{71680a^7}{b^7}+\ldots\ldots.$

35. $a^n-na^{n-1}b+\frac{n(n-1)}{2}a^{n-2}b^2-\frac{n(n-1)(n-2)}{6}a^{n-3}b^3$

$+\frac{n(n-1)(n-2)(n-3)}{24}a^{n-4}b^4-\ldots\ldots.$

36. 3.14−. **37.** 6.19+. **38.** 1.37−. **39.** 0.00+. **40.** 59604.64+.

41.

1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

Page 226.

1. $1-n+n^2-n^3+n^4-\ldots\ldots.$
2. $1+3t+6t^2+10t^3+15t^4+\ldots\ldots.$
3. $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4+\ldots\ldots.$
4. $1-\frac{1}{3}R-\frac{1}{9}R^2-\frac{5}{81}R^3-\frac{10}{243}R^4-\ldots\ldots.$
5. $1+\frac{1}{4}a-\frac{3}{32}a^2+\frac{7}{128}a^3-\frac{77}{2048}a^4+\ldots\ldots.$
6. $1+x^3+x^6+x^9+x^{12}+\ldots\ldots.$
7. $1-2k^2+3k^4-4k^6+5k^8-\ldots\ldots.$
8. $1-\frac{1}{2}P^2+\frac{3}{8}P^4-\frac{5}{16}P^6+\frac{35}{128}P^8-\ldots\ldots.$
9. $1-2a^2+6a^4-20a^6+70a^8-\ldots\ldots.$
10. $1+3y^3+18y^3+126y^9+945y^{12}+\ldots\ldots.$
11. $a^{-\frac{1}{2}}+\frac{1}{4}a^{-\frac{5}{2}}b^2+\frac{5}{32}a^{-\frac{9}{2}}b^4+\frac{15}{128}a^{-\frac{13}{2}}b^6+\frac{195}{2048}a^{-\frac{17}{2}}b^8+\ldots\ldots.$
12. $x^{\frac{1}{4}}+\frac{1}{2}x^{-\frac{1}{4}}y^{\frac{1}{2}}-\frac{1}{8}x^{-\frac{3}{4}}y+\frac{1}{16}x^{-\frac{5}{4}}y^{\frac{3}{2}}-\frac{5}{128}x^{-\frac{7}{4}}y^2+\ldots\ldots.$
13. $m^{\frac{1}{4}}+\frac{1}{2}m^{\frac{3}{4}}n^{-\frac{1}{2}}+\frac{3}{8}m^{\frac{5}{4}}n^{-1}+\frac{5}{16}m^{\frac{7}{4}}n^{-\frac{3}{2}}+\frac{35}{128}m^{\frac{9}{4}}n^{-2}+\ldots\ldots.$
14. $2^{\frac{2}{3}}v^{\frac{4}{9}}+\frac{2}{3}\times 2^{-\frac{1}{3}}v^{-\frac{2}{9}}t^{\frac{3}{2}}-\frac{1}{9}\times 2^{-\frac{4}{3}}v^{-\frac{8}{9}}t^3+\frac{4}{81}\times 2^{-\frac{7}{3}}v^{-\frac{14}{9}}t^{\frac{9}{2}}-\frac{7}{243}\times 2^{-\frac{10}{3}}v^{-\frac{20}{9}}t^6$
    $+\ldots\ldots.$
15. $3^{-\frac{1}{3}}p^{\frac{1}{4}}+\frac{2}{3}\times 3^{-\frac{4}{3}}pq^{-1}+\frac{5}{9}\times 3^{-\frac{7}{3}}p^{\frac{7}{4}}q^{-2}+\frac{112}{81}\times 3^{-\frac{10}{3}}p^{\frac{5}{2}}q^{-3}$
    $+\frac{560}{243}\times 3^{-\frac{13}{3}}p^{\frac{13}{4}}q^{-4}+\ldots\ldots.$

Page 227.

1. 1.968−.
2. 3.037−.
3. 4.041+.
4. 2.924+.
5. 5.066−.
6. 6.028−.
7. 2.080+.
8. 4.019+.
9. 2.993−.
10. 2.015+.
11. 4.998+.
12. 3.001−.
13. 3.036+.
14. 1.961−.
15. 2.030+.

Page 228.

1. $20a^6b^6.$
2. $35x^4.$
3. $-77520t^{14}.$
4. $-658008m^{35}n^5.$
5. $220x^{27}y^9.$
6. $A^{-40}B^{18}.$
7. $-\frac{21}{16}x^{18}.$
8. $\frac{88179}{256}V^{-11}.$
9. $\frac{4807}{262144}x^{-\frac{27}{8}}y^{\frac{7}{2}}.$
10. $\frac{5434}{59049}R^{\frac{25}{3}}r^{16}.$
11. $\dfrac{\sqrt{2}a^{-\frac{5}{2}}b^6}{128}.$
12. $\dfrac{17\times 35\times 29\times 37\times 41\times 53\times 57}{2\times 4^{20}}a^{-\frac{15}{2}}.$
13. $\frac{21}{256}k^5.$
14. $\dfrac{33\sqrt{2}n^{14}}{33554432}.$
15. $\dfrac{27170}{4782969}x^{-\frac{116}{9}}.$
16. $5e^{-5}.$

Page 229.

1. $x^4-8x^{\frac{7}{2}}y^{\frac{1}{2}}+28x^3y-56x^{\frac{5}{2}}y^{\frac{3}{2}}+70x^2y^2-56x^{\frac{3}{2}}y^{\frac{5}{2}}+28xy^3-8x^{\frac{1}{2}}y^{\frac{7}{2}}+y^4$.

2. $a^4+6a^{\frac{10}{3}}b^{\frac{2}{3}}+15a^{\frac{8}{3}}b^{\frac{4}{3}}+20a^2b^2+15a^{\frac{4}{3}}b^{\frac{8}{3}}+6a^{\frac{2}{3}}b^{\frac{10}{3}}+b^4$.

3. $m^{15}+10m^{\frac{27}{2}}n^{\frac{3}{2}}+45m^{12}n^3+120m^{\frac{21}{2}}n^{\frac{9}{2}}+210m^9n^6+252m^{\frac{15}{2}}n^{\frac{15}{2}}+210m^6n^9$
$+120m^{\frac{9}{2}}n^{\frac{21}{2}}+45m^3n^{12}+10m^{\frac{3}{2}}n^{\frac{27}{2}}+n^{15}$.

4. $e^{27}-12e^{\frac{99}{4}}+66e^{\frac{45}{2}}-220e^{\frac{81}{4}}+495e^{18}-792e^{\frac{63}{4}}+924e^{\frac{27}{2}}-792e^{\frac{45}{4}}+495e^9$
$-220e^{\frac{27}{4}}+66e^{\frac{9}{2}}-12e^{\frac{9}{4}}+1$.

5. $a^5b^{-10}+10a^{\frac{7}{2}}b^{-\frac{17}{2}}+45a^2b^{-7}+120a^{\frac{1}{2}}b^{-\frac{11}{2}}+210a^{-1}b^{-4}+252a^{-\frac{5}{2}}b^{-\frac{5}{2}}$
$+210a^{-4}b^{-1}+120a^{-\frac{11}{2}}b^{\frac{1}{2}}+45a^{-7}b^2+10a^{-\frac{17}{2}}b^{\frac{7}{2}}+a^{-10}b^5$.

6. $81m^4n^{-8}+108m^{\frac{7}{2}}n^{-5}+54m^3n^{-2}+12m^{\frac{5}{2}}n+m^2n^4$.

7. $x^3y^{-4}-6x^2y^{-\frac{17}{6}}+15xy^{-\frac{5}{3}}-20y^{-\frac{1}{2}}+15x^{-1}y^{\frac{2}{3}}-6x^{-2}y^{\frac{11}{6}}+x^{-3}y^3$.

8. $a^{\frac{16}{3}}b^{-1}+2a^3b^{-\frac{11}{20}}+\frac{3}{2}a^{\frac{2}{3}}b^{-\frac{1}{10}}+\frac{1}{2}a^{-\frac{5}{3}}b^{\frac{7}{20}}+\frac{1}{16}a^{-4}b^{\frac{4}{5}}$.

9. $1+4x+2x^2-8x^3-5x^4+8x^5+2x^6-4x^7+x^8$.

10. $x^6+3x^4y^2+3x^2y^4+y^6+3x^4z^2+6x^2y^2z^2+3y^4z^2+3x^2z^4+3y^2z^4+z^6$.

11. $16-32m+24m^2-8m^3+m^4+32n-48mn+24m^2n+4m^3n+24n^2-24mn^2$
$+6m^2n^2+8n^3-4mn^3+n^4$.

12. $a^4+b^4+c^4+d^4+6a^2b^2+6a^2c^2+6a^2d^2+6b^2c^2+6b^2d^2+6c^2d^2-4a^3b+4a^3c$
$-4a^3d-12a^2bc+12a^2bd-12a^2cd-4ab^3+12ab^2c-12ab^2d-4b^3c$
$+4b^3d-12b^2cd-12abc^2+4ac^3-12ac^2d-4bc^3+12bc^2d-4c^3d-12abd^2$
$+12acd^2-4ad^3-12bcd^2+4bd^3-4cd^3+24abcd$.

13. $1-3t+6t^2-10t^3+12t^4-12t^5+10t^6-6t^7+3t^8-t^9$.

14. $16a^4+b^4+81c^8+24a^2b^2+216a^2c^4+54b^2c^4-32a^3b+96a^3c^2-144a^2bc^2$
$-8ab^3+72ab^2c^2-12b^3c^2-216abc^4+216ac^6-108bc^6$.

15. $n^{-8}+4n^{-6}+10n^{-4}+16n^{-2}+19+16n^2+10n^4+4n^6+n^8$.

16. $\dfrac{n(n-1)(n-2)\ldots\ldots\text{to } r \text{ factors}}{1\,.\,2\,.\,3\ldots\ldots\text{to } r \text{ factors}}a^{n-r}b^r$.

17. $\dfrac{n(n-1)(n-2)\ldots\ldots\text{to } r-2 \text{ factors}}{1\,.\,2\,.\,3\ldots\ldots\text{to } r-2 \text{ factors}}a^{n-(r-2)}b^{r-2}$.

18. $-56w^{\frac{13}{2}}v^{\frac{11}{2}}$. 19. $210a^{-\frac{8}{5}}b^{-\frac{22}{5}}$. 20. 6,0368+.

Page 230.

2. $\log_2 64 = 6$.
3. $\log_5 25 = 2$.
4. $\log_3 81 = 4$.
5. $\log_4 64 = 3$.
6. $\log_5 125 = 3$.
7. $\log_7 343 = 3$.
8. $\log_{10} 10000 = 4$.
9. $\log_5 15625 = 6$.
10. $\log_6 \frac{1}{36} = -2$.
11. $\log_4 \frac{1}{64} = -3$.
12. $\log_{16} 2 = \frac{1}{4}$.
13. $\log_8 \frac{1}{4} = -\frac{2}{3}$.
15. $2^4 = 16$.
16. $4^2 = 16$.
17. $3^3 = 27$.
18. $10^3 = 1000$.
19. $6^3 = 216$.
20. $4^4 = 256$.
21. $8^{\frac{2}{3}} = 4$.
22. $27^{\frac{4}{3}} = 81$.
23. $7^{-2} = \frac{1}{49}$.
24. $3^{-5} = \frac{1}{243}$.
25. $10^{-3} = 0.001$.
26. $100^{-\frac{3}{2}} = 0.001$.
27. 4.
28. 3.
29. 2.
30. −4.
31. 3.
32. −1.
33. −2.
34. −2.
35. 3.
36. −3.
37. 0.
38. 0.
39. 4.
40. 64.
41. 1024.
42. $\frac{1}{16}$.
43. $\frac{1}{256}$.
44. 2.
45. $\frac{1}{3}$.
46. $\frac{1}{8}$.

Page 233.

1. 1.
2. 2.
3. −2.
4. −4.
5. 0.
6. 4.
7. 0.
8. 0.
9. −1.
10. 3.
11. 1.
12. −3.

Page 236.

1. 2.3324.
2. 2.8280.
3. 2.9731.
4. 2.8555.
5. 0.6646.
6. 1.2989.
7. 2.9191.
8. 1.2041.
9. 1.4624.
10. 0.9294.
11. 2.
12. 2.9542.
13. 0.
14. 0.3010.
15. 3.2263.
16. 1.5349.
17. 2.0022.
18. 2.9670.
19. $\bar{1}.9361$.
20. $\bar{3}.4983$.
21. $\bar{2}.8353$.
22. 3.6302.
23. 0.0398.
24. $\bar{4}.0792$.
25. 1.9999.
26. 3.3079.
27. $\bar{3}.9208$.
28. $\bar{1}.7025$.
29. 4.8363.
30. $\bar{2}.8739$.

Page 237.

1. 472.
2. 54.8.
3. 1210.
4. 6.18.
5. 0.0789.
6. 0.0095.
7. 0.415.
8. 1.69.
9. 57.3.
10. 7270.
11. 1.53.
12. 0.289.
13. 0.0427.
14. 488333.
15. 300.8.
16. 4007.
17. 0.005315.
18. 790.67.
19. 10223.8.
20. 29.53.
21. 0.09474.
22. 7.015.
23. 0.25541.
24. 0.00001052.
25. 1094.5.
26. 0.00029893.
27. 7.4733.
28. 37.445.

Page 239.

1. $\log_a x+\log_a y+\log_a z+\log_a w$.
2. $2\log_a x+2\log_a y$.
3. $\log_a 2+\log_a x+2\log_a y+3\log_a z$.
4. $2\log_a x+\log_a y-3\log_a z$.
5. $\log_a 5+\log_a x-\log_a y-\log_a z-\log_a w$.
6. $\log_a 8+2\log_a x+2\log_a y-3\log_a z-\log_a w$.
7. $2(\log_a x+\log_a y-\log_a z-\log_a w)$.
8. $\frac{2}{3}\log_a x-\frac{3}{4}\log_a y$.
9. $\frac{1}{3}\log_a x+\frac{1}{3}\log_a y-\frac{1}{4}\log_a z$.
10. $\frac{1}{2}\log_a x-\frac{1}{2}\log_a y$.
11. $\frac{1}{2}\log_a x+\frac{1}{2}\log_a y-\log_a z-\frac{1}{2}\log_a w$.
12. $\frac{1}{3}\log_a x+\frac{2}{3}\log_a y-\frac{1}{2}\log_a z-\frac{1}{3}\log_a w$.
13. $\frac{1}{4}\log_a z+\frac{3}{4}\log_a y-\frac{1}{3}\log_a x$.
14. $\frac{1}{2}\log_a x-\frac{1}{2}\log_a y+\frac{1}{2}\log_a z-\frac{1}{2}\log_a w$.
15. $\frac{1}{12}\log_a x-\frac{1}{12}\log_a y$.
16. $\frac{1}{3}\log 1728$.
17. $\frac{1}{2}\log 2+\frac{1}{3}\log 9$.
18. $\frac{1}{2}\log 98-\frac{1}{3}\log 56$.

Pages 241, 243.

1. 23.41.
2. 93.8.
3. 0.00417.
4. 248.43.
5. 25.635.
6. 0.1469.
7. 160.9.
8. 5428.7.
9. 0.08935.
10. 18.279.
11. 0.005164.
12. 9.8825.
13. 7.98.
14. 24.383.
15. 0.0060487.
16. 0.0030085.
17. 2.9538.
18. 0.9764.
19. 1.09325.
20. 18.175.
21. 0.00000000682.
22. 4.018.
23. 1.6727.
24. 4.537.
25. 1.5608.
26. 0.68114.
27. 4.116.
28. 0.001205.
29. 132.848.
30. 3025.71.
31. 428.
32. 5.00375.
33. 5.51.
34. 0.05639.
35. 0.6488.
36. 1.602.
37. 0.71016.
38. 0.063485.
39. 229.33 in.
40. 169.24 sq. in.
41. 19156.9 cu. in.
42. 5.411 ft.
43. 2.455 ft.
44. 22.805 in.
45. 103.92 sq. in.
46. 3827.5 sq. ft.
47. 0.5176.
48. 0.2610.
49. 67,500,000 mi.
50. 7.363.
51. 5.305 sq. ft.
52. $7590.
53. $7238.33.
54. $269.13.

Page 244.

1. $x=4$.
2. $x=2$.
3. $x=3$.
4. $x=2.47-$.
5. $x=1.55-$.
6. $x=8.63+$.
7. $x=2.16+$.
8. $x=3.39-$.
9. $x=3$.
10. $x=13$.
11. $x=5$.
12. $x=7.84+$.
13. 17.7+ yr.
14. 23.5+ yr.
15. 17.5 yr.
16. 20 yr.

Pages 244, 245.

4. 14.53.
5. 85.68.
6. $x=2.24-$.
7. $x=1.63+$.
8. $x=1$.
9. $x=\frac{1}{2}$.
10. $8937.50.
11. $1897.39.
12. $2120.

Pages 245, 251.

1. $(9n-a^2+a)(9n-a^2-a)$.
2. $(3a+b)(a-3b)$.
3. $(2x+2a-t)(2x-2a+t)$.
4. $(N^2+3N+9)(N^2-3N+9)$.
5. $V(V^2-5)(V^2-6)$.
6. $2(2k+1)(k+1)$.
7. $2(5R+6r)(3R+2r)$.
8. $(p^2+3pq-q^2)(p^2-3pq-q^2)$.
9. $(4N+1)(3N-2)$.
10. $(v^2+2)(2v^2-7)$.
11. $(x^4+1)(x^2+1)(x+1)(x-1)$.
12. $ab(a^2+b^2)(a+b)(a-b)$.
13. $(V^8+T^8)(V^4+T^4)(V^2+T^2)(V+T)(V-T)$.
14. $(m^4+n^4)(m^8-m^4n^4+n^8)$.
15. $(1+2a^3)(1-2a^3+4a^6)$.
16. $(a+1)(a-b-1)$.
17. $(t+1)(t-1)^2$.
18. $(y+2)(y-2)(3y+1)(3y-1)$.
19. $(n+1)(n-2)(n^4-2n^3+3n^2-2n+4)$.
20. $(H-1)(H+1)^2$. **21.** $(v-1)(v-2)(v-4)$. **22.** $(a+1)(a-1)(b+1)(b-1)$.
23. $(a+b+c)(a+b-c)(a-b+c)(a-b-c)$.
24. $(v^2+v-6w)(v^2-v+6w)$.
25. $(R+N)(R^2-RN+N^2+1)$.
26. $(h+t)(h-t-1)$.
27. $(n-1)(n-2)(n-3)$.
28. $(A+5)(A^2-5A-2)$.
29. $(m-2n^2)(m^4+2m^3n^2+4m^2n^4+8mn^6+16n^8)$.
30. $x(x-1)(x^3-x-1)$.
31. $(e^n+1)(e^{2n}+1)$.
32. $(G+a)(3G+b)$.
33. $(a+b)(a+c)(b+c)$.
34. $(5n-2+2a+3b)(5n-2-2a-3b)$.
35. $(6x^{n+3}-y^{n-1})(x^{n+3}-4y^{n-1})$.

36. $(x+y)(x-z)(y+z)(y-z-x)$.

37. $(n+a)(n+b+c)$.

38. $(a+x+b+c)(a+x-b-c)$.

39. $(m+n)(t+a)(t+b)(t-b)$.

40. $-3n^3+10n^2+2n+1$. **41.** $3a-1$. **42.** $a+2b-5c$.

43. $r^{3n}-r^{n+\frac{2}{n}}-r^n-r^{\frac{1}{n}}-r^{2n+\frac{1}{n}}+r^{\frac{3}{n}}$.

44. $t+2$.

45. $175m^3n^3(x-m)^2(x+m)^2(x+2m)$.

46. $x+y$; $(x^2+y^2)(x^3+y^3)(x^3-y^3)$.

47. Yes; No; Use rem. theorem.

48. $(p+1)(p-5)(2p+3)(3p^2+2)$.

49. $n=2$.

50. $3-n-5n^2+2n^3$.

51. $1-y^2+2y^3-y^4$.

52. $x^{-2}-x^{-1}y^{\frac{1}{2}}+y$.

53. $\dfrac{n+2}{n+1}$.

54. $\dfrac{2k^2-2}{4k^2+4k+1}$.

55. $\dfrac{r^4+r^2-2r+1}{r^7-r}$.

56. $(x-y)(y-z)(z-x)$.

57. $\dfrac{mn(n-m)(m^2-n^2)}{(m+n)(m^2+n^2)(m^2+mn+n^2)}$.

58. $\dfrac{1-x}{y}$.

59. $\dfrac{m}{1-n}$.

60. 1.

61. $\dfrac{m^4+6m^2n^2+n^4}{4m^3n+4mn^3}$.

62. 1.

64. $n=2\frac{11}{20}$.

65. $n=\frac{1}{12}$.

66. $x=1.239-$, $0.162-$.

67. $x=0.910-$.

68. $x^2-6x+4=0$.

69. Imaginary.

70. Impossible.

71. $n=5$.

72. $r=7$.

73. $v=2, t=5$; $v=-2, t=-5$; $v=\frac{9}{4}\sqrt{2}, t=-\frac{1}{2}\sqrt{2}$; $v=-\frac{9}{4}\sqrt{2}, t=\frac{1}{2}\sqrt{2}$.

74. $x=5, y=3$; $x=-3, y=-5$; $x=3, y=5$; $x=-5, y=-3$.

75. $a=-3$, $b=3$, $c=\frac{1}{2}$.

76. $n=\dfrac{d-2a\pm\sqrt{d^2-4ad+4a^2+8dS}}{2d}$.

77. $\sqrt{2}-1$.

78. $4-\sqrt{15}$.

79. $\dfrac{a-\sqrt{ab}+b}{a^2+ab+b^2}$.

80. 0.726.

81. $\frac{1}{2}\sqrt[6]{288}$.

82. $a^{\frac{5}{2}}c^{\frac{5}{6}}$.

83. $\frac{1}{64}m^{-\frac{1}{2}}n$.

84. $a^{-\frac{1}{6}}b^{-\frac{5}{4}}$.

85. $x^{15}y^{18}z^{-8}$.

86. $\dfrac{a}{bc^2}\sqrt[6]{b^4c^5}$.

87. $x=\sqrt{3}$.

92. 154.

93. 6, 7, 8, 9, 10.

94. $a, a+\frac{b-a}{n+1}, a+\frac{2b-2a}{n+1}, \ldots\ldots b.$

95. $-1860.$

96. 18th.

97. 24.

98. $34\frac{11}{64}.$

99. 6, 18, 54, 162, 486.

100. $a, a\sqrt[4]{\frac{b}{a}}, a\sqrt{\frac{b}{a}}, a\sqrt[4]{\frac{b^3}{a^3}}, b.$

101. $42+21\sqrt{2}.$

102. 154.9681956 cu. in.

103. $6\frac{5}{11}.$

104. 784 ft.

105. $\frac{1}{3}.$

106. $-\frac{117572a^5b^5\sqrt{ab}}{59049}.$

107. $4.946+.$

108. $1+\frac{1}{2}n^{-\frac{1}{2}}+\frac{3}{8}n^{-1}+\frac{5}{16}n^{-\frac{3}{2}}+\frac{35}{128}n^{-2}+\ldots\ldots$

109. $a^{2l}b^{2m}c^{2n}.$

110. $-2;\ \frac{1}{3};\ -1.$

111. $\sqrt[mn]{a^{my-nx}}.$

112. $a=1.$

113. $4m^{\frac{2}{3}}+n^{\frac{2}{3}}.$

114. $2n^2-8m.$

116. $x=15.05+.$

117. $\overline{1}.9112.$

118. $E=0.5029V^2D;\ D=\frac{1.9881E}{V^2}.$

119. $C'=\frac{EC}{E+Cr}.$

120. $h=\frac{R(\sqrt{g}-\sqrt{g'})}{\sqrt{g'}}.$

121. $u=\frac{t^2g+2s}{2t}.$

122. $v=7.64+.$

123. $r=\pm\frac{y}{\sqrt{4\pi y-4\pi^2R^2}};\ y=0.032+.$